# On the Classification of Four Dimensional Rank-1 N=2 Superconformal Field theories

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(based on 1505.04814, 1509.$*****$, 1509.$*****$,
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# Outline

- Generality of d=4 N=2 SCFTs
- Coulomb Branch & Seiberg-Witten Curves and 1-forms
- Higgs Branch and Enhanced Coulomb Branch & S-duality
- Conformal Central Charge & Topological Twisting
- Summary and Open Questions

## Generality of d=4 N=2 SCFTs

▶ The symmetries of N=2 SCFTs include d=4 N=2 superconformal algebra $\mathfrak{su}(2,2|2)$ and flavor symmetry algebra $\mathfrak{f}$:
unitary irreducible representations $\Longrightarrow$ operator spectrum:

- Primary and decedents
- Short and long multiplets

$\Rightarrow \{\Delta, (J_L, J_R), R, r, w_{\mathfrak{f}}\}$

▶ Operator algebras and OPE structure constants

- $\forall$ Chiral rings w/ non-singular OPEs
- $\{C_{ijk}\}$ determine 3-point correlation functions up to conformal covariance

$\Longrightarrow$ N=2 SCFT data: $\{\Delta, (J_L, J_R), R, r, w_{\mathfrak{f}}\} \cup \{C_{i,j,k}\}$

## Generality of d=4 N=2 SCFTs

▶ In principle, the consistency full set of N=2 SCFT data can be solved by conformal Bootstrap approach (4-point correlation functions and OPE associativity)

- Not exactly solvable, only certain numerical bounds
- Redundant for the purpose of classification

▶ In practice, we single out a sufficiently minimal set of N=2 SCFT data (exactly determined):

- Coulomb branch primary operators (CBO)
- Higgs branch primary operators (HBO)
- Energy-momentum tensor multiplets and conformal central charges $a$ and $c$: $T \cdot T \to c$, (Not sure how to define $a$ in this way; related to $a$-theorem of RG flow)
- Global internal symmetry current multiplets and flavor central charges $k$: $j \cdot j \to k\mathrm{I}$. ($j$ can couple with mass deformation parameter: relevant with scaling dimension 1)

# Generality of d=4 N=2 SCFTs

▶ CBO and HBO form scalar chiral rings, in turn the coordinate rings of certain algebraic varieties
$\Longrightarrow$ moduli space of vaccum: (chiral ring/moduli space correspondence)
• CBO $\Longrightarrow$ Coulomb branch (CB)
• HBO $\Longrightarrow$ Higgs branch (HB)
• CBO+HBO $\Longrightarrow$ enhanced Coulomb branch (ECB): Higgs factor fibering over CB

▶ Various central charges $a$, $c$, $k$ can be uniformly related to scale anomaly in the background metric and background gauge fields for the flavor symmetries, $\Theta \sim a\,(\text{Euler}) + c\,(\text{Weyl}) + k\,\text{Tr}(F^2)$, in turn to 't Hooft anomaly of global symmetries by N=2 superconformal symmetry
($\Theta \Longrightarrow \partial_\mu j^\mu_{U(1)_R} \propto$ gravitational and gauge instanton number density)

## Coulomb Branch & Seiberg-Witten Curves and 1-forms

▶ Coulomb branch is holomorphically parametrized by vev of CBOs

• $\dim_{\mathbb{C}}$CB = # of CBPOs $\equiv$ rank of N=2 SCFT = r

• Generic locus $P$: N=2 free abelian vector-multiplet and massive BPS dyons (and other massless neutral hypermultiplet; ECB)

• Singular locus $S$: $\text{codim}_{\mathbb{C}}$S = 1, characterized by EM monodromy on special coordinates and charge lattice

$\Longrightarrow$ in terms of Seiberg-Witten (or Donagi-Witten) geometry: complex-dim r torus fibering over CB

▶ For the case of rank 1, holomorphic coordinate $u$, $S \equiv pt$s

• w/n mass deformation and # $S$ =1: Kodaira classification of singular elliptic fibration over $u$ plane; N=2 SCFT at $S$.

• w/ mass deformation: depending on flavor symmetry, and $S$ splits $\mapsto \{S_i\}$, generic mass: IR free theories at $\{S_i\}$ or other interacting N=2 SCFT admitting no further relevant deformations (remark: This fact is crucial for our strategy)

## Coulomb branch & Seiberg-Witten Curves and 1-forms

▶ We have explored the cases in which the deformed CB is topologically $\mathbb{C}$: Planar CB

▶ Seiberg-Witten curves (elliptic curve as complex torus):

$$\Sigma(u, \mathbf{m}) : \qquad y^2 = x^3 + f(u, \mathbf{m})\, x + g(u, \mathbf{m}),$$

- Weyl-invariant mass terms as coefficient of $f$ and $g$
- Zeros of Discriminants: location of singularities on $u$ plane

▶ The meromorphic 1-form $\lambda(u, \mathbf{m})$ satisfies rigid special Kähler (RSK) conditions constraining its $u$- and $\mathbf{m}$-dependence,

$$\partial_u \lambda = \kappa \frac{\mathrm{d}x}{y} + \mathrm{d}\phi, \qquad \text{Residues}(\lambda) \in \{\boldsymbol{\omega}(\mathbf{m}) \mid \boldsymbol{\omega} \in \Lambda_F\}.$$

▶ The low energy u(1) gauge coupling and BPS central charges can be determined from the periods of 1-form over the curve.

## Coulomb branch & Seiberg-Witten Curves and 1-forms

▶ An ansatz for one-form satisfying RSK condition given by Minahan-Nemeschansky:

$$\lambda(\mathbf{m}) = \Bigg[2\Delta(u)\, a\, u + 6 b\, \mu\, x + 2W(M_d) + \sum_i r_i \sum_{\boldsymbol{\omega}_i \text{ orbit}} \frac{y_{\boldsymbol{\omega}_i}(u,\mathbf{m})}{\boldsymbol{\omega}_i(\mathbf{m})^2 x - x_{\boldsymbol{\omega}_i}(u,\mathbf{m})}\Bigg] \frac{dx}{y}.$$

▶ $a, b, W, r_i, x_{\boldsymbol{\omega}_i}$ are unknowns. Most difficult are $x_{\boldsymbol{\omega}_i}$ determined by factorization of curve. (Computer assistance required.)

▶ Sum over Weyl orbits of flavor algebra weights $\boldsymbol{\omega}_i$. Weyl group is determined by curve.

▶ Which weights appear and their coefficients $r_i$ determine together with Weyl group the flavor symmetry.

| Safe deformations of regular, rank 1, scale-invariant CBs | | | | | | | |
|---|---|---|---|---|---|---|---|
| Kodaira singularity | deformation pattern | flavor symmetry | central charges | | | Higgs branches | |
| | | | $k_F$ | $12 \cdot c$ | $24 \cdot a$ | $h_1$ | $h_0$ |
| $II^*$ | $\{{I_1}^{10}\}$ | $E_8$ | 12 | 62 | 95 | 0 | 29 |
| | $\{{I_1}^6, I_4\}$ | sp(10) | 7 | 49 | 82 | 5 | 16 |
| $III^*$ | $\{{I_1}^9\}$ | $E_7$ | 8 | 38 | 59 | 0 | 17 |
| | $\{{I_1}^5, I_4\}$ | sp(6) $\oplus$ sp(2) | $5 \oplus 8$ | 29 | 50 | 3 | 8 |
| $IV^*$ | $\{{I_1}^8\}$ | $E_6$ | 6 | 26 | 41 | 0 | 11 |
| | $\{{I_1}^4, I_4\}$ | sp(4) $\oplus$ u(1) | $4 \oplus ?$ | 19 | 34 | 2 | 4 |
| | $\{I_1, I_1^*\}$ | u(1) | ? | $14{+}h$ | $29{+}h$ | $h$ | ? |
| $I_0^*$ | $\{{I_1}^6\}$ | so(8) | 4 | 14 | 23 | 0 | 5 |
| | $\{{I_1}^2, I_4\} \simeq \{{I_2}^3\}$ | sp(2) | 3 | 9 | 18 | 1 | 1 |
| $IV$ | $\{{I_1}^4\}$ | su(3) | 3 | 8 | 14 | 0 | 2 |
| $III$ | $\{{I_1}^3\}$ | su(2) | 8/3 | 6 | 11 | 0 | 1 |
| $II$ | $\{{I_1}^2\}$ | – | – | 22/5 | 43/5 | 0 | 0 |
| **with the assumption of a frozen IV$^*$ SCFT with central charge $c'$** | | | | | | | |
| $II^*$ | $\{I_2, IV^*\}$ | su(2) | ? | $24c'{+}h{+}4$ | $24c'{+}h{+}37$ | $h$ | ? |
| or | | | | | | | |
| $II^*$ | $\{{I_1}^2, IV^*\}$ | $G_2$ | ? | $24c'{+}h{+}10$ | $24c'{+}h{+}43$ | $h$ | ? |
| $III^*$ | $\{I_1, IV^*\}$ | su(2) | ? | $16c'{+}h{+}\frac{10}{3}$ | $16c'{+}h{+}\frac{73}{3}$ | $h$ | ? |
| **with the assumption of a frozen III$^*$ SCFT with central charge $c''$** | | | | | | | |
| $II^*$ | $\{I_1, III^*\}$ | su(2) | ? | $18c''{+}h{+}5$ | $18c''{+}h{+}38$ | $h$ | ? |

## Coulomb branch & Seiberg-Witten Curves and 1-forms

▶ The well-known 4d rank-1 SCFTs with ADE flavor symmetries (ADE series):
- Minahan-Nemeschansky theories ($E_n$ series);
- $\mathfrak{su}(2)$ gauge theory with eight fundamental flavors ($D_4$ theory);
- Argyres-Douglas theories ($A_n$ series)

▶ The less-known 4d rank-1 SCFTs with non-ADE flavor symmetries with $\mathfrak{sp}$ factors (includes $N = 2^*$ $\mathfrak{su}(2)$ gauge theory): $\mathfrak{sp}$ series (we take $\mathfrak{sp}(1) \simeq \mathfrak{su}(2)$)

▶ Several other conjectured SCFTs are even exotic: Less constructed.

▶ Here different N=2 SCFTs are distinguished by their ($\Delta(u)$, $\mathfrak{g}$) $\Longrightarrow$ Classification ...Next we discuss various N=2 SCFT data of the ADE series and $\mathfrak{sp}$ series

## Higgs Branch and Enhanced Coulomb Branch & S-duality

▶ HB and ECB-fiber both take hyperkähler structures (i.e., take action under $SU(2)_R$ symmetry):
$\Longrightarrow$ counting the quaternionic dimensions:
$h_0 \equiv \dim_{\mathbb{H}} \text{HB}$, $h_1 \equiv \dim_{\mathbb{H}} \text{ECB-fiber}$

▶ HB and ECB-fiber both take action under flavor symmetry:
• e.g., for $F = ADE$ theories, co-adjoint actions: minimal nilpotent orbits: $\mathcal{O}_{\text{min}}(F) = F_{\mathbb{C}} \cdot (E_\theta)^*$
this can be understood in terms of N=2 chiral ring relations of HBOs.
• Conjecture: HB of $\mathfrak{sp}$ series are co-adjoint orbits of flavor symmetry, and can be understood from the N=2 chiral ring relations of HBOs
• For ECB-fiber, non-singular at the tip of the cone (for no charged massless hypermultiplets emerging);
$\Longrightarrow$ implying ECB-fiber is flat, e.g. $\mathbb{H}^n$

## Higgs Branch and Enhanced Coulomb Branch & S-duality

▶ The structure of HB and ECB fibers of $E_n$ and $\mathfrak{sp}$ can be deduced from the N=2 S-duality argument:
those SCFTs emerge at the cusp of conformal manifold of certain N=2 finite gauge theories.

▶ The HB and ECB of N=2 finite Gauge theories can be classically determined by hyperkähler quotient constructions, and receive no quantum corrections

• The vacuum moduli space structure $\Longrightarrow$ This fact can be used to extract the information about the Higgs branch and enhanced branch

• e.g., for $E_n$ series $h_0 = h^\vee_{\mathfrak{f}} - 1$ (actually application to $ADE$ theories, by F-theory construction)

• e.g., for $\mathfrak{sp}$ series, the ECB fibers are the defining module of $\mathfrak{sp}$ lie algebra: $\mathbb{H}^n \simeq \mathbb{C}^{2n}$ (in terms of $\mathfrak{so}(4n) \supset \mathfrak{sp}(1) \times \mathfrak{sp}(n)$)

## Conformal Central Charges & Topological Twisting

▶ Central charges $a$,$c$,$k$ can be determined in various ways:

• By $S$-duality argument (except the $k_{\mathfrak{u}(1)}$)) (for $E_n$ series, $\mathfrak{sp}$ series)

• By holographic methods (for $ADE$ series: admits F-theory construction, i.e. world volume theory of a D3 brane probing 7-brane backgrounds)

• By correspondence to 2d Chiral algebra (for theories admitting Class-S construction)

• By Topological twisting arguments (Shapere and Tachikawa): directly related to the low energy data on CB or EBC

• $SU(2)_L \times SU(2)_R \times SU(2)_I \to SU(2)_L \times SU(2)'_R$: changing spins

• On curved 4-manifold, $SU(2)_R$ connection = spin connection: relating $SU(2)_R$ instanton to gravitational instanton...

(note: twisting hypermultiplet need certain condition on spin structure)

## Conformal Central Charges & Topological Twisting

▶ The corresponding partition function of topological twisted sector takes $u(1)_R$ charge:

$$\Delta R = (2a - c) \cdot \chi + \frac{3}{2} c \cdot \sigma - \frac{1}{2} k_i \cdot n_i,$$

▶ In the IR at nonsingular locus on CB, the partition function is expressed as

$$Z = \int [dV][dH] A^{\chi} B^{\sigma} \prod_i C^{n_i} e^{S_{\mathrm{IR}}[V,H]}$$

with $u(1)_R$ charge

$$\Delta R = \left( R(A) + \frac{1}{4} n_V \right) \cdot \chi + \left( R(B) + \frac{1}{4} n_V + \frac{1}{8} n_H \right) \cdot \sigma + \sum_i \left( R(C_i) - \frac{1}{2} T_i(\mathbf{r}) \right) \cdot n_i,$$

## Conformal Central Charges & Topological Twisting

▶ Matching UV and IR results (by non-anomalous $U_R$ and 't Hooft anomaly matching) leads to:

$$
\begin{aligned}
24a &= 5n_V + n_H + 12R(A) + 8R(B) \\
12c &= 2n_V + n_H + 8R(B) \\
k_i &= T_i(\mathbf{r}) - 2R(C_i)
\end{aligned}
$$

▶ In the rank-1 case, $n_V = 1$, $n_H = h_1$, $\mathbf{r}$ has been determined, $\Longrightarrow$ scaling dimension of $A$, $B$, $C_i$ to be determined

▶ Holomorphy and EM duality of low energy physics and topological invariance lead to:

- $A(u) = \alpha\,\mathrm{det}(\frac{\partial u_i}{\partial a_j})$
- $B^8$ must be a single-valued holomorphic function of $u$ (for $\sigma \in 8\mathbb{Z}$ on spin 4-manifolds)
- $C$ must be a single-valued holomorphic function of $u$ (for instanton number $n \in \mathbb{Z}$)

## Conformal Central charges & Topological Twisting

- For the rank-1 case ($Z \equiv |\{S_i\}|$):

$$R(A) = \frac{\Delta - 1}{2}$$
$$R(B) = \frac{\Delta}{8} \sum_{i=1}^{Z} \frac{12c_i - 2 - h_i}{\Delta_i}$$

In turn the conformal central charge $a$ and $c$ can be determined:

$$24a = 5 + h_1 + 3\frac{\Delta - 1}{2} + \Delta \sum_{i=1}^{Z} \frac{12c_i - 2 - h_i}{\Delta_i},$$
$$12c = 2 + h_1 + \Delta \sum_{i=1}^{Z} \frac{12c_i - 2 - h_i}{\Delta_i}.$$

## Summary and open questions

- ▶ Classification of 4d rank-1 N=2 SCFTs with planar CB has been performed
- ▶ A minimal set of N=2 SCFT data has been computed exactly

- ▶ Further discussion on the flavor central charge $k$?
- ▶ Similar story of 4d rank-1 N=2 SCFTs with non-planar CB?
- ▶ Generalization to higher rank SCFTs?
- ▶ Stringy realization of $\mathfrak{sp}$ series?
- ▶ Possible d=5 N=1/d=6 N=(1,0) versions?

Thanks!