Thermal Corrections to Rényi Entropies on $\mathbb{S}^{d-1}$

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Outline

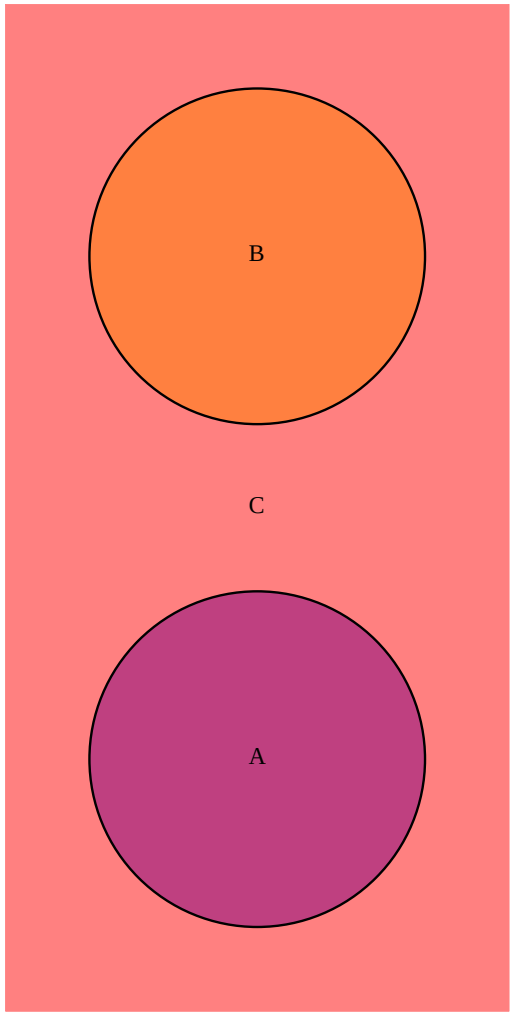
1. What are Entanglement and Rényi Entropy
2. Holographic EE and Ryu-Takayanagi
3. Thermal corrections for CFT's
4. Thermal Corrections for Scalars
5. Thermal Corrections for Fermions

Entanglement and Rényi Entropy

- The density matrix for a state ψ
  - $\rho = |\psi\rangle\langle\psi|$
- If the Hilbert space factors we can define the reduced density matrix
  - $H = H_A \otimes H_{\bar{A}}$
  - $\rho_A = \mathrm{tr}_{\bar{A}}\rho$
- Entanglement Entropy
  - $S_{EE} = -\mathrm{tr}\rho_A \log\, \rho_A$
- The Rényi Entropy
  - $S_n = \frac{1}{1-n}\log\, \mathrm{tr}\left(\rho_A^n\right)$
  - $S_{EE} = \lim_{n\to 1} S_n$

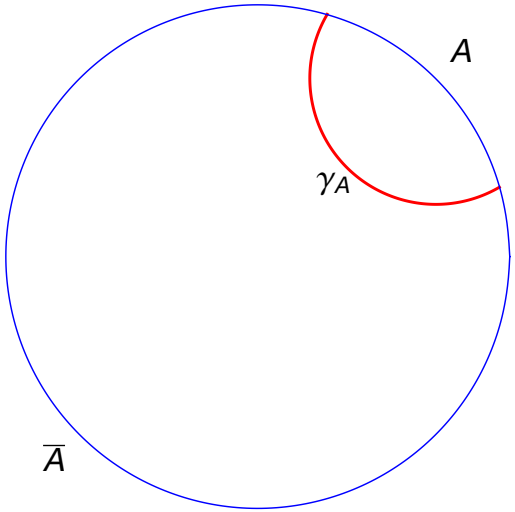
Properties of Entanglement Entropy

- For pure states
  - $S_A = S_{\bar{A}}$
  - This can be proven via Schmidt decomposition
- Strong Subadditivity
  - $S_{A+C} + S_{B+C} \geq S_{A+B+C} + S_C$
- Mutual information
  - $I(A, B) = S_A + S_B - S_{A+B}$
- Area Law
  - $S_A \sim \frac{\text{Area}(\partial A)}{\epsilon^{d-1}} + \ldots$



Holographic Entanglement Entropy

- Ryu-Takayanagi formula allows calculation for QFT's with holographic dual (arXiv:hep-th/0605073, arXiv:1102.0440)
  - $S_{EE}(A) = \frac{\text{Area}(\gamma_A)}{4G_N}$
- Proof of c-theorem (arXiv:1011.5819)
- Black holes (arXiv:1104.3712)
- RG flow (arXiv:1202.5650)
- Phase transitions (arXiv:hep-th/0510092)

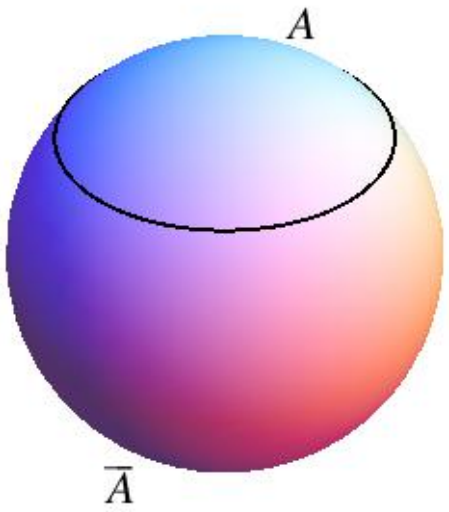


Entanglement Entropy Previous results

- For $d = 1 + 1$ CFT
  - $S_n = \frac{c(1+n)}{6n} \log(\frac{L}{\pi\epsilon} \sin(\pi\ell/L)) + c'$
- For $d = 1 + 1$ CFT at small finite temperature
  - $S_n(T) - S_n(0) = \frac{g}{1-n} \left[\frac{1}{n^{2\Delta}} \frac{\sin^{2\Delta}(\pi\ell/L)}{\sin^{2\Delta}(\pi\ell/nL)} - n \right] e^{-2\pi\Delta\beta/L} + o(e^{-2\pi\Delta\beta/L})$
- For CFT on S^{d-1}
  - $S_{EE}(T) - S_{EE}(0) = g\Delta I_d(\theta) e^{-\beta\Delta/R} + o(e^{-\beta\Delta/R})$
  - $I_d(\theta) = 2\pi \frac{\mathrm{Vol}(S^{d-2})}{\mathrm{Vol}(S^{d-1})} \int_0^\theta d\theta' \frac{\cos(\theta') - \cos(\theta)}{\sin(\theta)} \sin^{d-2}(\theta')$

CFT's on $\mathbb{R} \times S^{d-1}$

- We are interested in finding the EE and Rényi's on $\mathbb{R} \times S^{d-1}$
- We choose A to be a cap of open angle 2θ
- For small temperatures we can write the density matrix
  - $\rho = \frac{|0\rangle\langle 0| + \sum_i |\psi_i\rangle\langle\psi_i| e^{-\beta E_\psi} + \ldots}{1 + g e^{-\beta E_\psi} + \ldots}$
- Where ψ is the lowest excited state with energy E_ψ and degeneracy g
- Where $E_\psi = \Delta / R$



Rényi Entropy for small temperature

- We now need to calculate $\mathrm{tr}(\rho_A)^n$
  - $\mathrm{tr}(\rho_A)^n = \left(\frac{1}{1+ge^{-\beta E_\psi}+\ldots}\right)^n \mathrm{tr}\left[\mathrm{tr}_{\bar{A}}\left(|0\rangle\langle 0| + \sum_i |\psi_i\rangle\langle\psi_i| e^{-\beta E_\psi} + \ldots\right)^n\right]$

    $= \mathrm{tr}(\mathrm{tr}_{\bar{A}}|0\rangle\langle 0|)^n \left[1 + \left(\frac{\mathrm{tr}\left[\mathrm{tr}_{\bar{A}} \sum_i |\psi_i\rangle\langle\psi_i| (\mathrm{tr}_{\bar{A}}|0\rangle\langle 0|)^{n-1}\right]}{\mathrm{tr}(\mathrm{tr}_{\bar{A}}|0\rangle\langle 0|)^n} - g\right) ne^{-\beta E_\psi} + \ldots\right]$
- $\delta S_n \equiv S_n(T) - S_n(0)$
  - $\delta S_n = \frac{n}{1-n}\sum_i \left(\frac{\mathrm{tr}[\mathrm{tr}_{\bar{A}}|\psi_i\rangle\langle\psi_i|(\mathrm{tr}_{\bar{A}}|0\rangle\langle 0|)^{n-1}]}{\mathrm{tr}(\mathrm{tr}_{\bar{A}}|0\rangle\langle 0|)^n} - 1\right) e^{-\beta E_\psi} + o(e^{-\beta E_\psi})$

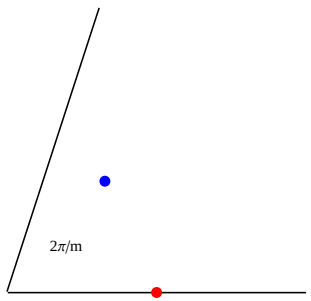
    $= \frac{n}{1-n}\sum_i \left(\frac{\langle\psi_i(z)\psi_i(z')\rangle_n}{\langle\psi_i(z)\psi_i(z')\rangle_1} - 1\right) e^{-\beta E_\psi} + o(e^{-\beta E_\psi})$
- Where we have defined $\langle\psi_i(z)\psi_i(z')\rangle_n$ as the 2pt function on n-sheeted cover of S^{d-1}

Conformal Transformations

- Calculating 2-pt functions on n-sheeted covers are hard
  - So we need a trick
- This comes in the form of a pair of conformal transformations
  - $$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$
    $$= \Lambda^2(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2)$$
    where
    $$t \pm r = \tan\left(\frac{\tau \pm \theta}{2}\right)$$
    $$\Lambda = \frac{1}{2}\sec\left(\frac{\tau+\theta}{2}\right)\sec\left(\frac{\tau-\theta}{2}\right)$$
- This is followed by a special conformal transformation $b^1 = 1/r$ and the rest of $b^\mu = 0$
  - $y^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$
- The combination of these two maps the n-sheeted cover of S^{d-1} to a cone of opening angle $2\pi n$

Method of Images

- We will analytically continue $n = 1/m$
- For free theories we can use the method of images
  - $\langle \psi(z)\psi(z')\rangle_{1/m} = \sum_{k=0}^{m-1} \langle \psi(z)\psi(z' + (0, 2\pi k/m, \vec{0}))\rangle_1$
- For our specific case z' $= (1,2\theta,\vec{0})$, $z = (1,0,\vec{0})$



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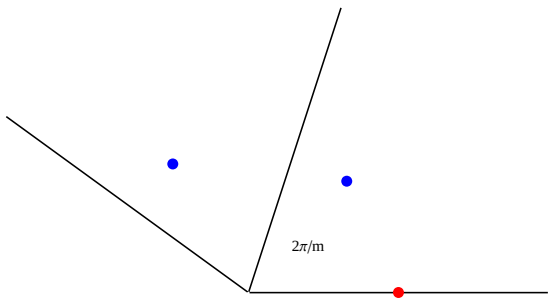

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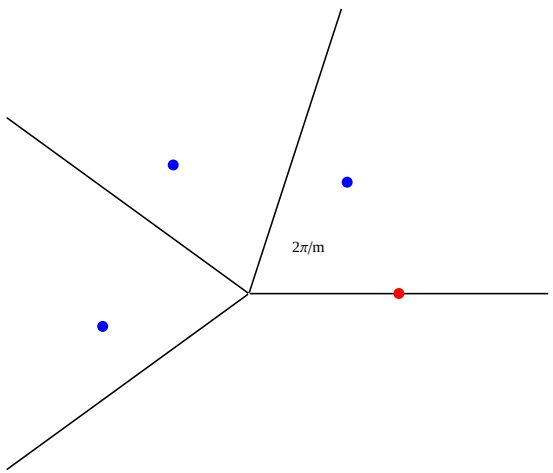

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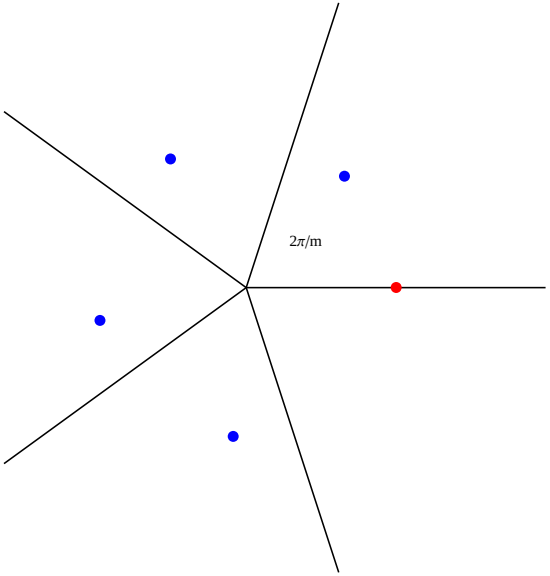

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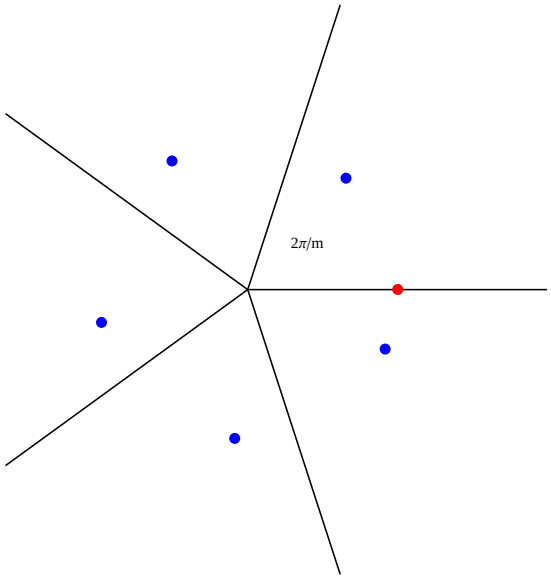


Conformally coupled Scalar

- Recall that for a free scalar
  - $\langle \phi(x)\phi(x')\rangle \sim (x-x')^{(2-d)/2}$
- The calculation we must do is then
  - $G^B_{(1/m,d)}(2\theta) \equiv \langle \phi(x)\phi(x')\rangle_{1/m} = \sum_{k=0}^{m-1} \csc^{d-2}(\theta + \pi k/m)$
- This formula satisfies the following recursion relation
  - $G^B_{(1/m,d+2)}(\theta) = \frac{1}{(d-1)(d-2)}\left[\left(\frac{d-2}{2}\right)^2 + \partial_\theta^2\right] G_{(1/m,d)}(\theta)$
- This means that we only have to calculate for $d=3$ and $d=4$ and then use the recursion relation

Scalar $d = 4$

- It turns out that even dimensions are easier (this will be true for fermions also)
- In fact, for $d = 4$ the answer is simply
  - $G^B_{(1/m,4)}(2\theta) = m^2 \csc^2(m\theta)$
  - $\delta S_n = (\csc(\theta/n)\sin(\theta)/n^2 - 1)/(1-n)e^{-\beta/R} + o(e^{-\beta/R})$
  - $\delta S_{EE} = 2 - 2\theta\cot(\theta)$
- These match the results above with one caveat $d \to d-2$

Scalar $d = 3$

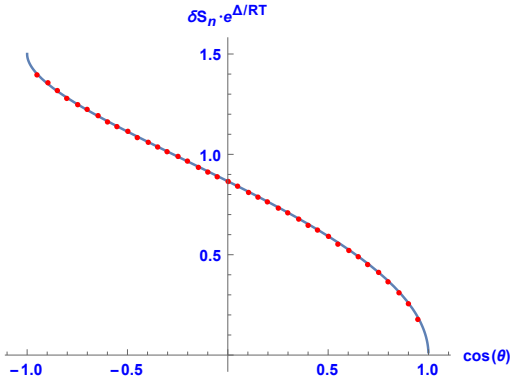
- We will use the following integral representation $\csc(y) = \int_0^\infty dx \frac{x^{y/\pi - 1}}{\pi(1+x)}$
- Plugging this in we get
  - $G^B_{(1/m,3)}(\theta) = \frac{1}{2\pi} \sum_{k=0}^{m-1} \int_0^\infty dx \frac{x^{\theta/2\pi + k/m - 1}}{1+x}$
- This integral does not have a closed form for general m
- But we can analytically continue the integral
  - $G^B_{(1,3)}(\theta) = \frac{1}{2\sin(\theta/2)}$
  - $G^B_{(2,3)}(\theta) = \frac{1-\theta/2\pi}{2\sin(\theta/2)}$
  - $S_2 = 2\theta/\pi$
- A Taylor expansion around $m = 1$ allows a calculation of the EE
  - $S_{EE} = \frac{\pi}{2}\tan(\theta/2)$

Numerical Check

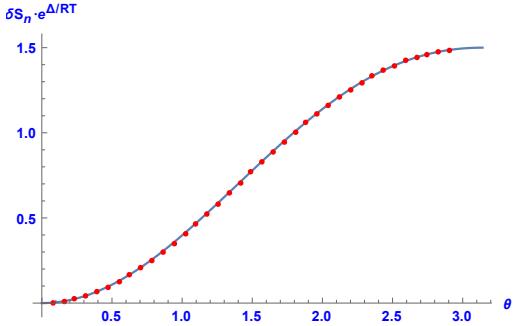
- We want to reduce the Hamiltonian to one dimension
  - $S = -\frac{1}{2}\int d^d x \sqrt{-g}\left[(\partial_\mu\phi)(\partial^\mu\phi) + \xi\,\mathcal{R}\,\phi^2\right]$
  - $\mathcal{H} = \frac{(R\sin\theta)^{d-2}\sqrt{h}}{2R}\left\{R^2\pi^2 + (\partial_\theta\phi)^2 + \frac{h^{ab}(\partial_a\phi)(\partial_b\phi)}{\sin^2\theta} + \frac{(d-2)^2}{4}\phi^2\right\}$
  - Where π is the conjugate momentum and h^{ab} is the metric on S^{d-1}
- Integrating over S^{d-2}
  - $H_{\vec{\ell}} =$
    $\frac{1}{2R^2}\int_0^\pi \left\{R^2\Pi^2_{\vec{\ell}} - \Phi_{\vec{\ell}}\partial^2_\theta\Phi_{\vec{\ell}} + \frac{1}{4}(2m+d-2)(2m+d-4)\frac{\Phi^2_{\vec{\ell}}}{\sin^2\theta}\right\}d\theta$
  - $H_{\vec{\ell}} = \frac{1}{2}\sum_{\ell=m}^{\infty}\left\{\tilde{\Pi}^2_\ell + \omega^2_\ell\tilde{\Phi}^2_\ell\right\}$
  - $\omega_\ell = \frac{1}{R}\left(\ell + \frac{d-2}{2}\right)$

Numerical Check

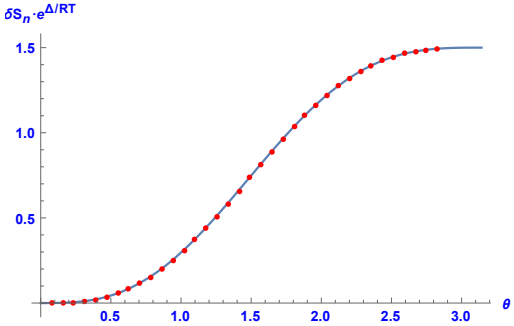
- We can use the discretized Hamiltonian to check our results



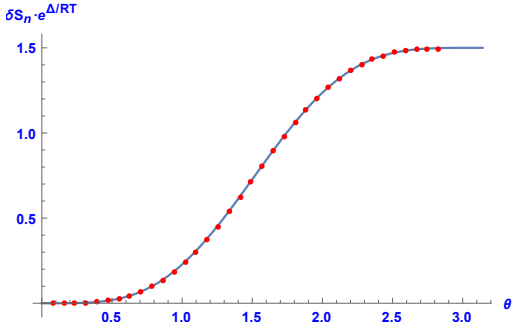
$\delta S_{n=3}$ in $(2+1)$ D



$\delta S_{n=3}$ in $(3+1)$ D



$\delta S_{n=3}$ in $(4+1)$ D



$\delta S_{n=3}$ in $(5+1)$ D

Free Fermions

- We want to repeat what we did for scalars for fermions
- First we need to know how fermions act under rotation (Euclidian space)
  - $\psi(x) \to \Lambda_{1/2}\psi(\Lambda^{-1}x)$
  - $\Lambda_{1/2} = \exp\left(\frac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]\right)$
  - $\omega_{01} = -\omega_{10} = \phi, \gamma^z = \gamma^0 + i\gamma^1$ and $\gamma^{\bar{z}} = \gamma^0 - i\gamma^1$,
  - $\Lambda_{1/2}(\phi) = \frac{1}{2}\gamma^0(e^{-i\phi/2}\gamma^z + e^{i\phi/2}\gamma^{\bar{z}})$
- Recall that in position space
  - $\langle\bar{\psi}(y')\psi(y)\rangle \sim -\frac{1}{d-2}\gamma^0\gamma^\mu\frac{\partial}{\partial y^\mu}\frac{1}{|y-y'|^{d-2}}$
- So $\langle\psi(z)\psi(z')\rangle_{1/m} = \sum_{k=0}^{m-1}\langle\psi(z)\psi(z' + (0, 2\pi k/m, \vec{0}))\rangle_1$ becomes
- $\langle\psi(z)\psi(z')\rangle_{1/m} = \sum_{k=0}^{m-1}(-1)^k\Lambda_{1/2}(\pi k/m)\langle\psi(z)\psi(z' + (0, 2\pi k/m, \vec{0}))\rangle_1$

Free Fermions

- Putting everything together we get the following formulation
- $G^F_{(1/m,d)}(2\theta) =$
  $-\frac{\gamma^0}{d-2}\lim_{z\to 1}\sum_{k=0}^{m-1}\left(e^{\frac{-\pi ik(m-1)}{m}}\gamma^z\partial_z + e^{\frac{\pi ik(m-1)}{m}}\gamma^{\bar{z}}\partial_{\bar{z}}\right)\frac{1}{|z-e^{2i(\pi k/m+\theta)}|^{d-2}}$
- As with bosons there is a recursion relation
- $G^F_{(1/m,d+2)}(2\theta) =$
  $\gamma^0\left((\partial_\theta^2 + d(d-2))(\gamma^z+\gamma^{\bar{z}}) + 2i(\gamma^z-\gamma^{\bar{z}})\partial_\theta\right)\frac{G^F_{(1/m,d)}(2\theta)}{8d(d-1)}$
- Again we only need to calculate two dimensions ($d=2$ and $d=3$)

Fermion $d = 2$

- In $d = 2$ the two point function simplifies to
  - $G^F_{(1/m,2)}(2\theta) = \frac{1}{2}\sum_{k=0}^{m-1}\gamma^0\left(\gamma^z\frac{\exp(\frac{-ik\pi(m-1)}{m})}{1-\exp(2i(k\pi/m+\theta))} + c.c.\right)$
    $= \gamma^0\left(\gamma^z\frac{mi}{4}e^{-i\theta}\csc(m\theta) - \gamma^{\bar{z}}\frac{mi}{4}e^{i\theta}\csc(m\theta)\right)$
- This leads to
  - $\delta S_n(\theta) = \frac{2}{1-n}\left(\sin(\theta)\csc(\theta/n) - n\right)e^{-\beta/(2R)} + o(e^{-\beta/(2R)})$
  - $\delta S_{EE} = 2(1-\theta\cot(\theta))e^{-\beta/(2R)} + o(e^{-\beta/(2R)})$
- These match the previous results mentioned above

Fermion $d = 3$

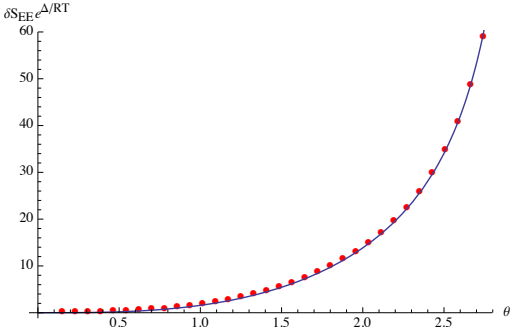
- In $d = 3$ we have a problem
  - The integral we used before does not analytically continue $n = 1/m$
  - $\sum_{k=0}^{m-1} \frac{e^{-i\pi k(m-1)/m}}{\sin(\pi k/m+\theta)} = \frac{1}{\pi}\int_0^\infty dx \frac{x^{\theta/\pi-1}}{1+x}\frac{(1+e^{-i\pi m}x)}{1+e^{i\pi/m}x^{1/m}}$
- While we cannot use this form to get the Rényi Entropies, but we can get the EE
  - $\delta S_{EE} = 4\pi \csc(\theta)\sin^4(\theta/2)e^{-\beta/R} + o(e^{-\beta/R})$
- This matches the previous results above

Numerical Check

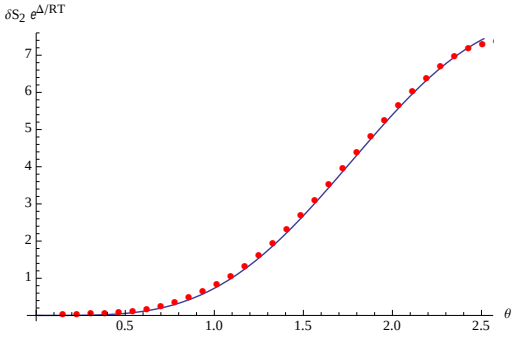
- Again we can use the discretized Hamiltonian to check our results
  - $\gamma^\mu = \gamma^I e^\mu_I$
  - $D_\mu = \partial_\mu + \frac{1}{8}\omega_{\mu IJ}[\gamma^I, \gamma^J]$
  - $\mathcal{H} = \sqrt{-g}\bar{\Psi}(i\gamma^j D_j)\Psi$
  - $\omega^j{}_i = \cos(\theta_i)\left(\prod_{k=i+1}^{j-1}\sin\theta_k\right)d\theta_j$
- If we integrate over S^{d-2}
  - $H_d = \int_0^\pi d\theta_1 \psi^\dagger \left(\gamma_0\gamma_1\partial_{\theta_1} + \frac{(d-2)\gamma_0}{2\sin(\theta_1)}\right)\psi$
- This can be discretized and the Rényi and EE can be found numerically

Numerical Check

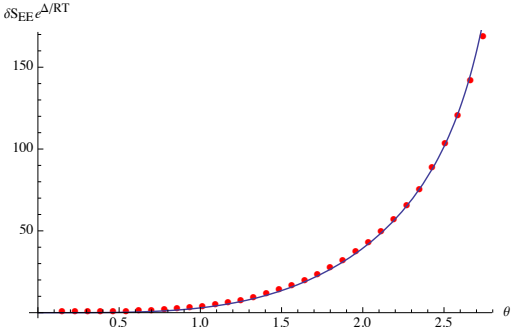
- We can also numerically check our results



δS_{EE} in $(2+1)$ D



$\delta S_{n=2}$ in $(3+1)$ D



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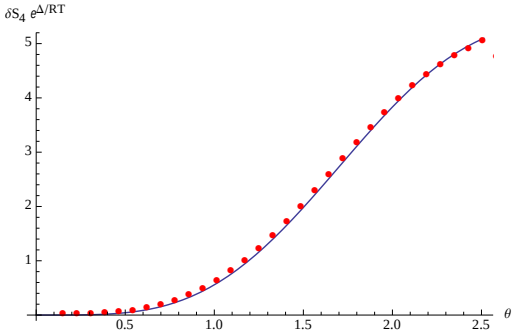

$\delta S_{n=4}$ in $(3+1)$ D

Conclusion and Outlook

- Entanglement entropy is applicable to many fields of physics
- It has a good holographic description
- Fermions without boundary terms match previous results
- Other free fields can be studied in a similar manner
- Can other measures of entanglement be studied similarly

Entanglement for general CFT's

- We can define an operator called the modular Hamiltonian such that
  - $\rho_A = e^{-H_M}$
- In hyperbolic space
  - $H_M = 2\pi R^d \int_0^\infty du \int_{S^{d-2}} \mathrm{vol}(S^{d-2}) T_{\tau\tau} \sinh^{d-2}(u)$
- After a conformal transformation to the sphere after transforming the stress tensor
  - $H_M = 2\pi R^d \int_0^\theta d\theta' \int_{S^{d-2}} \mathrm{vol}(S^{d-2}) T_{tt} \frac{\cos(\theta')-\cos(\theta)}{\sin(\theta)} \sin^{d-2}(\theta')$
- Then $\delta S_{EE} = \mathrm{tr}[(\mathrm{tr}_{\bar{A}} |\psi\rangle\langle\psi| - \mathrm{tr}_{\bar{A}} |0\rangle\langle 0|) H_M] e^{-\beta E_\psi} + \ldots$
- For spaces with boundary
  - $S = -\frac{1}{2} \int d^d x \sqrt{-g} \left[(\partial_\mu \phi)(\partial^\mu \phi) + \xi \mathcal{R} \phi^2\right] - \xi \int_{\partial M} d^{d-1}x \sqrt{-\gamma} K \phi^2$
- K of the pull back of the boundary corresponding to A does not match the natural one on H^{d-1}