On the Classification of Four Dimensional Rank-1 N=2 Superconformal Field theories

Yongchao Lü

(based on 1505.04814, 1509.***** , 1509.***** , with Philip Argyres, Matteo Lotito, Mario Martone)
• Generality of $d=4 \ N=2$ SCFTs

• Coulomb Branch & Seiberg-Witten Curves and 1-forms

• Higgs Branch and Enhanced Coulomb Branch & S-duality

• Conformal Central Charge & Topological Twisting

• Summary and Open Questions
The symmetries of $N=2$ SCFTs include $d=4$ $N=2$ superconformal algebra $su(2,2|2)$ and flavor symmetry algebra $\mathfrak{f}$: unitary irreducible representations $\Rightarrow$ operator spectrum:
- Primary and decedents
- Short and long multiplets
$\Rightarrow \{\Delta, (J_L, J_R), R, r, w_f\}$

Operator algebras and OPE structure constants
- $\forall$ Chiral rings w/ non-singular OPEs
- $\{C_{ijk}\}$ determine 3-point correlation functions up to conformal covariance

$\Rightarrow$ $N=2$ SCFT data: $\{\Delta, (J_L, J_R), R, r, w_f\} \cup \{C_{i,j,k}\}$
In principle, the consistency full set of N=2 SCFT data can be solved by conformal Bootstrap approach (4-point correlation functions and OPE associativity)
• Not exactly solvable, only certain numerical bounds
• Redundant for the purpose of classification

In practice, we single out a sufficiently minimal set of N=2 SCFT data (exactly determined):
• Coulomb branch primary operators (CBO)
• Higgs branch primary operators (HBO)
• Energy-momentum tensor multiplets and conformal central charges $a$ and $c$: $T \cdot T \rightarrow c$, (Not sure how to define $a$ in this way; related to $a$-theorem of RG flow)
• Global internal symmetry current multiplets and flavor central charges $k$: $j \cdot j \rightarrow k\,I$. ($j$ can couple with mass deformation parameter: relevant with scaling dimension 1)
Generality of d=4 N=2 SCFTs

CBO and HBO form scalar chiral rings, in turn the coordinate rings of certain algebraic varieties
\[ \Rightarrow \] moduli space of vacuum: (chiral ring/moduli space correspondence)
- CBO \[\Rightarrow\] Coulomb branch (CB)
- HBO \[\Rightarrow\] Higgs branch (HB)
- CBO+HBO \[\Rightarrow\] enhanced Coulomb branch (ECB): Higgs factor fibering over CB

Various central charges \( a, c, k \) can be uniformly related to scale anomaly in the background metric and background gauge fields for the flavor symmetries, \( \Theta \sim a \text{(Euler)} + c \text{(Weyl)} + k \text{Tr}(F^2) \), in turn to 't Hooft anomaly of global symmetries by N=2 superconformal symmetry
\( \Theta \[\Rightarrow\] \partial_\mu j^\mu_{U(1)_R} \propto \text{gravitational and gauge instanton number density} \)
Coulomb Branch & Seiberg-Witten Curves and 1-forms

- Coulomb branch is holomorphically parametrized by vev of CBPOs
  - $\dim_{\mathbb{C}} \text{CB} = \# \text{ of CBPOs} \equiv \text{rank of N=2 SCFT} = r$
  - Generic locus $P$: N=2 free abelian vector-multiplet and massive BPS dyons (and other massless neutral hypermultiplet; ECB)
  - Singular locus $S$: codim$_{\mathbb{C}} S = 1$, characterized by EM monodromy on special coordinates and charge lattice
    $\implies$ in terms of Seiberg-Witten (or Donagi-Witten) geometry: complex-dim $r$ torus fibering over CB
- For the case of rank 1, holomorphic coordinate $u$, $S \equiv \text{pts}$
  - w/n mass deformation and $\# S = 1$: Kodaira classification of singular elliptic fibration over $u$ plane; N=2 SCFT at $S$.
  - w/ mass deformation: depending on flavor symmetry, and $S$ splits $\rightarrow \{S_i\}$, generic mass: IR free theories at $\{S_i\}$ or other interacting N=2 SCFT admitting no further relevant deformations
  (remark: This fact is crucial for our strategy)
We have explored the cases in which the deformed CB is topologically $\mathbb{C}$: **Planar CB**

- Seiberg-Witten curves (elliptic curve as complex torus):
  \[ \Sigma(u, m) : \quad y^2 = x^3 + f(u, m)x + g(u, m), \]

- Weyl-invariant mass terms as coefficient of $f$ and $g$
- Zeros of Discriminants: location of singularities on $u$ plane

- The meromorphic 1-form $\lambda(u, m)$ satisfies rigid special Kähler (RSK) conditions constraining its $u$- and $m$-dependence,
  \[ \partial_u \lambda = \kappa \frac{dx}{y} + d\phi, \quad \text{Residues}(\lambda) \in \{ \omega(m) \mid \omega \in \Lambda_F \}. \]

- The low energy $u(1)$ gauge coupling and BPS central charges can be determined from the periods of 1-form over the curve.
Coulomb branch & Seiberg-Witten Curves and 1-forms

- An ansatz for one-form satisfying RSK condition given by Minahan-Nemeschansky:

\[
\lambda(m) = \left[ 2\Delta(u) a u + 6b \mu x + 2W(M_d) \right]

+ \sum_{i} r_i \sum_{\omega_i \text{ orbit}} \frac{y\omega_i(u, m)}{\omega_i(m)^2 x - x\omega_i(u, m)} \int \frac{dx}{y}.
\]

- \( a, b, W, r_i, x\omega_i \) are unknowns. Most difficult are \( x\omega_i \) determined by factorization of curve. (Computer assistance required.)

- Sum over Weyl orbits of flavor algebra weights \( \omega_i \). Weyl group is determined by curve.

- Which weights appear and their coefficients \( r_i \) determine together with Weyl group the flavor symmetry.
<table>
<thead>
<tr>
<th>Kodaira singularity</th>
<th>deformation pattern</th>
<th>flavor symmetry</th>
<th>$k_F$</th>
<th>central charges</th>
<th>Higgs branches</th>
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<td>$h_1$</td>
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<td>$E_8$</td>
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<td>${I_1^6, I_4}$</td>
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<td>${I_1^5, I_4}$</td>
<td>$\text{sp}(6) \oplus \text{sp}(2)$</td>
<td>5 $\oplus$ 8</td>
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<td>$IV^*$</td>
<td>${I_1^8}$</td>
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<td>${I_1^4, I_4}$</td>
<td>$\text{sp}(4) \oplus \text{u}(1)$</td>
<td>4 $\oplus$ ?</td>
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<td></td>
<td>${I_1, I_1^*}$</td>
<td>$\text{u}(1)$</td>
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<td>14+$h$</td>
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<td>$I_0^*$</td>
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with the assumption of a frozen $IV^*$ SCFT with central charge $c'$

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Coulomb branch & Seiberg-Witten Curves and 1-forms

▶ The well-known 4d rank-1 SCFTs with ADE flavor symmetries (ADE series):
  - Minahan-Nemeschansky theories ($E_n$ series);
  - $su(2)$ gauge theory with eight fundamental flavors ($D_4$ theory);
  - Argyres-Douglas theories ($A_n$ series)

▶ The less-known 4d rank-1 SCFTs with non-ADE flavor symmetries with $sp$ factors (includes $N = 2^*$ $su(2)$ gauge theory): $sp$ series (we take $sp(1) \cong su(2)$)

▶ Several other conjectured SCFTs are even exotic: Less constructed.

▶ Here different $N=2$ SCFTs are distinguished by their ($\Delta(u), g)$ $\implies$ Classification ...Next we discuss various $N=2$ SCFT data of the ADE series and $sp$ series
HB and ECB-fiber both take hyperkähler structures (i.e., take action under $SU(2)_R$ symmetry):

\[ h_0 \equiv \dim_{\mathbb{H}} \text{HB}, \ h_1 \equiv \dim_{\mathbb{H}} \text{ECB-fiber} \]

HB and ECB-fiber both take action under flavor symmetry:

- e.g., for $F = ADE$ theories, co-adjoint actions: minimal nilpotent orbits: $O_{\text{min}}(F) = F_C \cdot (E_\theta)^*$
  
  this can be understood in terms of $N=2$ chiral ring relations of HBOs.

- Conjecture: HB of $Sp$ series are co-adjoint orbits of flavor symmetry, and can be understood from the $N=2$ chiral ring relations of HBOs

- For ECB-fiber, non-singular at the tip of the cone (for no charged massless hypermultiplets emerging);

  \[ \implies \text{implying ECB-fiber is flat, e.g. } \mathbb{H}^n \]
The structure of HB and ECB fibers of $E_n$ and $\mathfrak{sp}$ can be deduced from the $N=2$ S-duality argument: those SCFTs emerge at the cusp of conformal manifold of certain $N=2$ finite gauge theories.

The HB and ECB of $N=2$ finite Gauge theories can be classically determined by hyperkähler quotient constructions, and receive no quantum corrections

- The vacuum moduli space structure $\Rightarrow$ This fact can be used to extract the information about the Higgs branch and enhanced branch
- e.g., for $E_n$ series $h_0 = h_{\frac{1}{2}}^\vee - 1$ (actually application to $ADE$ theories, by F-theory construction)
- e.g., for $\mathfrak{sp}$ series, the ECB fibers are the defining module of $\mathfrak{sp}$ lie algebra: $\mathbb{H}^n \simeq \mathbb{C}^{2n}$ (in terms of $\mathfrak{so}(4n) \supset \mathfrak{sp}(1) \times \mathfrak{sp}(n)$)
Central charges $a, c, k$ can be determined in various ways:

- By $S$-duality argument (except the $k_{u(1)}$) (for $E_n$ series, $sp$ series)
- By holographic methods (for $ADE$ series: admits F-theory construction, i.e. world volume theory of a D3 brane probing 7-brane backgrounds)
- By correspondence to 2d Chiral algebra (for theories admitting Class-S construction)
- By Topological twisting arguments (Shapere and Tachikawa): directly related to the low energy data on CB or EBC
- $SU(2)_L \times SU(2)_R \times SU(2)_I \rightarrow SU(2)_L \times SU(2)'_R$: changing spins
- On curved 4-manifold, $SU(2)_R$ connection = spin connection: relating $SU(2)_R$ instanton to gravitational instanton...
  (note: twisting hypermultiplet need certain condition on spin structure)
The corresponding partition function of topological twisted sector takes $u(1)_R$ charge:

$$\Delta R = (2a - c) \cdot \chi + \frac{3}{2} c \cdot \sigma - \frac{1}{2} k_i \cdot n_i,$$

In the IR at nonsingular locus on CB, the partition function is expressed as

$$Z = \int [dV][dH] A^\chi B^\sigma \prod_i C^{n_i} e^{S_{IR}[V,H]}$$

with $u(1)_R$ charge

$$\Delta R = \left( R(A) + \frac{1}{4} n_V \right) \cdot \chi + \left( R(B) + \frac{1}{4} n_V + \frac{1}{8} n_H \right) \cdot \sigma$$

$$+ \sum_i \left( R(C_i) - \frac{1}{2} T_i(r) \right) \cdot n_i,$$
Matching UV and IR results (by non-anomalous $U_R$ and 't Hooft anomaly matching) leads to:

$$24a = 5n_V + n_H + 12R(A) + 8R(B)$$
$$12c = 2n_V + n_H + 8R(B)$$
$$k_i = T_i(r) - 2R(C_i)$$

In the rank-1 case, $n_V = 1$, $n_H = h_1$, $r$ has been determined, implying scaling dimension of $A$, $B$, $C_i$ to be determined.

Holomorphy and EM duality of low energy physics and topological invariance lead to:

- $A(u) = \alpha \det(\frac{\partial u_i}{\partial a_j})$
- $B^8$ must be a single-valued holomorphic function of $u$ (for $\sigma \in 8\mathbb{Z}$ on spin 4-manifolds)
- $C$ must be a single-valued holomorphic function of $u$ (for instanton number $n \in \mathbb{Z}$)
For the rank-1 case ($Z \equiv |\{S_i\}|$):

\[
R(A) = \frac{\Delta - 1}{2} \\
R(B) = \frac{\Delta}{8} \sum_{i=1}^{Z} \frac{12c_i - 2 - h_i}{\Delta_i}
\]

In turn the conformal central charge $a$ and $c$ can be determined:

\[
24a = 5 + h_1 + 3\frac{\Delta - 1}{2} + \Delta \sum_{i=1}^{Z} \frac{12c_i - 2 - h_i}{\Delta_i},
\]

\[
12c = 2 + h_1 + \Delta \sum_{i=1}^{Z} \frac{12c_i - 2 - h_i}{\Delta_i}.
\]
Classification of 4d rank-1 $N=2$ SCFTs with planar CB has been performed

A minimal set of $N=2$ SCFT data has been computed exactly

Further discussion on the flavor central charge $k$?

Similar story of 4d rank-1 $N=2$ SCFTs with non-planar CB?

Generalization to higher rank SCFTs?

Stringy realization of $\mathfrak{sp}$ series?

Possible $d=5$ $N=1$/$d=6$ $N=(1,0)$ versions?
Thanks!