GLSMs on the Ω-deformed S^2

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Observables of $\mathcal{N} = (2, 2)$ GLSMs

Consider a two-dimensional $\mathcal{N} = (2, 2)$ GLSM with at least one $U(1)$ factor. We have the complexified FI parameter

$$\tau = \frac{\theta}{2\pi} + i\xi$$

which is classicaly marginal.

Schematically, expectation values of appropriately supersymmetric local operators $\mathcal{O}$ have the expansion

$$\langle \mathcal{O} \rangle \sim \sum_k q^k Z_k(\mathcal{O}) \; , \qquad q = e^{2\pi i \tau} \; .$$

The 2d instantons are *gauge vortices*.

(In the case of $\mathcal{O} = \mathbf{1}$ on the plane with the Ω-background, this is sometimes known as “vortex counting”.)

$S^2_{\epsilon_\Omega}$ correlators

We will consider a generic gauged linear sigma model (GLSM) on a sphere with an $U(1)$-equivariant deformation of the A-twist —also known as Ω-deformation.

Consider S^2 with a $U(1)$ isometry.

$$V = i(z\partial_z - \bar{z}\partial_{\bar{z}}) = \partial_\varphi$$

We preserve two supercharges $\mathcal{Q}$, $\tilde{\mathcal{Q}}$ such that

$$\mathcal{Q}^2 = 0 \,, \qquad \tilde{\mathcal{Q}}^2 = 0 \,, \qquad \{\mathcal{Q}, \tilde{\mathcal{Q}}\} = Z + \epsilon_\Omega \mathcal{L}_V \,.$$

Note that $Q_A = \mathcal{Q} + \tilde{\mathcal{Q}}$ is the nilpotent supercharge in the topological A-model when $\epsilon_\Omega = 0$. [Witten, 1988]

$S^2_{\epsilon_\Omega}$ correlators

We will derive a formula for GLSM supersymmetric observables on S^2_Ω of the schematic form:

$$\langle \mathcal{O} \rangle = \sum_k q^k \oint_{\mathcal{C}} d^r\sigma \, Z_k^{1-loop} \mathcal{O}(\sigma) \, ,$$

valid for any standard GLSM. The simplifies the computations of [Morrison, Plesser, 1994] and generalizes them to non-Abelian theories.

This can also be seen as a gauge-theory generalization of Vafa's formula [Vafa, 1990] for A-twisted Landau-Ginzburg models of twisted chiral multiplets Ω (equivalently, B-twisted LG of chirals):

$$\langle \mathcal{O} \rangle = \oint \frac{d\omega_1 \wedge \cdots \wedge d\omega_n \, \mathcal{O}(\omega)}{\partial_1 \hat{W} \cdots \partial_n \hat{W}}$$

Some further motivations

In field theory:

- These 2d theories appear on the worldvolume of *surface operators* in 4d $\mathcal{N}=2$ theories.
- Our 2d setup can also be uplifted to 4d $\mathcal{N}=1$ on $S^2 \times T^2$.
  [C.C., Shamir, 2013; Honda, Yoshida, 2015; Benni, Zaffaroni, 2015]

In string theory or "quantum geometry":

- Think in term of a target space X_d with $\xi \sim \mathrm{vol}(X_d)$. New localization results can give new tools for enumerative geometry.
  [Jockers, Kumar, Lapan, Morrison, Romo, 2012]

Supersymmetric backgrounds in 2d

Rigid supersymmetry on Σ: Topological twist [Witten, 1988].
Is there anything else?

Assume the $\mathcal{N} = (2,2)$ theory has a $U(1)_V$ R-symmetry.
Supergravity background:

$$g_{\mu\nu}\,, \qquad A_\mu^{(R)}\,, \qquad C_\mu\,, \qquad \tilde{C}_\mu$$

We consider $\Sigma = S^2$ with a $U(1)$ isometry, and:

$$ds^2 = \sqrt{g}(|z|^2)dzd\bar{z}\,, \qquad A_\mu^{(R)} = \frac{1}{2}\omega_\mu\,, \qquad C_\mu = \frac{1}{2}\epsilon_\Omega V_\mu\,, \qquad \tilde{C}_\mu = 0\,.$$

It preserves two supercharges, giving a one-parameter deformation of the A-twist, or "Ω-background". [C.C., Cremonesi, 2014]

GLSMs: Lightning review

Let us consider 2d $\mathcal{N} = (2,2)$ supersymmetric GLSM on this S^2_Ω.

We have the following field content:

- Vector multiplets $\mathcal{V}_a$ for a gauge group G, with Lie algebra $\mathfrak{g}$.
- Chiral multiplets Φ_i in representations $\mathfrak{R}$ of $\mathfrak{g}$.

We also have interactions dictated by:

- A superpotential $W(\Phi)$
- A twisted superpotential $\hat{W}(\sigma)$, where $\sigma \subset \mathcal{V}$.

Assumption: The classical twisted superpotential is linear in σ:

$$\hat{W} = \sum_a \tau_a \sigma_a \ .$$

That is, we turn on one FI parameter for each $U(1)$ factor in G.

For simplicity of presentation, we can focus on $G = U(1)$, with chiral multiplets Φ of gauge charge Q_i and R-charge r_i.

The coupling τ has a one-loop running

$$\tau(\mu) = \tau(\mu_0) - \frac{b_0}{2\pi i} \log\left(\frac{\mu}{\mu_0}\right) , \qquad b_0 \equiv \sum_i Q_i \, .$$

Classically, the theory has an axial R-symmetry $U(1)_A$, which is anomalous:

$$\varphi \to e^{i\alpha r_A}\varphi, \qquad \theta \to \theta + 2b_0\alpha \, ,$$

so $U(1)_A \to \mathbb{Z}_{2b_0}$ at one-loop. We can define the RG invariant scale

$$\Lambda = \mu \, \exp\left(\frac{2\pi i\tau(\mu)}{b_0}\right) \sim \mu \, q^{\frac{1}{b_0}} \, .$$

At $\xi \gg 0$, Λ is small and perturbative physics is reliable.

Examples with $G = U(1)$

Example 1: $\mathbb{C}P^{n-1}$ model. With n chirals with $Q_i = 1$, $r_i = 0$.
The dynamical scale is

$$\Lambda = \mu q^{\frac{1}{n}} \ .$$

$b_0 = n$. For $\xi \gg 0$, target space is $\mathbb{C}P^{n-1}$.

For $n = 1$: Abelian Higgs model (AHM). Target is a point.

Example 2: The quintic model. 5 chirals x_i with $Q_i = 1$, $r_i = 0$, and one chiral p with $Q_p = -5$, $r_p = 2$, with a superpotential

$$W = pF(x_i)$$

F is homogeneous of degree 5.

$b_0 = 0$. For $\xi \gg 0$: quintic CY_3 in $\mathbb{C}P^4$.

A non-Abelian example

More recently, non-Abelian cases have been studied in greater detail.
[Hori, Tong, 2006; Jockers, Kumar, Morrison, Lapan, Romo, 2012]

Example 3: The Rødland CY_3 model. Consider $G = U(2)$ with 7 chirals Φ_i in the fundamental with $r_i = 0$ and 7 chirals P_α in the $\det^{-1}$ rep. with $r_\alpha = 2$. We have the baryons

$$B_{ij} = \epsilon_{a_1 a_2} \Phi_i^{a_1} \Phi_j^{a_2} \ ,$$

charged under the diagonal $U(1) \subset U(2)$. Let $G^\alpha(B)$ be polynomials of degree one in B_{ij}. We have a superpotential

$$W = \sum_{\alpha=1}^{7} P_\alpha G^\alpha(B)$$

The target space for $\xi \gg 0$ is a complete intersection in the Grassmanian $G(2,7)$ known as the Rødland CY_3.

The Coulomb branch formula

Supersymmetric observables: Gauge-invariant operators in σ inserted at the north or south poles of S^2_Ω:

$$\langle \mathcal{O}_N(\sigma)\mathcal{O}_S(\sigma)\rangle$$

This is what we shall compute explicitly, as a function of q and ϵ_Ω.

It can be computed by supersymmetric localization, following previous works—in particular [Benini, Eager, Hori, Tachikawa, 2013; Hori, Kim, Yi, 2014]. The basic idea is to localize "on the Coulomb branch" of the $\mathcal{N} = (2,2)$ theory.

This can be contrasted with "Higgs branch" localization, where we have a sum over vortices. [Morrison, Plesser, 1994]

We still have a sum over fluxes (instanton sectors), however.

The Coulomb branch formula

We find:

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{\mathcal{C}^{(JK)}} \prod_{a=1}^{\mathrm{rank}(G)} \left[d\hat{\sigma}_a \, q_a^{k_a} \right] Z_k^{1-\mathrm{loop}}(\hat{\sigma}) \, \mathcal{O}_{N,S}\left(\hat{\sigma} \mp \frac{1}{2} \epsilon_\Omega k \right)$$

- $|W|$ denotes the order of the Weyl group.
- The contour $\mathcal{C}^{(JK)}$ essentially picks all the poles from "positively charged" fields. More precisely, it is a Jeffrey-Kirwan residue.
- The result depends on the FI parameters explicitly and through the definition of the contour.
- The sum is over all fluxes k_a's. However, only some chambers in $\{k_a\}$ effectively contribute residues.

The Coulomb branch formula

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{\mathcal{C}^{(JK)}} \prod_{a=1}^{\text{rank}(G)} \left[d\hat{\sigma}_a \, q_a^{k_a} \right] Z_k^{1-\text{loop}}(\hat{\sigma}) \, \mathcal{O}_{N,S}\left(\hat{\sigma} \mp \frac{1}{2} \epsilon_\Omega k \right)$$

- The distinct q_a's are somewhat formal. We have as many actual q's as the number of $U(1)$ factors in $\mathfrak{g}$.
  For instance, for $G = U(N)$ we have $q_a = q$ for $a = 1, \cdots, N$.
- The one-loop term reads

$$Z_k^{1-\text{loop}}(\hat{\sigma}) = \prod_{\alpha \in \mathfrak{g}} Z_k^W(\alpha(\hat{\sigma})) \prod_{\rho \in \mathfrak{R}} Z_k^\Phi(\rho(\hat{\sigma}))$$

from the W-bosons and chiral multiplets.

The Coulomb branch formula

$$\langle \mathcal{O}_{N,S}(\sigma) \rangle = \frac{1}{|W|} \sum_k \oint_{\mathcal{C}(JK)} \prod_{a=1}^{\text{rank}(G)} \left[d\hat{\sigma}_a \, q_a^{k_a} \right] Z_k^{1-\text{loop}}(\hat{\sigma}) \, \mathcal{O}_{N,S} \left(\hat{\sigma} \mp \frac{1}{2} \epsilon_\Omega k \right)$$

- For chiral multiplet of $U(1)$ charge Q and R-charge r, we have

$$Z_k^{\Phi}(\hat{\sigma}) \propto \frac{\Gamma\left(Q\frac{\hat{\sigma}}{\epsilon_\Omega} - Q\frac{k}{2} + \frac{r}{2}\right)}{\Gamma\left(Q\frac{\hat{\sigma}}{\epsilon_\Omega} + Q\frac{k}{2} - \frac{r}{2} + 1\right)} = \frac{1}{\left(Q\frac{\hat{\sigma}}{\epsilon_\Omega} - Q\frac{k}{2} + \frac{r}{2}\right)_{Qk-r+1}} .$$

- The W-boson W^α contributes exactly like a chiral of R-charge $r = 2$ and gauge charges α.
- Twisted masses m_i for global symmetries can be introduced in the obvious way.

A-model Coulomb branch formula ($\epsilon_\Omega = 0$)

When $\epsilon_\Omega \to 0$, the Coulomb branch formula simplifies and we can perform the sum over fluxes explicitly, leading to:

$$\langle \mathcal{O}(\sigma) \rangle_{\epsilon_\Omega=0} = \frac{1}{|W|} \oint \prod_{a=1}^{\text{rank}(G)} \left[d\hat{\sigma}_a \, \frac{1}{1 - e^{2\pi i \partial_{\sigma_a} \hat{W}_{\text{eff}}}} \right] Z_0^{1-\text{loop}}(\hat{\sigma}) \, \mathcal{O}(\hat{\sigma})$$

Here $\hat{W}_{\text{eff}}(\sigma)$ is the (one-loop exact) effective twisted superpotential on the Coulomb branch, and

$$Z_0^{1-\text{loop}}(\hat{\sigma}) = \prod_{\alpha \in \mathfrak{g}} \alpha(\hat{\sigma}) \prod_{\rho \in \mathfrak{R}} \rho(\sigma)^{r_\rho - 1}$$

A-model Coulomb branch formula ($\epsilon_\Omega = 0$)

Finally, if the critical locus

$$e^{2\pi i \partial_{\sigma_a} \hat{W}_{\rm eff}} = 1$$

consists of isolated points (such as typically happens for massive theories), we can write the contour integral as

$$\langle \mathcal{O}(\sigma) \rangle_{\epsilon_\Omega = 0} = \sum_{\hat{\sigma}^* | d\hat{W} = 0} \frac{Z_0^{1-\text{loop}}(\hat{\sigma}^*)\, \mathcal{O}\,(\hat{\sigma}^*)}{H(\hat{\sigma}^*)}\,, \qquad H = \det \partial_{\sigma_a} \partial_{\sigma_b} \hat{W}$$

This same formula appeared in [Nekrasov, Shatashvili, 2014] and also in [Melnikov, Plesser, 2005].

Abelian examples

Example 1. In the $\mathbb{C}P^{n-1}$ model, we have

$$\begin{aligned}\langle \mathcal{O}_{N,S}(\sigma)\rangle &= \sum_{k=0}^{\infty} q^k \oint \partial\hat{\sigma} \left(\frac{\Gamma\left(\hat{\sigma}-\frac{k}{2}\right)}{\Gamma\left(\hat{\sigma}+\frac{k}{2}+1\right)}\right)^n \mathcal{O}\left(\hat{\sigma}\mp\frac{k}{2}\right) \\ &= \sum_{k=0}^{\infty} q^k \oint \partial\hat{\sigma} \left(\prod_{p=0}^{k} \frac{1}{\hat{\sigma}-k/2+p}\right)^n \mathcal{O}\left(\hat{\sigma}\mp\frac{k}{2}\right)\end{aligned}$$

For $n=1$, this gives the Abelian Higgs model result above.

In the A-model limit, this simplifies to

$$\langle \mathcal{O}(\sigma)\rangle_{\epsilon_\Omega=0} = \oint d\hat{\sigma} \left(\frac{1}{1-q\hat{\sigma}^{-n}}\right)\frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^n} = \oint d\hat{\sigma}\frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^n-q}$$

Abelian examples

Example 2. For the quintic model, we can massage the result to

$$\langle \mathcal{O}_N(\sigma) \rangle = \frac{1}{{\epsilon_\Omega}^3} \sum_{k=0}^{\infty} q^k \oint ds \frac{\prod_{l=0}^{5k}(-5s-l)}{\prod_{p=0}^{k}(s+p)^5} \mathcal{O}(\epsilon_\Omega s)$$

In the A-model limit, we obtain

$$\langle \mathcal{O}(\sigma) \rangle_{\epsilon_\Omega=0} = \sum_{k=0}^{\infty} (-5^5 q)^k \oint d\hat{\sigma} \frac{5\hat{\sigma}\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^5} = \frac{5}{1+5^5 q} \oint d\hat{\sigma} \frac{\mathcal{O}(\hat{\sigma})}{\hat{\sigma}^4}$$

For any ϵ_Ω, we find $\langle \sigma^n \rangle = 0$ if $n = 0, 1, 2$, and

$$\left\langle \sigma^3 \right\rangle = \frac{5}{1+5^5 q}\,, \qquad \left\langle \sigma^4 \right\rangle = 10\epsilon_\Omega \frac{5^5 q}{(1+5^5 q)^2}\,, \cdots$$

in perfect agreement with [Morrison, Plesser, 1994].

Non-Abelian example

For simplicitly and in order to compare to known results, let us focus on the $\epsilon_\Omega = 0$ limit here.

Example 3. For the Rødland CY_3 model, our formula reads

$$\frac{1}{2}\sum_{k_1,k_2=0}^{\infty} q^{k_1+k_2} \oint_{(\hat{\sigma}_a=0)} d\hat{\sigma}_1 d\hat{\sigma}_2 (\hat{\sigma}_1-\hat{\sigma}_2)^2 \frac{(-\hat{\sigma}_1-\hat{\sigma}_2)^{7(1+k_1+k_2)}}{\hat{\sigma}_1^{7(1+k_1)}\hat{\sigma}_2^{7(1+k_2)}} \mathcal{O}(\hat{\sigma}) \; .$$

The observables are polynomials in the gauge invariants

$$u_1(\sigma) = \mathrm{Tr}(\sigma) = \sigma_1 + \sigma_2 \; , \qquad u_2(\sigma) = \mathrm{Tr}(\sigma^2) = \sigma_1^2 + \sigma_2^2 \; .$$

Non-Abelian example

Example 3, continued. The only non-vanishing correlators are given by:

$$\begin{aligned}\langle u_1(\sigma)^3\rangle &= \frac{42-14q}{1-57q-289q^2+q^3}\,,\\ \langle u_2(\sigma)u_1(\sigma)\rangle &= \frac{14+126q}{1-57q-289q^2+q^3}\,.\end{aligned}$$

Note:

- The Yukawa $\langle u_1(\sigma)^3\rangle$ was computed by mirror symmetry in [Batyrev *et al.*, 1998]. The second correlator is a new result.
- Many more examples can be considered. In particular, we can study the $PAX/PAXY$ models of [Jockers, Kumar, Morrison, Lapan, Romo, 2012] for determinantal CY varieties.

Conclusions and Outlook

Conclusions:

- We considered $\mathcal{N} = (2, 2)$ supersymmetric GLSMs on the Ω-deformed sphere, S^2_Ω.
- We derived a simple Coulomb branch formula for the S^2_Ω observables.
- When $\epsilon_\Omega = 0$, this gives a simple, general formula for A-twisted GLSM correlation functions.

Outlook:

- A more satisfactory understanding of the physics of the ϵ_Ω deformation is needed.
- A similar computation can be performed on $\mathbb{R}^2_\Omega$. The "Coulomb branch" formula in that case simplifies the study of vortex partition functions considerably.
- Our new tools might help to better understand mirror symmetry for Calabi-Yau's engineered by non-Abelian GLSMs.