Thermal Corrections to Rényi Entropies on $S^{d-1}$

Michael Spillane
with Chris Herzog

Stony Brook YITP

michael.spillane@stonybrook.edu

August 7, 2015
Outline

1. What are Entanglement and Rényi Entropy
2. Holographic EE and Ryu-Takayanagi
3. Thermal corrections for CFT’s
4. Thermal Corrections for Scalars
5. Thermal Corrections for Fermions
The density matrix for a state $\psi$
- $\rho = |\psi\rangle\langle\psi|$

If the Hilbert space factors we can define the reduced density matrix
- $H = H_A \otimes H_{\bar{A}}$
- $\rho_A = \text{tr}_{\bar{A}}\rho$

Entanglement Entropy
- $S_{EE} = -\text{tr}\rho_A \log \rho_A$

The Rényi Entropy
- $S_n = \frac{1}{1-n} \log \text{tr} (\rho_A^n)$
- $S_{EE} = \lim_{n \to 1} S_n$
For pure states

- $S_A = S_{\bar{A}}$
- This can be proven via Schmidt decomposition

Strong Subadditivity

$S_{A+C} + S_{B+C} \geq S_{A+B+C} + S_C$

Mutual information

$I(A, B) = S_A + S_B - S_{A+B}$

Area Law

$S_A \sim \frac{\text{Area}(\partial A)}{\epsilon^{d-1}} + \ldots$

\[ S_{EE}(A) = \frac{\text{Area}(\gamma_A)}{4G_N} \]

- Proof of c-theorem (arXiv:1011.5819)
- Black holes (arXiv:1104.3712)
- RG flow (arXiv:1202.5650)
- Phase transitions (arXiv:hep-th/0510092)
Entanglement Entropy Previous results

- For $d = 1 + 1$ CFT
  - $S_n = \frac{c(1+n)}{6n} \log\left( \frac{L}{\pi \epsilon} \sin(\pi \ell/L) \right) + c'$

- For $d = 1 + 1$ CFT at small finite temperature
  - $S_n(T) - S_n(0) = \frac{g}{1-n} \left[ \frac{1}{n^2} \frac{\sin^2(\Delta \pi \ell/L)}{\sin^2(\pi \ell/nL)} - n \right] e^{-2\pi \Delta \beta/L} + o\left(e^{-2\pi \Delta \beta/L}\right)$

- For CFT on $S^{d-1}$
  - $S_{EE}(T) - S_{EE}(0) = g \Delta I_d(\theta) e^{-\beta \Delta / R} + o\left(e^{-\beta \Delta / R}\right)$
  - $I_d(\theta) = 2\pi \frac{\text{Vol}(S^{d-2})}{\text{Vol}(S^{d-1})} \int_0^\theta d\theta' \frac{\cos(\theta') - \cos(\theta)}{\sin(\theta)} \sin^{d-2}(\theta')$
We are interested in finding the EE and Rényi’s on $\mathbb{R} \times S^{d-1}$.

We choose $A$ to be a cap of open angle $2\theta$.

For small temperatures we can write the density matrix

$$\rho = \frac{|0\rangle\langle 0| + \sum_i |\psi_i\rangle\langle \psi_i|e^{-\beta E_\psi} + \ldots}{1 + ge^{-\beta E_\psi} + \ldots}$$

Where $\psi$ is the lowest excited state with energy $E_\psi$ and degeneracy $g$.

Where $E_\psi = \Delta/R$. 
We now need to calculate $\text{tr}(\rho_A)^n$

$$\text{tr}(\rho_A)^n = \left(\frac{1}{1 + ge^{-\beta E_{\psi}} + \ldots}\right)^n \text{tr} \left[ \text{tr}_A (|0\rangle\langle0| + \sum_i |\psi_i\rangle\langle\psi_i|e^{-\beta E_{\psi}} + \ldots)^n \right]$$

$$= \text{tr}(\text{tr}_A |0\rangle\langle0|)^n \left[ 1 + \left( \frac{\text{tr}[\text{tr}_A \sum_i |\psi_i\rangle\langle\psi_i| (\text{tr}_A |0\rangle\langle0|)^{n-1}]}{\text{tr}(\text{tr}_A |0\rangle\langle0|)^n} - g \right) ne^{-\beta E_{\psi}} + \ldots \right]$$

$\delta S_n \equiv S_n(T) - S_n(0)$

$$\delta S_n = \frac{n}{1-n} \sum_i \left( \frac{\text{tr}[\text{tr}_A |\psi_i\rangle\langle\psi_i| (\text{tr}_A |0\rangle\langle0|)^{n-1}]}{\text{tr}(\text{tr}_A |0\rangle\langle0|)^n} - 1 \right) e^{-\beta E_{\psi}} + o(e^{-\beta E_{\psi}})$$

$$= \frac{n}{1-n} \sum_i (\frac{\langle\psi_i(z)|\psi_i(z')\rangle_n}{\langle\psi_i(z)|\psi_i(z')\rangle_1} - 1) e^{-\beta E_{\psi}} + o(e^{-\beta E_{\psi}})$$

Where we have defined $\langle\psi_i(z)|\psi_i(z')\rangle_n$ as the 2pt function on n-sheeted cover of $S^{d-1}$
Calculating 2-pt functions on n-sheeted covers are hard

So we need a trick

This comes in the form of a pair of conformal transformations

\[ ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \]
\[ = \Lambda^2 (-d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega^2) \]

where

\[ t \pm r = \tan \left( \frac{\tau \pm \theta}{2} \right) \]
\[ \Lambda = \frac{1}{2} \sec \left( \frac{\tau + \theta}{2} \right) \sec \left( \frac{\tau - \theta}{2} \right) \]

This is followed by a special conformal transformation \( b^1 = 1/r \) and the rest of \( b^\mu = 0 \)

\[ y^\mu = \frac{x^\mu - b^\mu x^2}{1-2b \cdot x + b^2 x^2} \]

The combination of these two maps the n-sheeted cover of \( S^{d-1} \) to a cone of opening angle \( 2\pi n \)
Method of Images

- We will analytically continue $n = 1/m$
- For free theories we can use the method of images
  \[ \langle \psi(z)\psi(z') \rangle_{1/m} = \sum_{k=0}^{m-1} \langle \psi(z)\psi(z' + (0, 2\pi k/m, \vec{0})) \rangle_1 \]
- For our specific case $z' = (1, 2\theta, \vec{0}), z = (1, 0, \vec{0})$
Method of Images

- We will analytically continue $n = 1/m$
- For free theories we can use the method of images
  $$\langle \psi(z)\psi(z') \rangle_{1/m} = \sum_{k=0}^{m-1} \langle \psi(z)\psi(z' + (0, 2\pi k/m, \vec{0})) \rangle_1$$
- For our specific case $z' = (1, 2\theta, \vec{0})$, $z = (1, 0, \vec{0})$
We will analytically continue $n = 1/m$

For free theories we can use the method of images

$$\langle \psi(z)\psi(z') \rangle_{1/m} = \sum_{k=0}^{m-1} \langle \psi(z)\psi(z' + (0, 2\pi k/m, \vec{0})) \rangle_{1}$$

For our specific case $z' = (1, 2\theta, \vec{0})$, $z = (1, 0, \vec{0})$
Method of Images

- We will analytically continue $n = 1/m$
- For free theories we can use the method of images
  \[
  \langle \psi(z) \psi(z') \rangle_{1/m} = \sum_{k=0}^{m-1} \langle \psi(z) \psi(z' + (0, 2\pi k/m, \vec{0})) \rangle_1
  \]
- For our specific case $z' = (1, 2\theta, \vec{0})$, $z = (1, 0, \vec{0})$
We will analytically continue $n = 1/m$

For free theories we can use the method of images

$$\langle \psi(z)\psi(z') \rangle_{1/m} = \sum_{k=0}^{m-1} \langle \psi(z)\psi(z' + (0,2\pi k/m,\vec{0})) \rangle_1$$

For our specific case $z' = (1,2\theta,\vec{0})$, $z = (1,0,\vec{0})$
Conformally coupled Scalar

- Recall that for a free scalar
  \[ \langle \phi(x)\phi(x') \rangle \sim (x - x')^{(2-d)/2} \]
- The calculation we must do is then
  \[ G_{1/m,d}(2\theta) \equiv \langle \phi(x)\phi(x') \rangle_{1/m} = \sum_{k=0}^{m-1} \csc^{d-2}(\theta + \pi k/m) \]
- This formula satisfies the following recursion relation
  \[ G_{1/m,d+2}(\theta) = \frac{1}{(d-1)(d-2)} \left[ \left( \frac{d-2}{2} \right)^2 + \partial_\theta^2 \right] G_{1/m,d}(\theta) \]
- This means that we only have to calculate for \( d = 3 \) and \( d = 4 \) and then use the recursion relation
It turns out that even dimensions are easier (this will be true for fermions also)

In fact, for $d = 4$ the answer is simply

- $G_{(1/m,4)}^B(2\theta) = m^2 \csc^2(m\theta)$
- $\delta S_n = (\csc(\theta/n)\sin(\theta)/n^2 - 1)/(1 - n)e^{-\beta/R} + o(e^{-\beta/R})$
- $\delta S_{EE} = 2 - 2\theta \cot(\theta)$

These match the results above with one caveat $d \to d - 2$
Scalar $d = 3$

- We will use the following integral representation
  \[
  \csc(y) = \int_0^\infty dx \frac{x^{y/\pi - 1}}{\pi(1+x)}
  \]
- Plugging this in we get
  \[
  G^B_{(1/m,3)}(\theta) = \frac{1}{2\pi} \sum_{k=0}^{m-1} \int_0^\infty dx \frac{x^{\theta/2\pi + k/m - 1}}{1+x}
  \]
- This integral does not have a closed form for general $m$
- But we can analytically continue the integral
  \[
  G^B_{(1,3)}(\theta) = \frac{1}{2\sin(\theta/2)}
  \]
  \[
  G^B_{(2,3)}(\theta) = \frac{1-\theta/2\pi}{2\sin(\theta/2)}
  \]
  \[
  S_2 = 2\theta/\pi
  \]
- A Taylor expansion around $m = 1$ allows a calculation of the EE
  \[
  S_{EE} = \frac{\pi}{2} \tan(\theta/2)
  \]
We want to reduce the Hamiltonian to one dimension

\[
S = -\frac{1}{2} \int d^d x \sqrt{-g} \left[ (\partial_\mu \phi)(\partial^\mu \phi) + \xi \mathcal{R} \phi^2 \right]
\]

\[
\mathcal{H} = \frac{(R \sin \theta)^{d-2}}{2R} \sqrt{h} \left\{ R^2 \pi^2 + (\partial_\theta \phi)^2 + \frac{h^{ab}(\partial_a \phi)(\partial_b \phi)}{\sin^2 \theta} + \frac{(d-2)^2}{4} \phi^2 \right\}
\]

Where \( \pi \) is the conjugate momentum and \( h^{ab} \) is the metric on \( S^{d-1} \)

Integrating over \( S^{d-2} \)

\[
H_{\ell} = \frac{1}{2R^2} \int_0^\pi \left\{ R^2 \Pi_{\ell}^2 - \Phi_{\ell} \partial_\theta \Phi_{\ell} + \frac{1}{4} (2m + d - 2)(2m + d - 4) \frac{\Phi_{\ell}^2}{\sin^2 \theta} \right\} d\theta
\]

\[
H_{\ell} = \frac{1}{2} \sum_{\ell=m}^\infty \left\{ \tilde{\Pi}_{\ell}^2 + \omega_{\ell}^2 \tilde{\Phi}_{\ell}^2 \right\}
\]

\[
\omega_{\ell} = \frac{1}{R} \left( \ell + \frac{d-2}{2} \right)
\]
We can use the discretized Hamiltonian to check our results.

- $\delta S_{n=3}$ in $(2 + 1)$ D
- $\delta S_{n=3}$ in $(3 + 1)$ D
- $\delta S_{n=3}$ in $(4 + 1)$ D
- $\delta S_{n=3}$ in $(5 + 1)$ D
Free Fermions

- We want to repeat what we did for scalars for fermions
- First we need to know how fermions act under rotation (Euclidian space)
  \[ \psi(x) \rightarrow \Lambda_{1/2}\psi(\Lambda^{-1}x) \]
  \[ \Lambda_{1/2} = \exp\left(\frac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]\right) \]
  \[ \omega_{01} = -\omega_{10} = \phi, \gamma^z = \gamma^0 + i\gamma^1 \text{ and } \gamma^\bar{z} = \gamma^0 - i\gamma^1, \]
  \[ \Lambda_{1/2}(\phi) = \frac{1}{2}\gamma^0(e^{-i\phi/2}\gamma^z + e^{i\phi/2}\gamma^\bar{z}) \]
- Recall that in position space
  \[ \langle \bar{\psi}(y')\psi(y) \rangle \sim -\frac{1}{d-2}\gamma^0\gamma^\mu \frac{\partial}{\partial y^\mu} \frac{1}{|y-y'|^{d-2}} \]
  So \( \langle \psi(z')\psi(z') \rangle_{1/m} = \sum_{k=0}^{m-1} \langle \psi(z)\psi(z' + (0, 2\pi k/m, \vec{0})) \rangle_{1} \)
becomes
  \[ \langle \psi(z)\psi(z') \rangle_{1/m} = \sum_{k=0}^{m-1} (-1)^k \Lambda_{1/2}(\pi k/m) \langle \psi(z)\psi(z' + (0, 2\pi k/m, \vec{0})) \rangle_{1} \]
Putting everything together we get the following formulation

\[ G^F_{(1/m,d)}(2\theta) = -\frac{\gamma^0}{d-2} \lim_{z \to 1} \sum_{k=0}^{m-1} \left( e^{-\frac{\pi i k (m-1)}{m}} \gamma^z \partial_z + e^{\frac{\pi i k (m-1)}{m}} \gamma^{\bar{z}} \partial_{\bar{z}} \right) \frac{1}{|z - e^{2i(\pi k/m + \theta)}|^{d-2}} \]

As with bosons there is a recursion relation

\[ G^F_{(1/m,d+2)}(2\theta) = \gamma^0 \left( (\partial^2_\theta + d(d-2))(\gamma^z + \gamma^{\bar{z}}) + 2i(\gamma^z - \gamma^{\bar{z}}) \partial_\theta \right) \frac{G^F_{(1/m,d)}(2\theta)}{8d(d-1)} \]

Again we only need to calculate two dimensions (\( d = 2 \) and \( d = 3 \))
Fermion $d = 2$

- In $d = 2$ the two point function simplifies to
  \[
  G^F_{(1/m, 2)}(2\theta) = \frac{1}{2} \sum_{k=0}^{m-1} \gamma^0 \left( \gamma^z \frac{\exp(-ik\pi(m-1)/m)}{1-\exp(2i(k\pi/m+\theta))} + \text{c.c.} \right)
  = \gamma^0 \left( \gamma^z \frac{mi}{4} e^{-i\theta} \csc(m\theta) - \gamma^\bar{z} \frac{mi}{4} e^{i\theta} \csc(m\theta) \right)
  \]

- This leads to
  - $\delta S_n(\theta) = \frac{2}{1-n} (\sin(\theta) \csc(\theta/n) - n) e^{-\beta/(2R)} + o(e^{-\beta/(2R)})$
  - $\delta S_{EE} = 2(1 - \theta \cot(\theta)) e^{-\beta/(2R)} + o(e^{-\beta/(2R)})$

- These match the previous results mentioned above
Fermion $d = 3$

- In $d = 3$ we have a problem
  - The integral we used before does not analytically continue $n = 1/m$
  - \[ \sum_{k=0}^{m-1} \frac{e^{-i\pi k(m-1)/m}}{\sin(\pi k/m + \theta)} = \frac{1}{\pi} \int_0^{\infty} \frac{dx}{1+x} \frac{x^{\theta/\pi-1} (1+e^{-i\pi m x})}{1+e^{i\pi/m x^{1/m}}} \]

- While we cannot use this form to get the Rényi Entropies, but we can get the EE
  - \[ \delta S_{EE} = 4\pi \csc(\theta) \sin^4(\theta/2)e^{-\beta/R} + o(e^{-\beta/R}) \]

- This matches the previous results above
Again we can use the discretized Hamiltonian to check our results

- $\gamma^\mu = \gamma^I e^\mu_I$
- $D_\mu = \partial_\mu + \frac{1}{8} \omega_{\mu IJ} [\gamma^I, \gamma^J]$
- $\mathcal{H} = \sqrt{-g} \bar{\Psi} (i \gamma^j D_j) \Psi$
- $\omega^j_i = \cos(\theta_i) \left( \prod_{k=i+1}^{j-1} \sin \theta_k \right) d\theta_j$

If we integrate over $S^{d-2}$

- $H_d = \int_0^\pi d\theta_1 \psi^\dagger \left( \gamma_0 \gamma_1 \partial_\theta_1 + \frac{(d-2)\gamma_0}{2 \sin(\theta_1)} \right) \psi$

This can be discretized and the Rényi and EE can be found numerically.
We can also numerically check our results.

\[ \delta S_{EE} \text{ in } (2 + 1) \text{ D} \]

\[ \delta S_{n=2} \text{ in } (3 + 1) \text{ D} \]

\[ \delta S_{EE} \text{ in } (3 + 1) \text{ D} \]

\[ \delta S_{n=4} \text{ in } (3 + 1) \text{ D} \]
Entanglement entropy is applicable to many fields of physics
It has a good holographic description
Fermions without boundary terms match previous results
Other free fields can be studied in a similar manner
Can other measures of entanglement be studied similarly
Entanglement for general CFT’s

- We can define an operator called the modular Hamiltonian such that
  \[ \rho_A = e^{-H_M} \]

- In hyperbolic space
  \[ H_M = 2\pi R^d \int_0^\infty du \int_{S^{d-2}} \text{vol}(S^{d-2}) T_{\tau\tau} \sinh^{d-2}(u) \]

- After a conformal transformation to the sphere after transforming the stress tensor
  \[ H_M = 2\pi R^d \int_0^\theta d\theta' \int_{S^{d-2}} \text{vol}(S^{d-2}) T_{tt} \frac{\cos(\theta')-\cos(\theta)}{\sin(\theta)} \sin^{d-2}(\theta') \]

- Then \( \delta S_{EE} = \text{tr}[(\text{tr} \bar{A} | \psi \rangle \langle \psi | - \text{tr} \bar{A} | 0 \rangle \langle 0 |) H_M] e^{-\beta E_\psi} + ... \)

- For spaces with boundary
  \[ S = -\frac{1}{2} \int d^d x \sqrt{-g} \left[ (\partial_\mu \phi)(\partial^\mu \phi) + \xi R \phi^2 \right] - \xi \int_{\partial M} d^{d-1}x \sqrt{-\gamma} K \phi^2 \]

- K of the pull back of the boundary corresponding to A does not match the natural one on \( H^{d-1} \)