Space charge and Impedances

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Contents:

- Introduction: Interplay of space charge and impedance related effects.
- Betatron tune shifts from space charge and impedances.
- Beam response and stability with space charge and impedances (coasting beams).
- Instability thresholds with space charge: Resistive wall, TMCI.

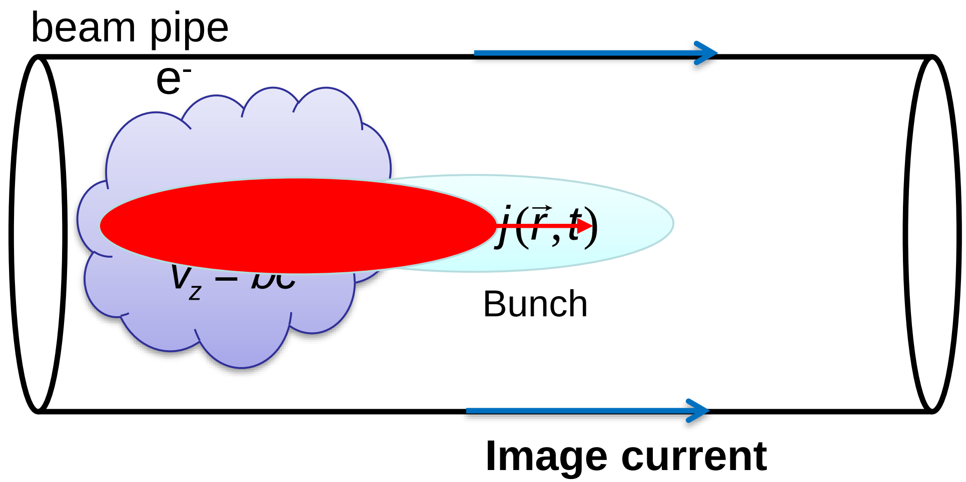
Introduction

Incoherent space charge:

$\varepsilon_0 \nabla \cdot \vec{E} = \rho$ (in the rest system of the beam)

tune shift: $\Delta Q_y^{sc} \propto -\frac{q^2}{m}\frac{N}{B_f}\frac{4}{\varepsilon_y \beta_0^2 \gamma_0^3}\frac{1}{1+\sqrt{\varepsilon_y/\varepsilon_x}} \lesssim 0.3-0.5$

-> beam intensity and emittance limits



Wakefields and impedances:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$ (lab system)

- image currents in the beam pipe

-> heat load and resistive wall instability

Electron clouds:
created by residual gas or wall emission.

Beam-beam interaction:
Can be incoherent and/or coherent

The interplay of collective mechanisms can lead to complex effects, usually studied in computer simulations.
Here we will focus on space charge and impedances and on analytical models (mostly).

Literature

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Books:

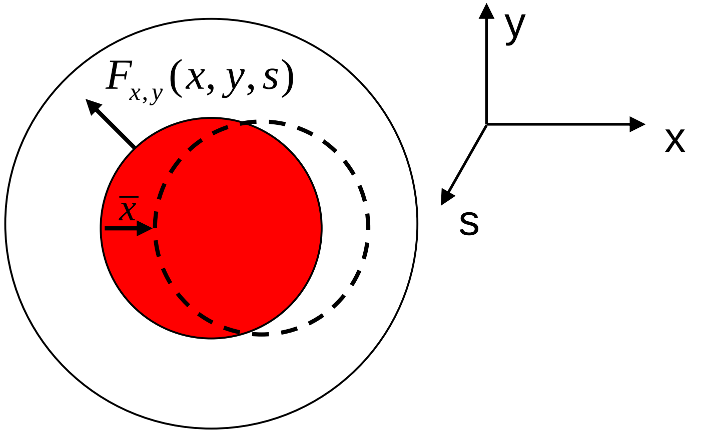
- Alex Chao, *Physics of collective beam instabilities in high energy accelerators* (1993)
- K.Y. Ng, *Physics of intensity dependent beam instabilities* (2006)

Selected articles (in refereed journals and conference proceedings):

- D. Möhl, H. Schönauer, Part. Accel. (1974, 1995)
- D. Pestrikov, Nucl. Inst. Meth. A (2006, 2007)
- M. Blaskiewicz, Phys. Rev. ST-AB (1998, 2001)
- A. Burov, V. Lebedev, Phys. Rev. ST-AB (2009)
- O. Boine-F., V. Kornilov, S. Paret, Phys. Rev. ST-AB (2000,2010)
- …..

Betatron tune shifts

Betatron oscillations with beam induced forces:



$$x'' + \frac{Q_{x0}^2}{R^2} x = \frac{F_x(x, \overline{x}, s)}{m\gamma(\beta c)^2} \quad \text{(only horizontal)}$$

Assuming small offsets:

$$x'' + \frac{Q_{x0}^2}{R^2} x = \frac{1}{m\gamma(\beta c)^2} \left(\left.\frac{\partial F_x}{\partial x}\right|_{\overline{x}=0} x + \left.\frac{\partial F_x}{\partial \overline{x}}\right|_{x=0} \overline{x} \right)$$

Incoherent tune shift: $Q_x = Q_{x0} + \Delta Q_x^{\text{inc}}$ $\quad \Delta Q_x^{\text{inc}} = -\frac{R^2}{2Q_{x0}\gamma m(\beta c)^2} \left.\frac{\partial F_x}{\partial x}\right|_{\overline{x}=0}$

Beam offset oscillations (coherent):

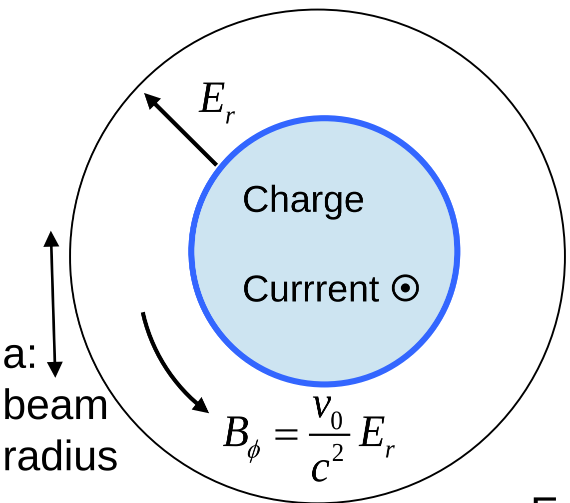
$$\overline{x}'' + \frac{Q_{x0}^2}{R^2} \overline{x} = \frac{1}{m\gamma(\beta c)^2} \left(\left.\frac{\partial F_x}{\partial x}\right|_{\overline{x}=0} \overline{x} + \left.\frac{\partial F_x}{\partial \overline{x}}\right|_{x=0} \overline{x} \right)$$

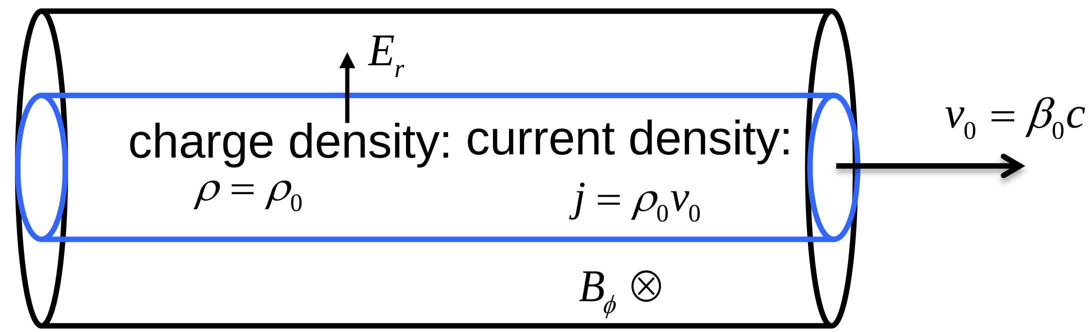
Coherent tune shift:

$$\Delta Q_x^{\text{coh}} = -\frac{R^2}{2Q_{x0} m\gamma(\beta c)^2} \left(\left.\frac{\partial F_x}{\partial x}\right|_{\overline{x}=0} + \left.\frac{\partial F_x}{\partial \overline{x}}\right|_{x=0} \right)$$

Transverse space charge force in a coasting beam

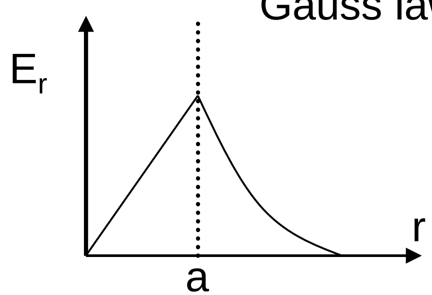
Beam in a vacuum pipe





Constant beam density:

$$\rho(r) = \begin{cases} \rho_0(s), r \le a \\ 0, r > a \end{cases}$$



Gauss law: $\varepsilon_0 \int E_r dA = \int \rho dV$

$$E_r = \begin{cases} \frac{\rho_0 r}{2\epsilon_0}, & r < a \\ \frac{\rho_0 a}{2\epsilon_0 r}, & r \ge a \end{cases}$$

Stokes: $\int B_\phi ds = \mu_0 v_0 \int \rho dA \quad \Rightarrow \quad B_\phi = \frac{v_0}{c^2} E_r$

beam current: $I(s) = \pi a^2 v_0 \rho_0$

line density: $I(s) = q v_0 \lambda(s)$

Defocusing force on a beam particle:

$$F_x = q(E_x - v_0 B_x) = q(1-\beta^2)E_x = \frac{qE_x}{\gamma^2} = \frac{q\lambda}{2\pi\epsilon_0 a^2 \gamma^2} x$$

Incoherent space charge tune shift

Transverse space charge force:

$$F_x(x,\bar{x}) = \frac{qE_x(x-\bar{x})}{\gamma^2} = \frac{q^2\lambda}{2\pi\varepsilon_0\gamma^2 a^2}(x-\bar{x})$$

Incoherent tune shift:

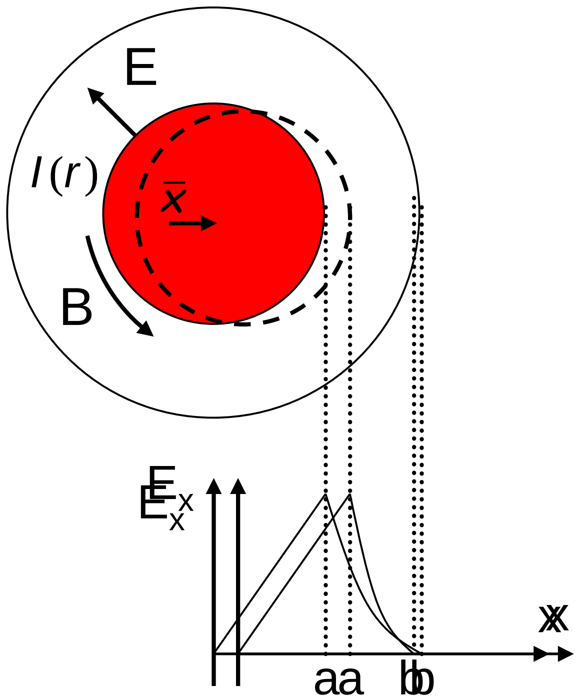
$$\Delta Q_x^{sc} = -\frac{R^2}{2Q_{x0}\gamma m(\beta c)^2}\left.\frac{\partial F_x}{\partial x}\right|_{\bar{x}=0} = -\frac{Nr_pR}{2\pi\beta^2\gamma^3 a^2 Q_{x0}}$$

Maximum space charge tune shift:

$$\Delta Q_x^{\mathrm{inc}} = -\frac{Nr_0}{2\pi\beta^2\gamma^3\varepsilon_x B_f} \qquad B_f = \frac{I_0}{I_{\max}} \qquad a^2 \approx \frac{R}{Q_{x0}}\varepsilon_x$$

(bunching factor) (beam envelope)

Oscillation of a beam inside a pipe.

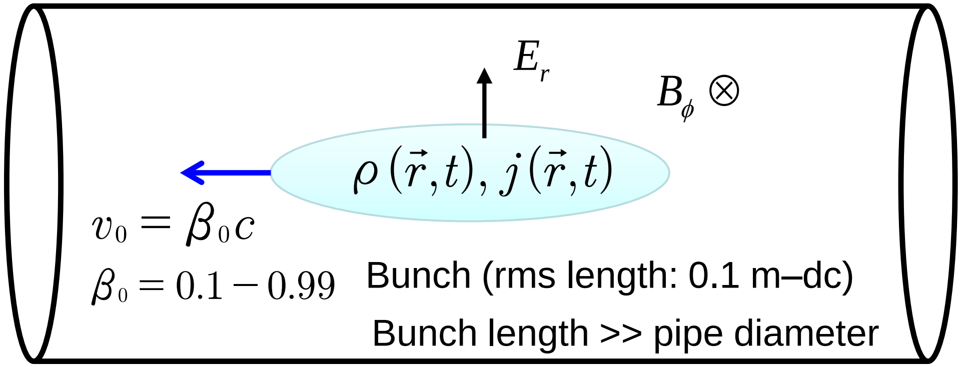


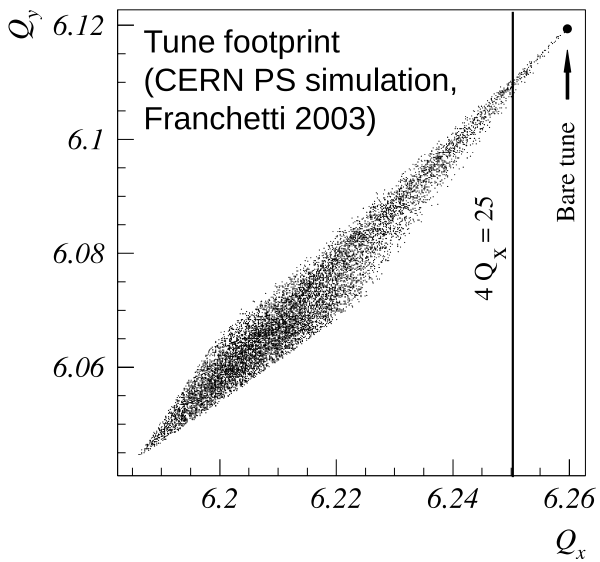
Space charge field moves with the beam center

Space charge tune spread in bunches

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho$$

(in the rest system of the beam)





Maximum space charge tune shift:

$$\Delta Q_x^{\mathrm{inc}} = -\frac{N r_0}{\pi \beta^2 \gamma^3 B_f \varepsilon_x} \frac{g_f}{\left(1+\sqrt{\frac{\varepsilon_y \hat{\beta}_y}{\varepsilon_x \hat{\beta}_x}}\right)}$$

-> Lecture by G. Franchetti (Saturday) !

Image currents and force in a cylindrical pipe

beam with a horizontal offset

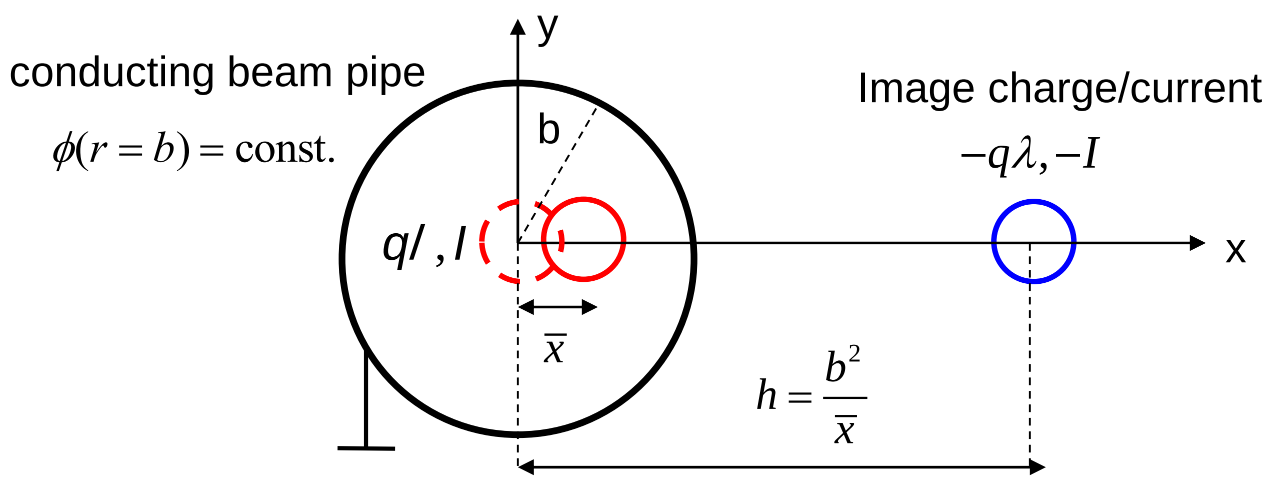


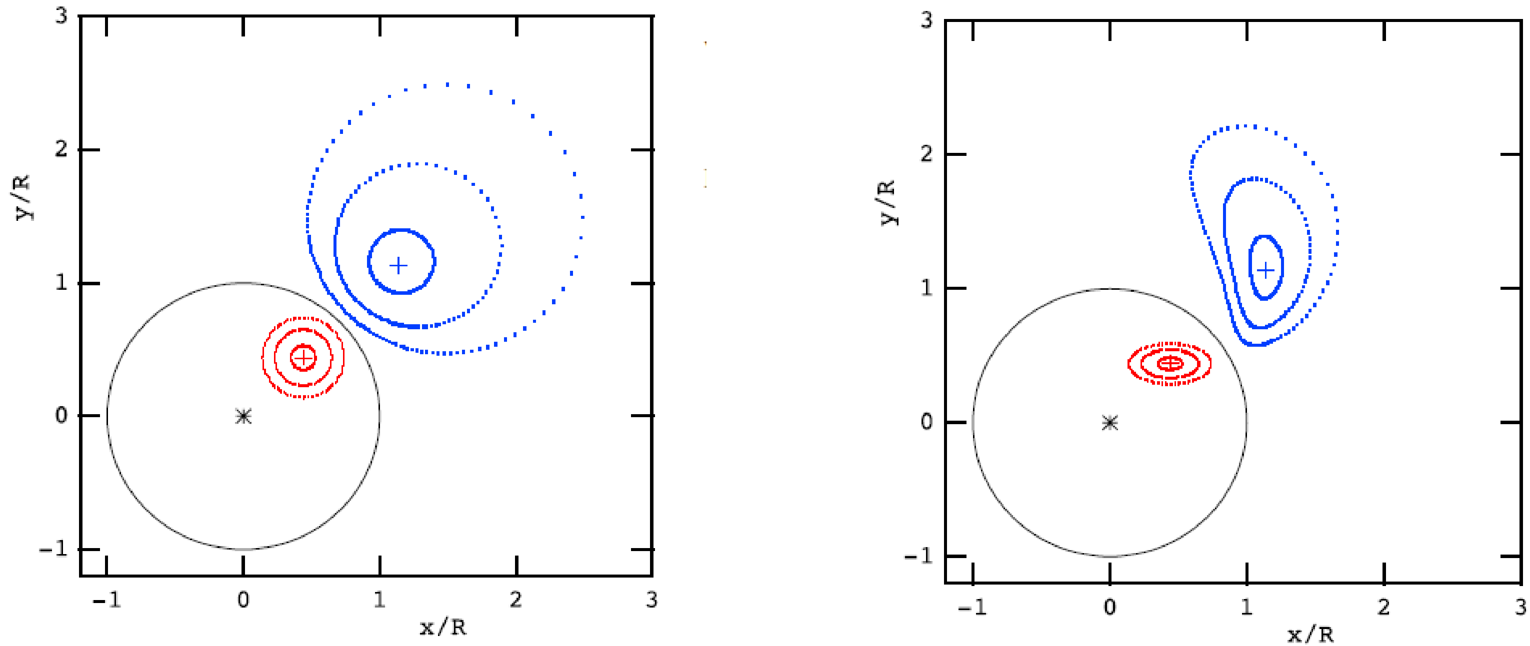
Image fields in the beam pipe:

$$E_x = \frac{q\lambda}{2\pi\varepsilon_0 h} \qquad B_y = \frac{\mu_0 I}{2\pi h}$$

Force on the beam center (for small offset):

$$\Rightarrow \quad \bar{F}_x = q(E_x - v_0 B_y) = \frac{qE_x}{\gamma^2} = \frac{q^2\lambda\bar{x}}{2\pi\varepsilon_0\gamma^2 b^2}$$

Remark: Electric field of 2D charge distributions in a cylinder



M. Furman, Phys. Rev. ST-AB (2007)

Transverse impedance of a cylindrical beam pipe

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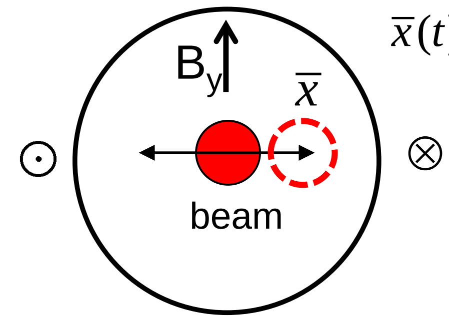
Transverse impedance (definition)

$$Z_x(\omega) = \frac{-i}{q\beta I\overline{x}} F_x = \frac{-i}{\beta I\overline{x}}\left(E_x + \left[\vec{v}_0 \times \vec{B}\right]_x\right) \quad [\Omega / \mathrm{m}^2]$$

$$\left.\frac{\partial F_x}{\partial \overline{x}}\right|_{x=0} = iqZ_x\beta I = iq^2 Z_x \beta^2 c\lambda$$

beam pipe



Offset oscillations

$$\overline{x}(t) = \hat{x}\exp(-i\omega t)$$

image currents

Example: Ideally conducting pipe

$$F_x = q(E_x - v_0 B_y) = \frac{qE_x}{\gamma^2} = \frac{q^2\lambda\overline{x}}{2\pi\varepsilon_0\gamma^2 b^2} \quad \Rightarrow \quad Z_x = -i\frac{Z_0}{2\pi(\beta_0\gamma_0)^2 b^2}$$

$$Z_0 = \left(\varepsilon_0 c\right)^{-1} = 377\,\Omega$$

(imaginary impedance)

Resistive pipe (low frequencies):

$$Z_x(\omega) = \frac{c}{b^3 d\sigma_w\omega}, \quad d \ll \delta_w$$

(resistive wall impedance)

$$\delta_w = \sqrt{\frac{2}{\mu_0\sigma_w\omega}}$$

(skin depth)

Coherent tune shift in a cylindrical pipe (ideal conductor)

Force: $$F_x(x,\bar{x}) = \frac{q^2\lambda}{2\pi\varepsilon_0\gamma_0^2 a^2}(x-\bar{x}) + \frac{q^2\lambda}{2\pi\varepsilon_0\gamma^2 b^2}\bar{x}$$

(incoherent: space charge) (coherent: images)

Coherent tune shift: $$\Delta Q_x^{\mathrm{coh}} = -\frac{R^2}{2Q_{x0}\gamma m(\beta c)^2}\left(\left.\frac{\partial F_x}{\partial x}\right|_{\bar{x}=0} + \left.\frac{\partial F_x}{\partial \bar{x}}\right|_{x=0}\right) = -\frac{Nr_pR}{2\pi\beta^2\gamma^3 b^2 Q_{x0}B_f}$$

Transverse “space charge” impedance:

$$Z_\perp = i\frac{4\pi Q_0\gamma mc}{q^2\lambda R}\left(\Delta Q_x^{sc} - \Delta Q_x^{coh}\right) \Rightarrow \quad Z_\perp^{sc} = i\frac{2Z_0R}{(\beta\gamma)^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

General pipe geometries

$$\Delta Q_x^{inc} = -\frac{N r_p R}{\pi\gamma\beta^2 Q_{0x}}\left(\frac{\epsilon_{1x}}{h^2} + \beta^2\frac{\epsilon_{2x}}{h^2} + \left(1-\beta^2\right)\frac{\epsilon_x^{sc}}{a^2}\right)$$

$$\Delta Q_x^{\mathrm{coh}} = -\frac{N r_p R}{\pi\gamma\beta^2 Q_{0x}}\left((1-\beta^2)\frac{\xi_1}{h^2} + \beta^2\frac{\epsilon_{1x}}{h^2} + F\beta^2\frac{\epsilon_{2x}}{g^2}\right)$$

See book by K.Ng, Chap. 3.

Beam force components	Images in vacuum chamber: electric	Images in vacuum chamber: magnetic	Images in pole faces: magnetic	Comments
$\left.\frac{\partial\langle F_{\mathrm{beam}}\rangle}{\partial y}\right\|_{\bar{y}=0}$	$\frac{\epsilon_{1y,x}}{h^2}$	$-\beta^2\frac{\epsilon_{1y,x}}{h^2}$	$\beta^2\frac{\epsilon_{2y,x}}{g^2}$	incoherent dc coherent
$\left.\frac{\partial\langle F_{\mathrm{beam}}\rangle}{\partial y}\right\|_{\bar{y}=0} + \left.\frac{\partial\langle F_{\mathrm{beam}}\rangle}{\partial \bar{y}}\right\|_{y=0}$	$\frac{\xi_{1y,x}}{h^2}$	$-\beta^2\frac{\xi_{1y,x}}{h^2}$	$\beta^2\frac{\xi_{2y,x}}{g^2}$	coherent
$\left.\frac{\partial\langle F_{\mathrm{beam}}\rangle}{\partial \bar{y}}\right\|_{y=0}$		$-\beta^2\frac{\xi_{1y,x}-\epsilon_{1y,x}}{h^2}$	$\beta^2\frac{\xi_{2y,x}-\epsilon_{2y,x}}{g^2}$	ac coherent

Transverse beam stability (with space charge)

Simplified transverse particle equation of motion:

$$x'' + \frac{Q_x^2}{R^2} x = \frac{1}{\gamma_0 m v_0^2} \frac{\partial F_x}{\partial \overline{x}}\bigg|_{x=0} \overline{x} \qquad x'' + \frac{\omega_\beta^2}{(\beta c)^2} x = \frac{2Q_{x0}}{R^2}\left(\Delta Q_x^{coh} - \Delta Q_x^{sc}\right)\overline{x}$$

$$\overline{x}(s,\theta_0) = \overline{A}\exp\left(i\Omega s/(\beta c)\right)$$

(coherent betatron frequency)

$$\omega_\beta = \omega_0 Q_x$$

(incoherent betatron frequency)

Ansatz: $x(s,\theta) = A\exp\left(in\theta - i\Omega s/(\beta c)\right) \qquad \theta = \theta_0 + \omega_0 t$

$$A = (\beta c)^2 \frac{2Q_0\left(\Delta Q_x^{coh} - \Delta Q_x^{sc}\right)\overline{A}}{R^2\left[\omega_\beta^2 - (n\omega_0 - \Omega)^2\right]} \approx \omega_0 \frac{\left(\Delta Q_x^{coh} - \Delta Q_x^{sc}\right)\overline{A}}{\Omega - n\omega_0 - \omega_\beta}$$

Dispersion relation (wave number n):

$$1 = \omega_0\left(\Delta Q_x^{coh} - \Delta Q_x^{sc}\right)\int \frac{f(\omega_\beta)}{\Omega - n\omega_0 - \omega_\beta} d\omega_\beta \qquad \Omega = \Omega_R + i\Omega_I$$

$$1 = \frac{\Delta Q_x^{coh} - \Delta Q_x^{sc}}{S\delta_{rms}} \int \frac{g(\delta)}{\hat{\Omega} - \delta} d\delta \qquad \hat{\Omega} = \frac{\Omega - \omega_\beta - n\omega_0}{S\delta_{rms}\omega_0}$$

$$\omega_\beta = \omega_0 Q_{x0} + \omega_0 \Delta Q^{sc}$$

Frequency spread: $\omega_n = (n+Q)\omega_0$

$$\frac{d\omega_n}{(dp/p)} = (n+Q)\frac{d\omega_0}{(dp/p)} + \frac{dQ}{(dp/p)}\omega_0$$

$$\xi = \frac{\Delta Q}{\Delta p/p_0} \qquad \frac{\Delta\omega}{\omega_0} = -\eta_0 \frac{\Delta p}{p_0}$$

(chromaticity) (frequency slip)

$$\Rightarrow \quad \delta\omega_n = (\xi - \eta_0(n+Q))\omega_0 \frac{\Delta p}{p} = S\omega_0\delta$$

(Stable) unbunched Gaussian beams

Gaussian momentum distribution:

$$g(\delta) = \frac{1}{\sqrt{2\pi\delta_{rms}}} \exp\left(-\frac{\delta^2}{2\delta_{rms}^2} \right)$$

Dispersion function:

$$1 = \frac{\Delta Q_x^{coh} - \Delta Q_x^{sc}}{S\delta_{rms}} \int \frac{g(\delta)}{\hat{\Omega} - \delta} d\delta =: D(\hat{\Omega})$$

$$D(\hat{\Omega}) = (\Delta U + i\Delta V - \Delta U_{sc}) r_0\left(\hat{\Omega}\right)$$

$$r_0(z) = i\sqrt{\frac{\pi}{2}} \exp(-z^2/2)(1 - \mathrm{erf}(-iz/\sqrt{2}))$$ (complex error function)

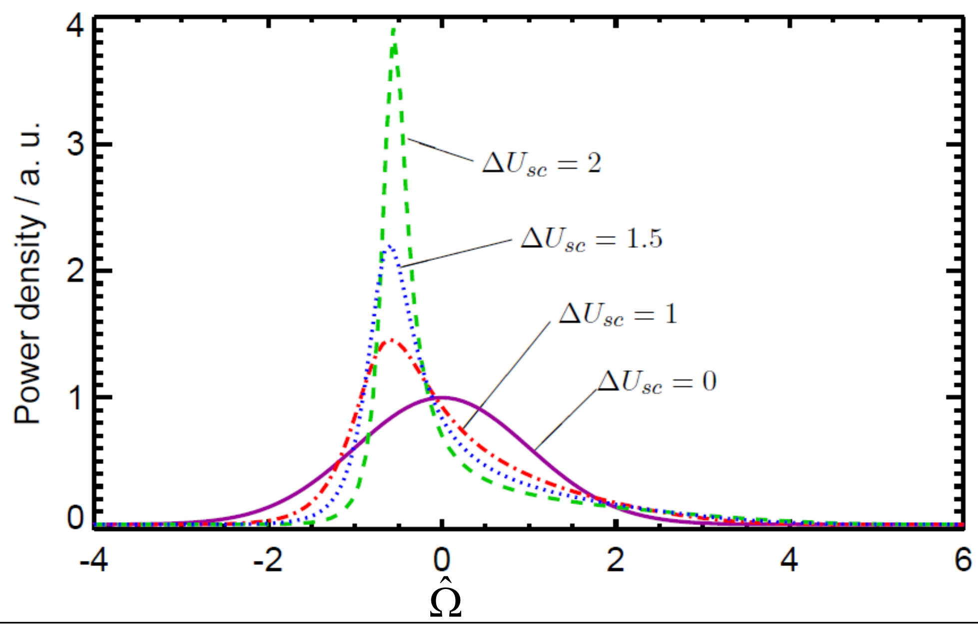
Normalised tune shifts:

$$\Delta U = -\frac{\Delta Q^{coh}}{S\delta_{rms}} \qquad \Delta U_{sc} = -\frac{\Delta Q^{sc}}{S\delta_{rms}}$$

(impedance parameter) (space charge parameter)

Schottky power spectrum:

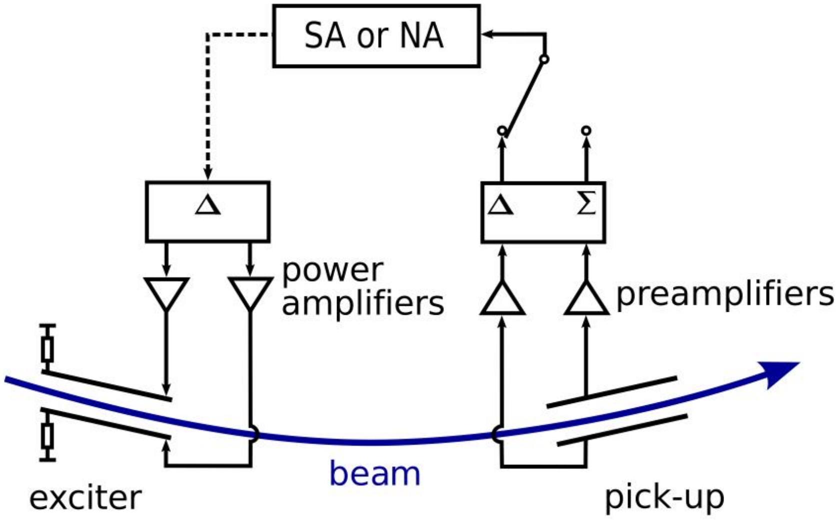
$$P(\hat{\Omega}) = \frac{P_0(\hat{\Omega})}{\left|1 - D(\hat{\Omega})\right|^2}$$



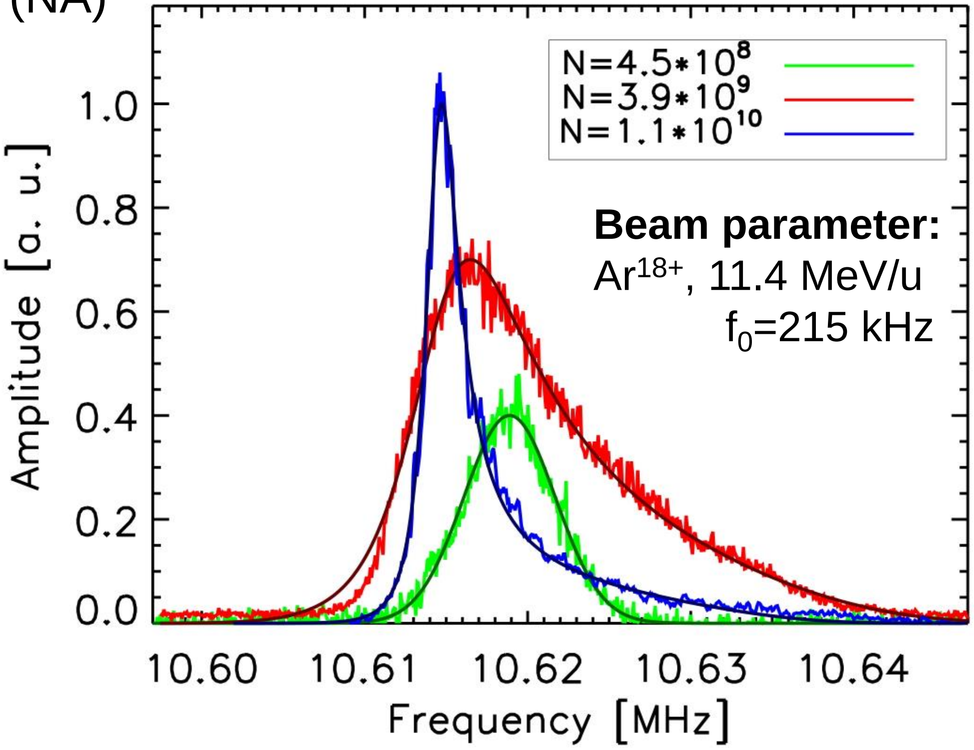
Measurement of transverse offset oscillations

Schottky signals:
Using a pick-up + spectrum analyzer (SA)
BTFs:
Using exciter/pick-up + network analyzer (NA)

$$P(f) = \frac{A\exp\left(-z^2\right)}{\left|1 - B\Delta U_{sc} w(z)\right|^2}$$



Measurement time: ≈ 100 msecs



Paret, Boine-F.,Kornoliv, Phys. Rev. ST-AB (2010)

Measurement of transverse offset oscillations II

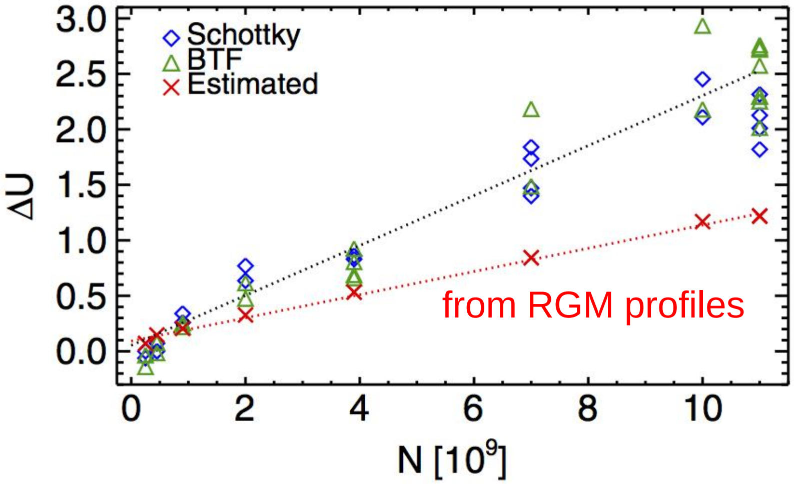
Space charge parameter:

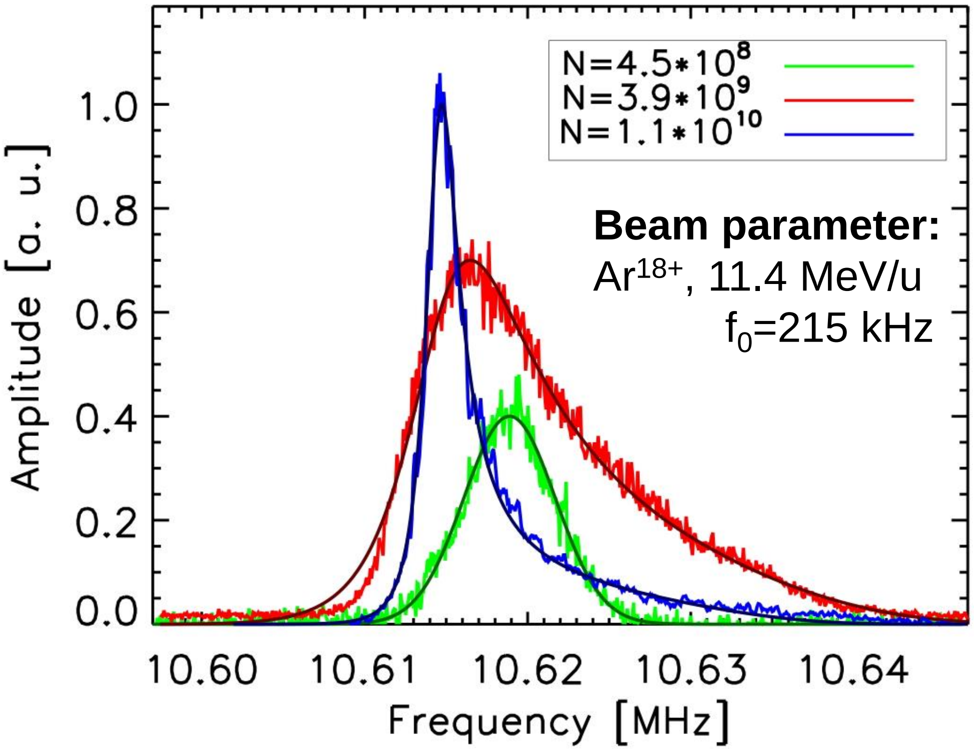
$$\Delta U_{sc} = \frac{\Delta Q_{sc}}{S\delta_{rms}}$$

Fit to a measured, modified Schottky band:

$$S(f) = \frac{A\exp\left(-u^2\right)}{\left|1 - B\Delta U_{sc} w(u)\right|^2}$$

Measurement time: ≈ 100 msecs





Beam parameter:
Ar^{18+}, 11.4 MeV/u
f_0=215 kHz

Example measurement:
Space charge tune shift from Schottky/BTF is systematically larger than the one from RGM profiles.

Beam transfer function (unbunched beams)

Response function:

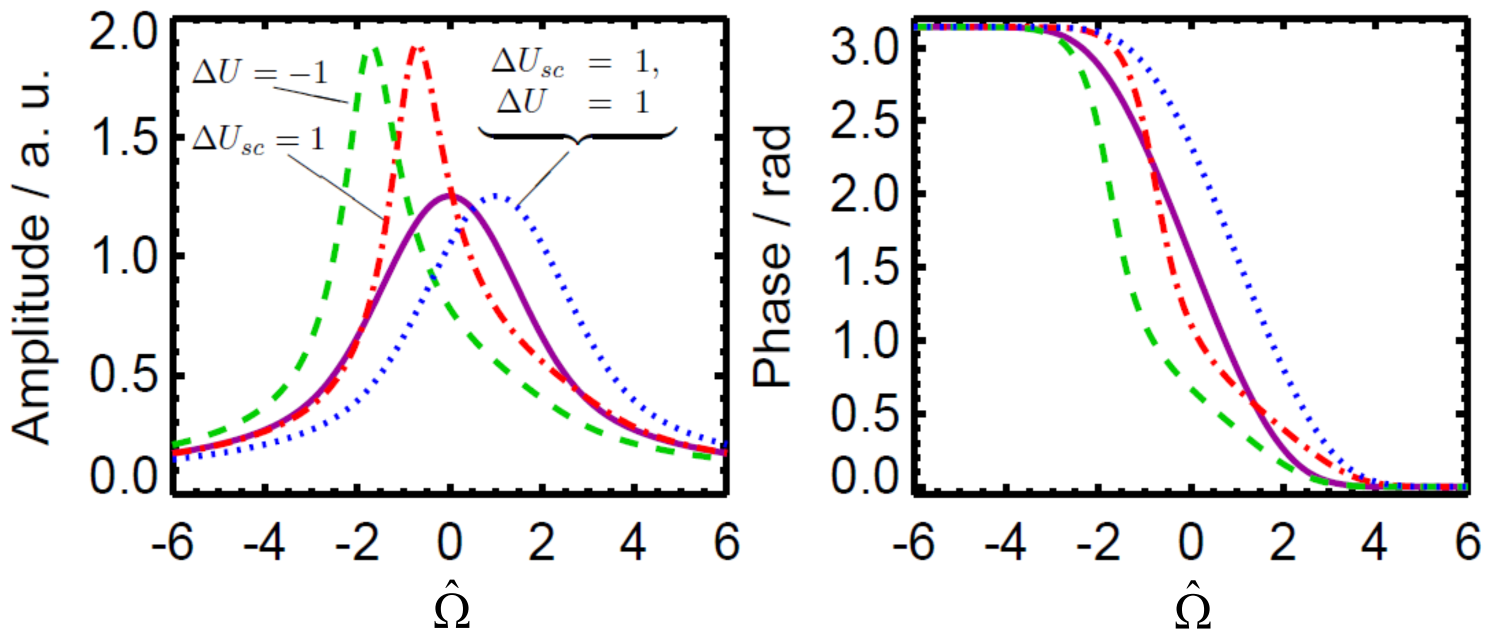
$$r(f) = \frac{A}{G} = \frac{\text{Offset amplitude}}{\text{External exciter}}$$

$$r(\hat{\Omega}) = \frac{r_0(\hat{\Omega})}{1 - D(\hat{\Omega})}$$

$$D(\hat{\Omega}) = (\Delta U - \Delta U_{sc}) r_0\left(\hat{\Omega}\right)$$

$$\hat{\Omega} = \frac{\Omega - \omega_\beta - n\omega_0}{S\delta_{rms}\omega_0}$$

$$\omega_\beta = \omega_0 Q_{x0} + \omega_0 \Delta Q^{sc}$$



Measurement of Beam Transfer Functions (unbunched beams)

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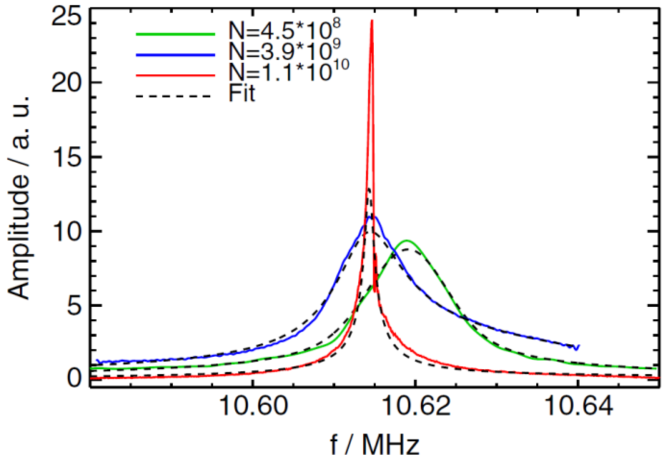
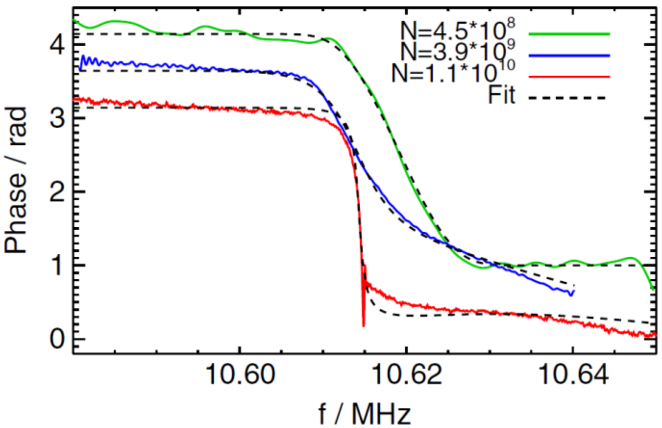
Beam transfer function:

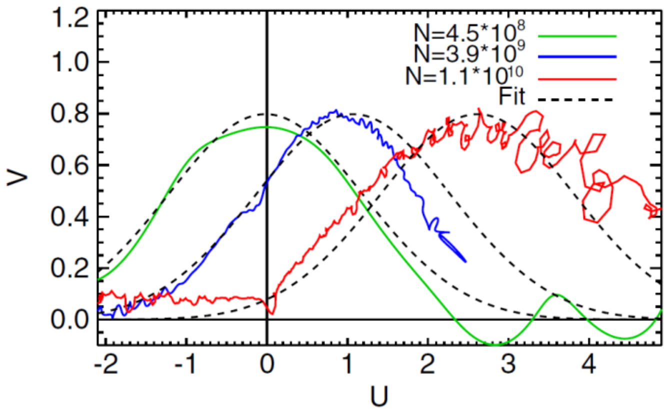
$$r(\hat{\Omega}) = \frac{r_0(\hat{\Omega})}{1 - D(\hat{\Omega})}$$





Beam stability function

$$\frac{1}{r(\hat{\Omega})} = U + iV$$



Resistive wall instability with space charge (unbunched beams)

$$(\Delta U + i\Delta V - \Delta U_{sc}) = \frac{1}{r_0(\hat{\Omega})}$$

$$\tau^{-1} = \omega_0 S \delta_{rms} \hat{\Omega}_I$$

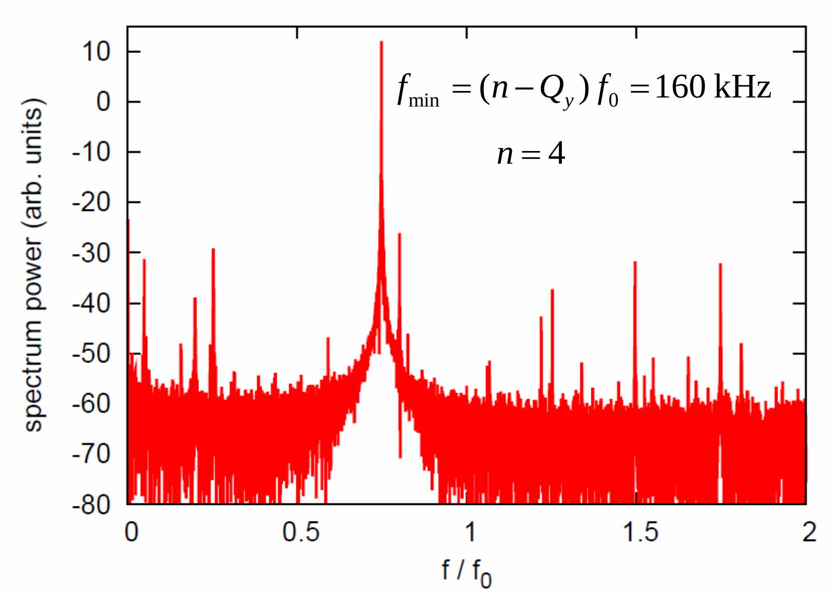
$$\tau^{-1} = \omega_0 \mathrm{Im}\left(\Delta Q_x^{coh}\right) \approx \frac{qIR^2c}{2\pi Q\beta_0 E_0 b^3 d\sigma_w \omega}$$

$$\Rightarrow \quad \left|\Delta U + i\Delta V - \Delta U_{sc}\right| \leq F$$

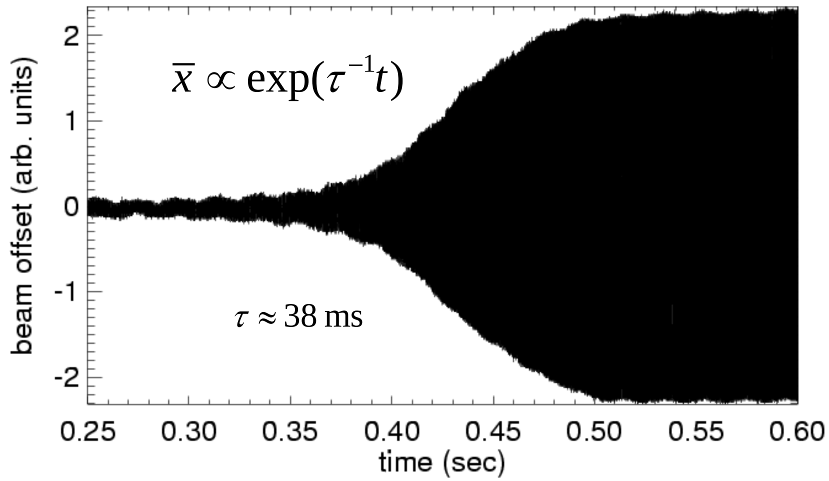
(circle approximation)

$$Z_x^R(\omega) = \frac{c}{\pi b^3 d\sigma_w \omega}$$

"Loss of Landau damping"

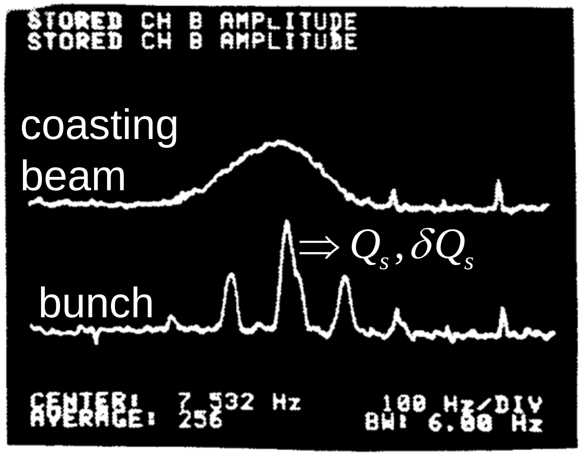


Beam offset vs. time observed in SIS 18

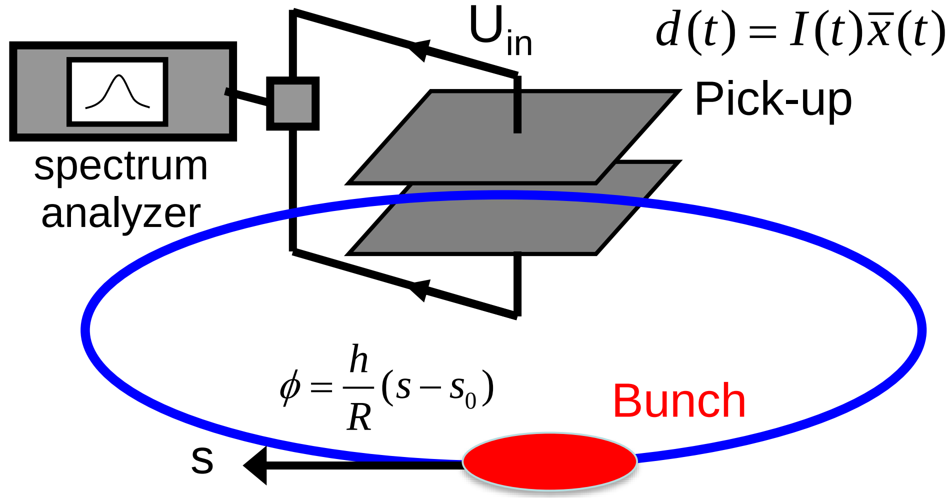


Tune spectra from bunches

Transverse spectra



CERN/SPS measurement (Linnecar, PAC 1981)



$$d(t) = I(t)\bar{x}(t)$$

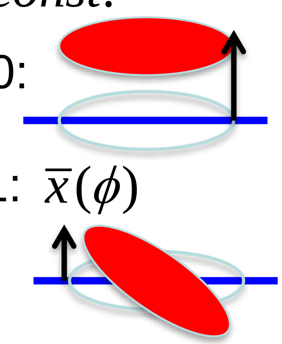
Synchrotron satellites (synchrotron tune Q_s):

$$Q_k = Q_0 + kQ_s \quad \text{or} \quad \Delta Q_k = kQ_s$$

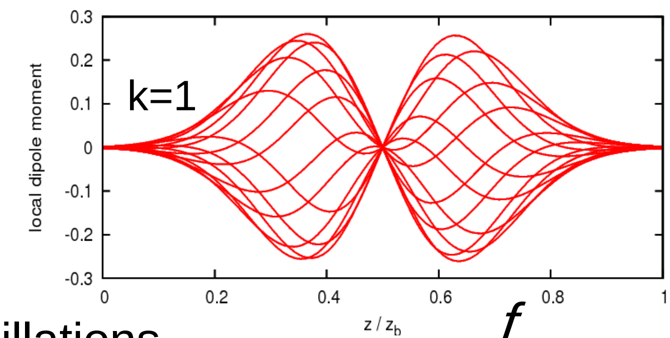
Tune spread in a rf bucket (rms bunch length σ):

$$\delta Q \approx kQ_s \frac{\sigma_\phi^2}{4}$$

$\bar{x} = const.$



$$I(\phi)\bar{x}_k(\phi) \propto e^{-i\chi\phi/\phi_b} e^{-iQ_k t}$$



head-tail oscillations

Bunches modes with space charge

Blaskiewicz, Phys. Rev. ST Accel. Beams (1998)

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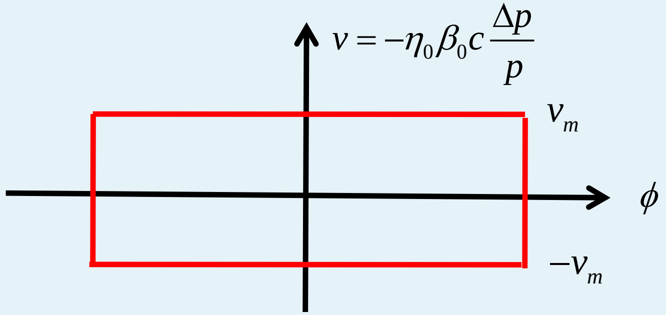
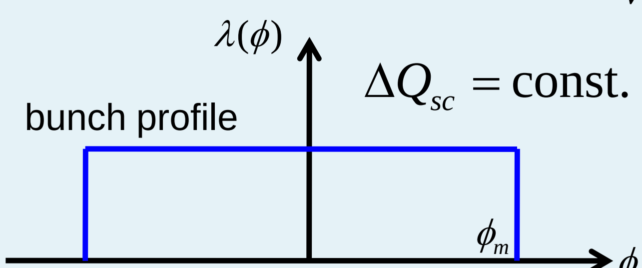
Longitudinal bunch distribution:

$$f(v,\phi) = A\left[\delta(v_m - v) + \delta(v_m + v)\right]$$

Synchrotron tune: $Q_s = \frac{1}{f_0}\frac{v_m}{2l}$ (bunch length l)



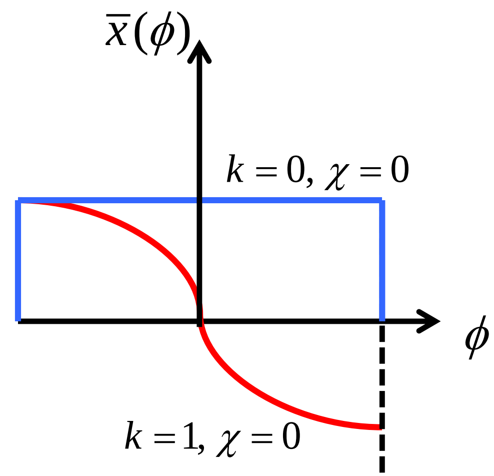
$x_{\pm}(\phi,t)$: transverse offset of particles with $v = \pm v_m$



head-tail eigenmodes and transverse offset:

$$\bar{x}_k(\phi) = (x_{+,k} + x_{-,k})/2 = \cos(k\pi\phi/\phi_b)e^{-i\chi\phi/\phi_b}e^{-iQ_k t}$$

Chromatic phase: $\chi = -\xi\phi_b/\eta_0$



head-tail tune shifts for the airbag distribution:

$$\Delta Q_k = -\frac{\Delta Q_{sc}}{2} \pm \sqrt{(\Delta Q_{sc}/2)^2 + (kQ_s)^2}$$

for $\Delta Q^{sc} = 0$: $\Delta Q_k = kQ_s$

Tune spectra in bunched beams with space charge

Weak space charge: $\Delta Q_{sc} \lesssim Q_s$ or $q_{sc} \lesssim 1$

space charge parameter:

$$q_{sc} = \frac{\Delta Q^{sc}}{Q_{s0}}$$

SIS-18/100: q_{sc}=10-20

CERN PSB/PS: q_{sc} ≥100

Shift of synchrotron satellites (synchrotron tune Q_s):

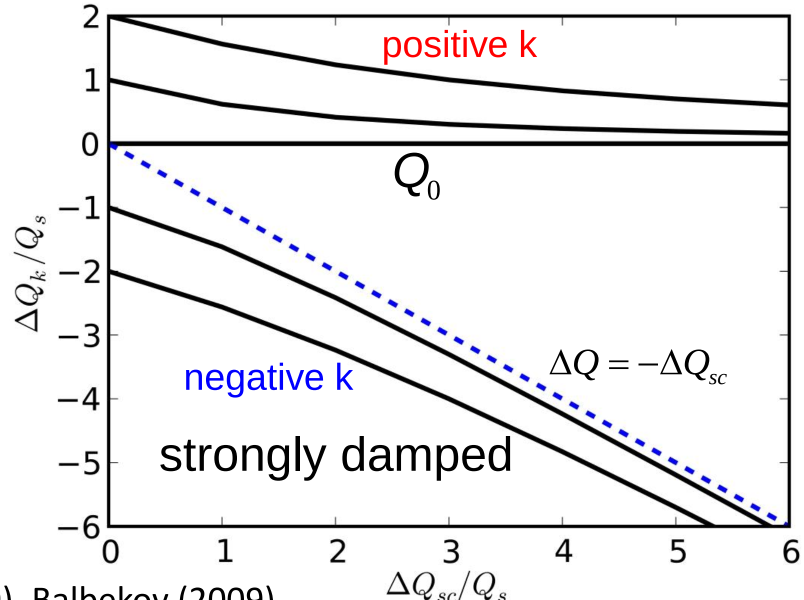
$$\Delta Q_k \approx kQ_s - \frac{1}{2}\Delta Q_{sc}$$

$$\Delta Q_k = Q_k - Q_0 = -\frac{\Delta Q_{sc}}{2} \pm \sqrt{(\Delta Q_{sc}/2)^2 + (kQ_s)^2}$$

Strong space charge $\Delta Q_{sc} \gg Q_s$ or $q_{sc} \gg 1$

positive k: $\Delta Q_k = \frac{(kQ_s)^2}{\Delta Q_{sc}} \to 0$

negative k: $\Delta Q_k = -\Delta Q_{sc}$

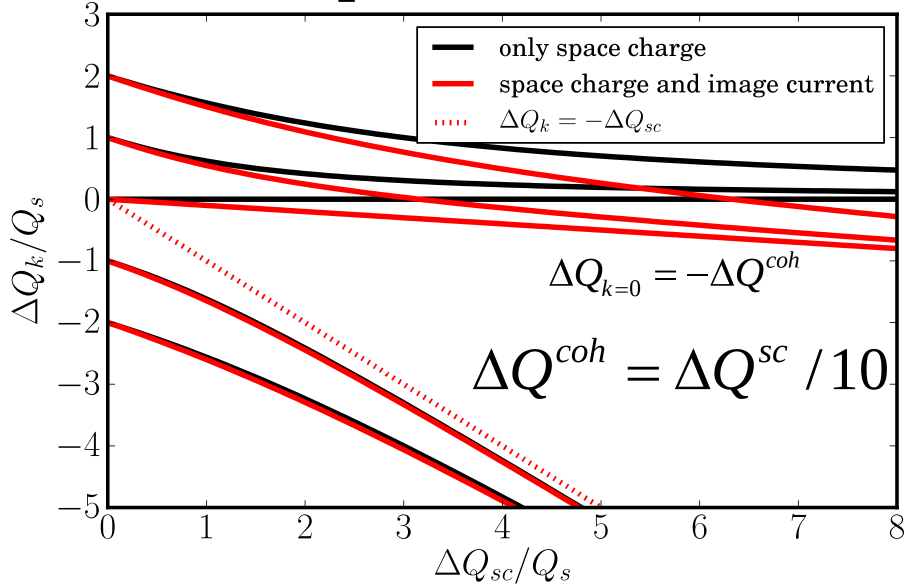


Blaskiewicz, Phys. Rev. ST Accel. Beams (1998)

Boine-F., Kornilov, Phys. Rev. ST Accel. Beams (2009), Burov (2009), Balbekov (2009)

Head tail modes with space charge and image currents

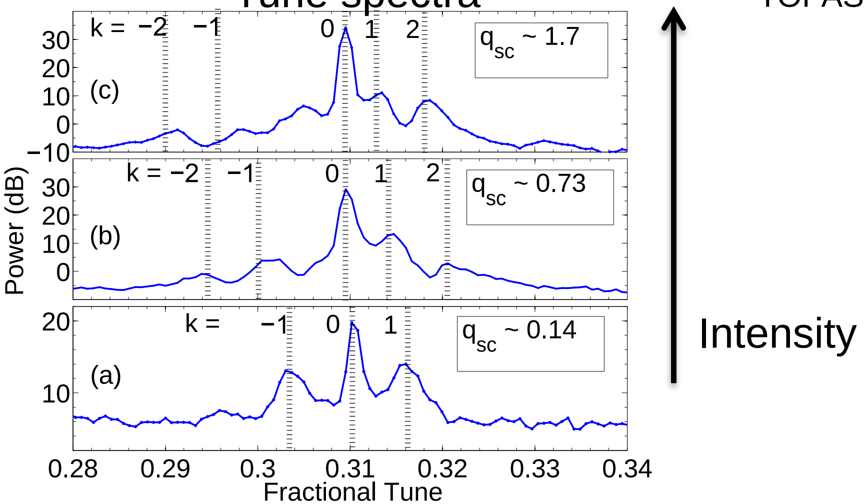
$$\Delta Q_k = -\frac{\Delta Q_{sc} + \Delta Q_c}{2} \pm \sqrt{(\Delta Q_{sc} - \Delta Q_c)^2 / 4 + (kQ_s)^2}$$



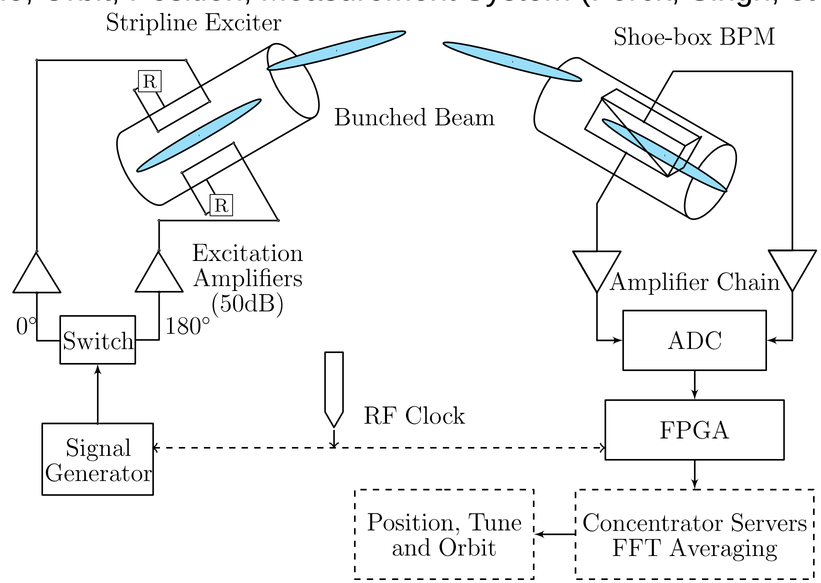
Measurements/Simulations: Airbag model describes the tune spectra for short bunches.

Measurement of tune spectra in bunches

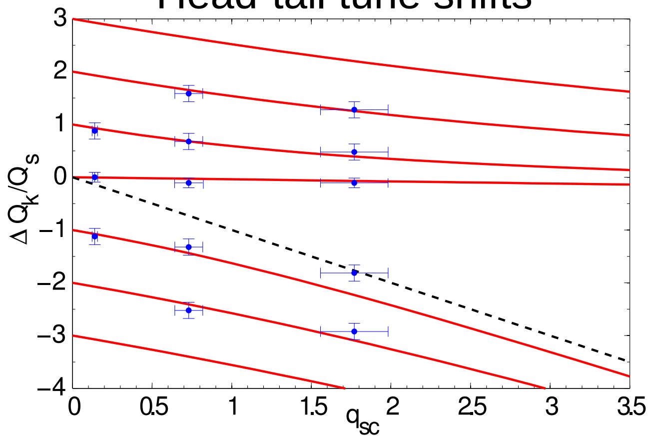
Tune spectra



TOPAS: Tune, Orbit, Position, Measurement System (Forck, Singh, et al.)



Head-tail tune shifts



$$\Delta Q_k = -\frac{\Delta Q_{sc}}{2} \pm \sqrt{(\Delta Q_{sc}/2)^2 + (kQ_s)^2}$$

Weak space charge:
Width of lines caused by nonlinear synchrotron motion.
Moderate space charge:
Width of the lines caused by nonlinear space charge

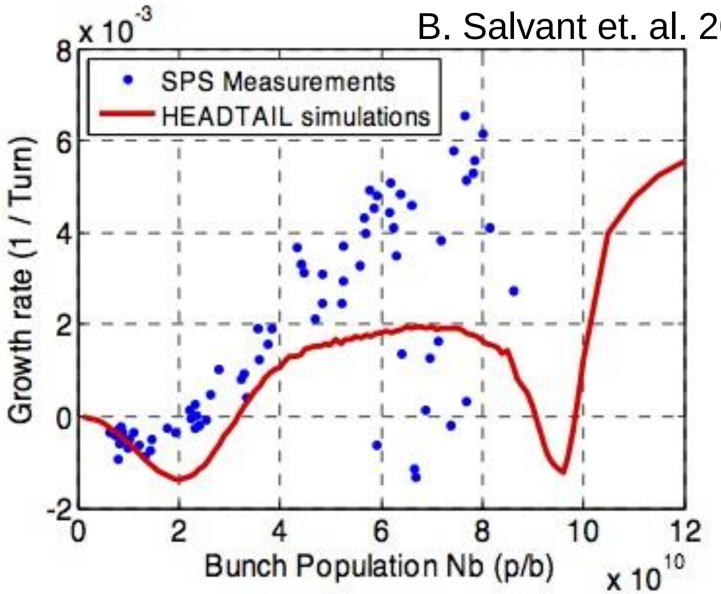
Singh, Boine-F., Chorny, Forck, et al., Phys. Rev. ST-AB (2013)

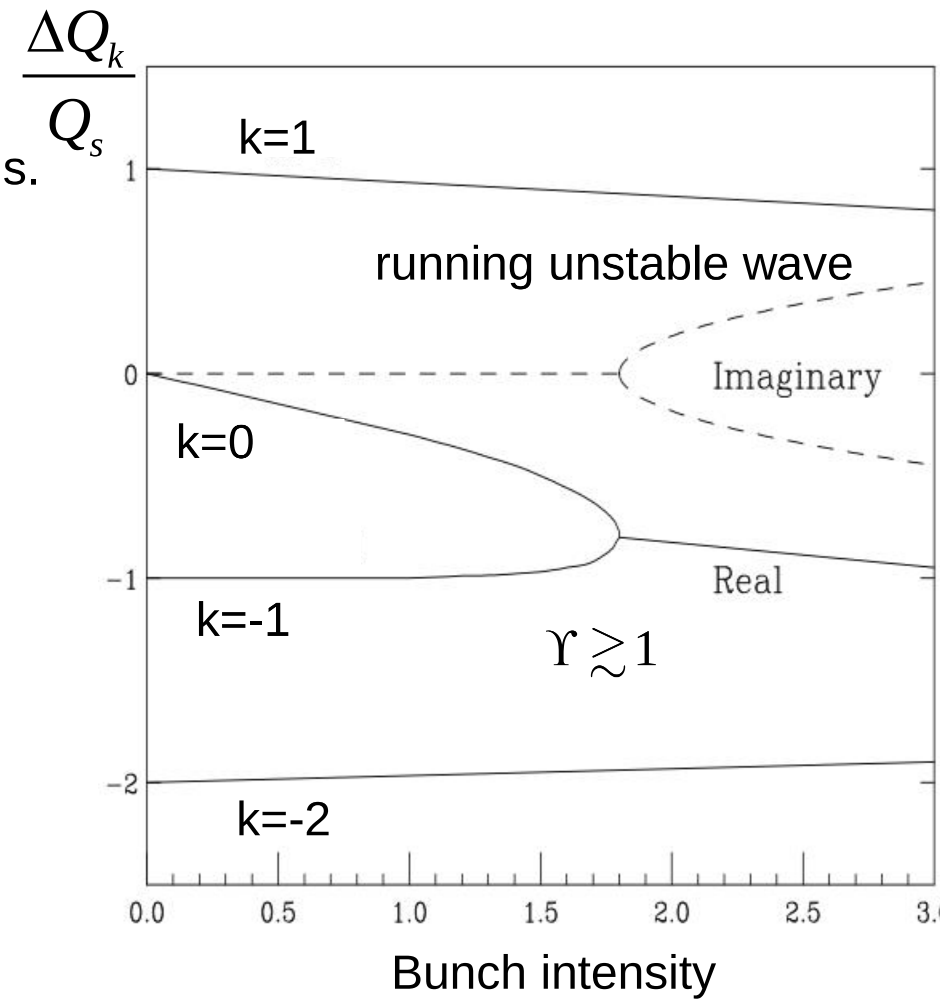
Transverse Mode-Coupling Instability (TMCI)

Observed in the CERN SPS/LHC.

Caused by the kicker impedance or e-clouds.

B. Salvant et. al. 2008

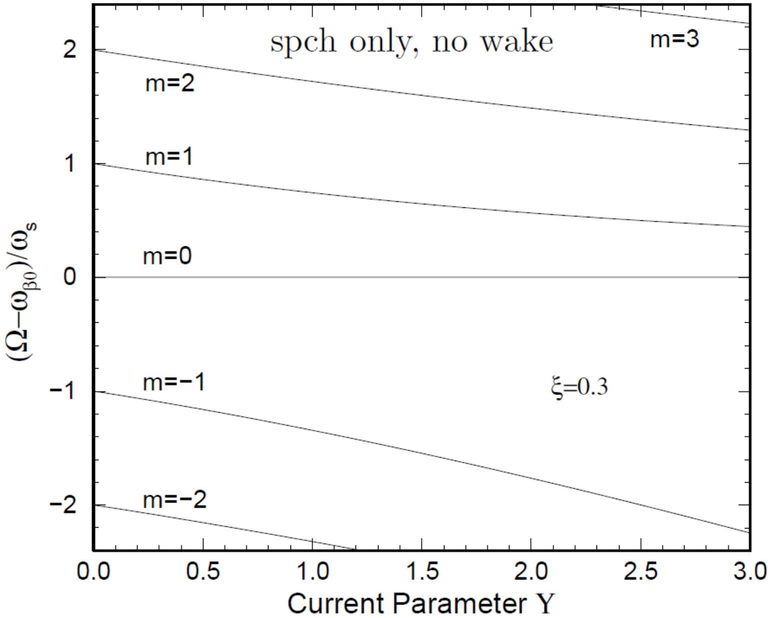


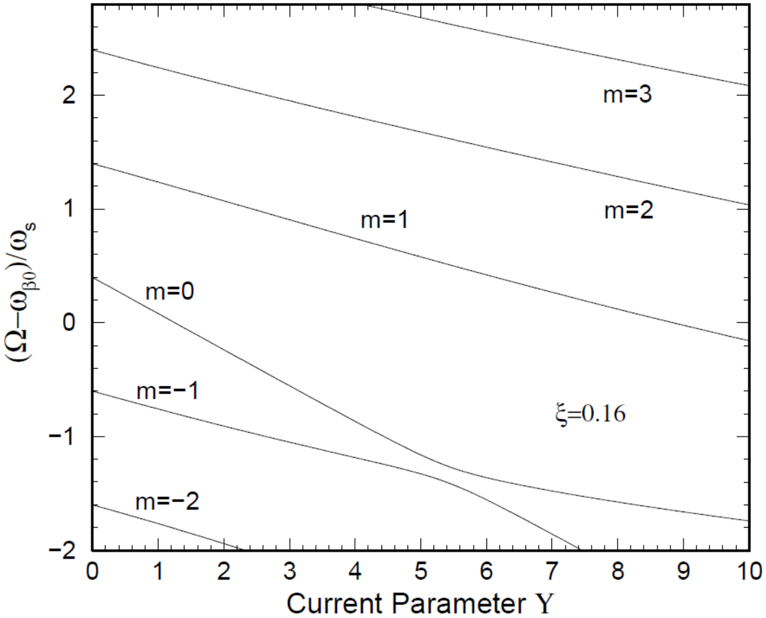


Intensity parameter: $\Upsilon = \frac{qI_b\hat{\beta}_x W_x}{4\beta^2 E_0 \omega_s}$

TMCI and space charge: Airbag model

$$\Delta Q_m = -\frac{\Delta Q_{sc}}{2} \pm \sqrt{(\Delta Q_{sc}/2)^2 + (mQ_s)^2}$$





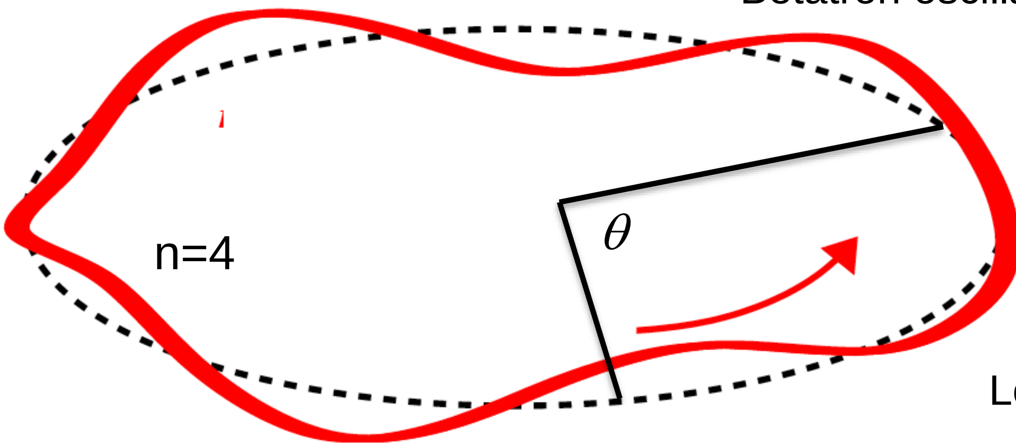
Summary and conclusions

- Example cases for the interplay of the (transverse) space charge force, image forces and wakefields have been given.
- Space charge does modify the coherent oscillation modes and instability thresholds.It can lower (Loss of Landau damping) or increase (suppress mode coupling) the instability threshold currents.
- The study of the interplay of space charge, impedances, electron clouds, beam-beam, ... becomes even more relevant as the machines are operated close to the intensity limits.
- To study „interplay“ computer models are usually the tools of choice, as analytical expressions can only be obtained for very reduced models (as in this lecture).

Backup

Reminder: Transverse spectrum for coasting beams

Betatron oscillation of one beam slice: $\bar{x}(\theta_0, t) = \hat{x}e^{\pm i\omega_0 Q_0 t}$



Seen by a stationary observer (pick-up):

$$\bar{x}(\theta, t) = \hat{x}e^{i(n\theta - \omega t)}$$

fast/slow mode: $\omega_{n,\pm} = (n \pm Q_0)\omega_0$

Lowest mode in SIS-18 ($Q_{0y} = 3.24$):

$$f_{\min,-} = (4 - Q_{y0}) f_0 \approx 0.4 f_0 \approx 80 \text{ kHz}$$

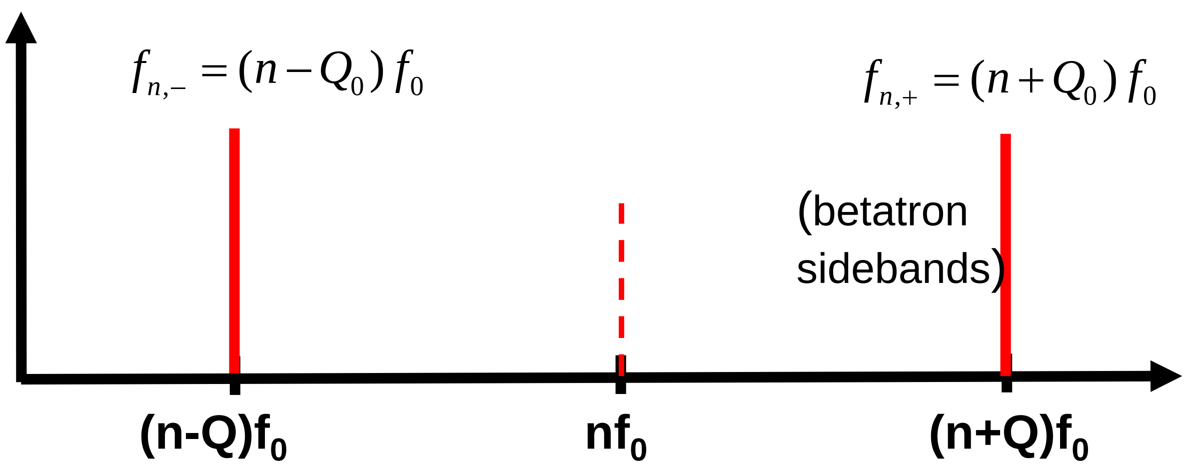
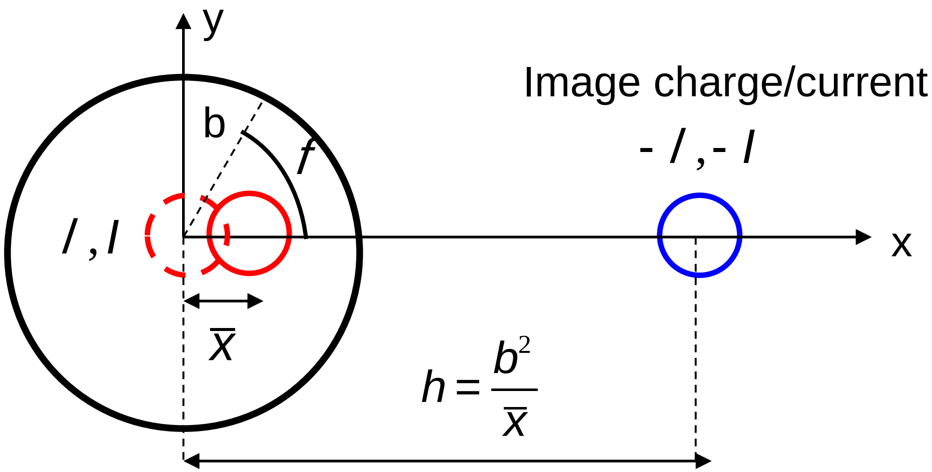
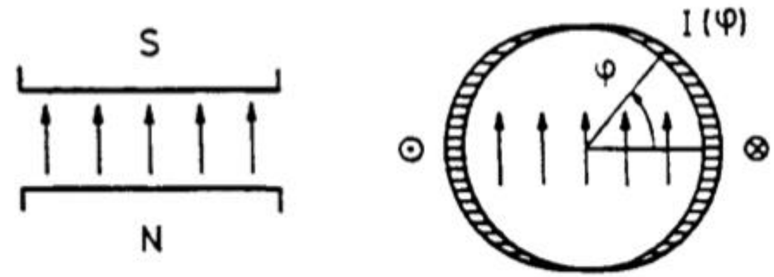


Image wall charge and current



cos(ϑ)-superconducting magnet



surface image current: $j_s(\phi) = j_s(0)\cos(\phi)$

Surface charge density:

$$\rho_s(\phi) \approx \frac{q\lambda\bar{x}}{\pi b^2}\cos\phi \qquad \lambda_s = \int_{-\pi/2}^{\pi/2} \rho_s(\phi) b\, d\phi = \frac{2}{\pi}\frac{\lambda\bar{x}}{b}$$

(line charge on one half of the pipe)

Surface current density:

$$j_s(\phi) \approx \frac{I\bar{x}}{\pi b^2}\cos\phi \qquad I_s = \int_{-\pi/2}^{\pi/2} j_s(\phi) b\, d\phi = \frac{2}{\pi}\frac{I\bar{x}}{b}$$

Dipole electric field:

$$E_x = \frac{1}{2\pi\epsilon_0}\int_0^{2\pi} \rho_s(\phi)\cos\phi\, d\phi = \frac{q\lambda\bar{x}}{2\pi\varepsilon_0 b^2}$$

Dipole magnetic field:

$$B_y = \frac{\mu_0 I\bar{x}}{2\pi b^2}$$

Transverse impedance of a resistive pipe

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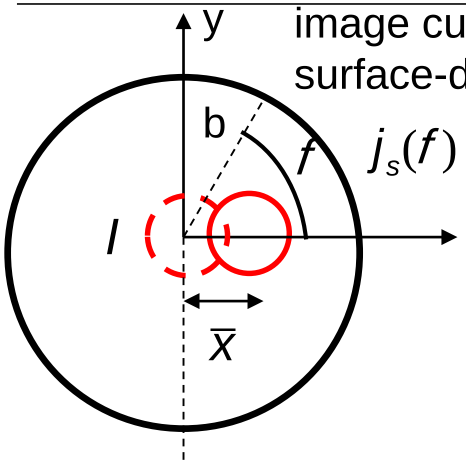


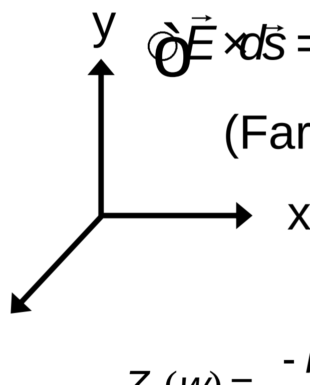
image current surface-density:

$$j_s(\phi) = j_s(0)\cos(\phi)$$

Vertical B-field:

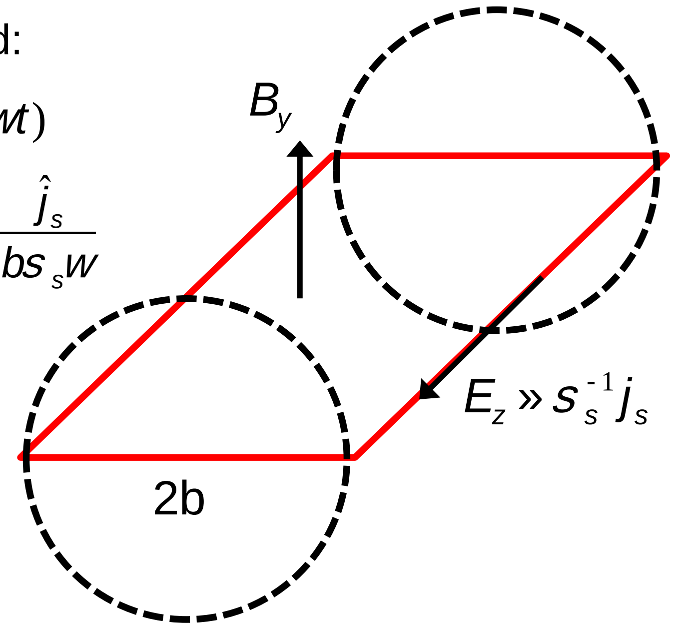
$$B_y = \hat{B}_y \exp(-i\omega t)$$

$$\hat{B}_y = -i\frac{\hat{E}_z}{\omega b} = -i\frac{\hat{j}_s}{b\sigma_s\omega}$$



$$\oint \vec{E}\cdot d\vec{s} = -\oint \dot{\vec{B}}\cdot d\vec{A}$$

(Faraday)



beam offset:

$$\bar{x}(t) = \hat{x}\exp(-i\omega t)$$

image surface current density:

$$j_s = \hat{j}_s \exp(-i\omega t) \qquad j_s(0) \approx \frac{I\bar{x}}{\pi b^2}$$

$$Z_x(\omega) = \frac{-i}{\beta_0 I\hat{x}}\left(\hat{E}_x - v_0\hat{B}_y\right) \Rightarrow$$

longitudinal electric field:

$$E_z = \hat{E}_z \exp(-i\omega t) \qquad E_z = \sigma_s^{-1} j_s \qquad \sigma_s = \sigma_w \delta$$

(surface conductivity)

$$Z_x(\omega) = \frac{c}{b^3 \delta \sigma_w \omega}, \quad \delta \ll \delta_w$$

(resistive wall impedance)

$$\delta_w = \sqrt{\frac{2}{\mu_0 \sigma_w \omega}}$$

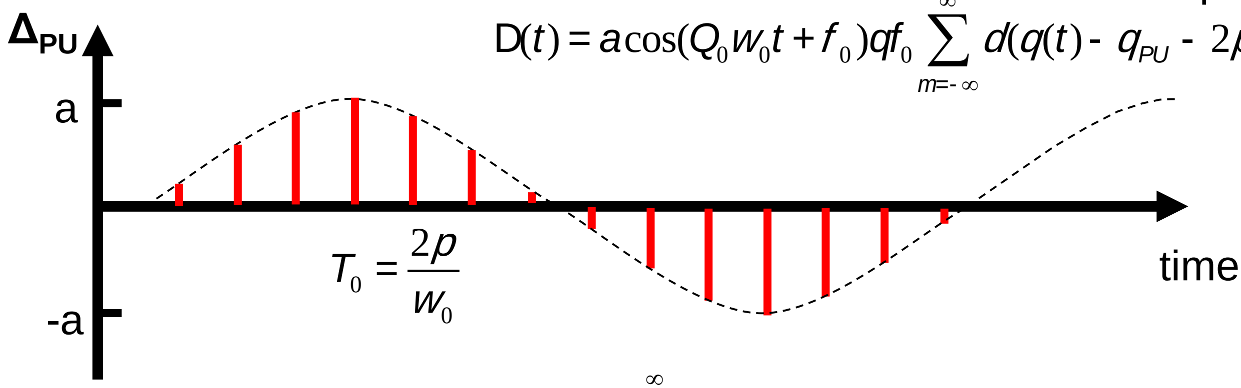
(skin depth)

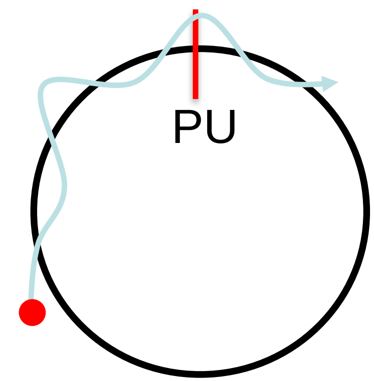
Single particle oscillation spectrum

Betatron oscillation: $x(t) = a\cos(Q_0\omega_0 t + \varphi_0)$

Dipole moment: $D(t) = x(t)\,I(t)$

$$D(t) = a\cos(Q_0\omega_0 t + \varphi_0)\,q f_0 \sum_{m=-\infty}^{\infty} \delta(\theta(t) - \theta_{PU} - 2\pi m)$$

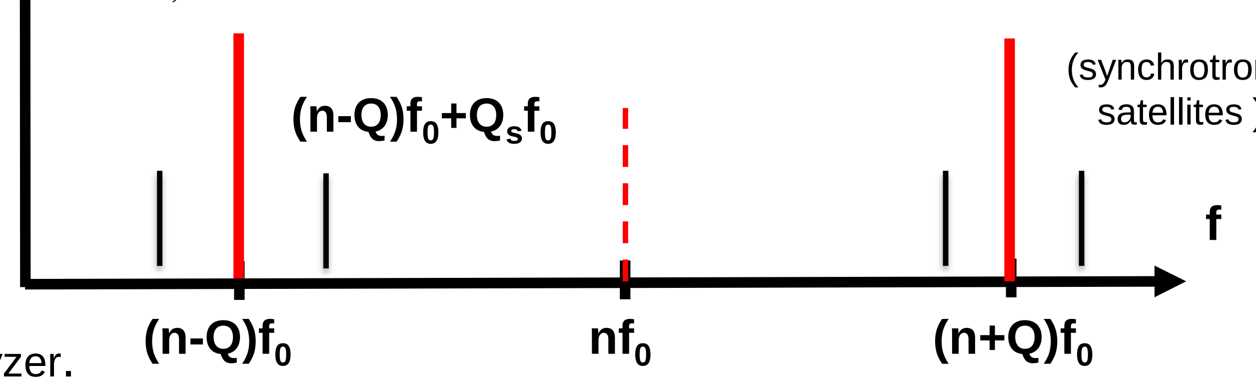




Frequency domain: $D(t) = q a f_0 \Re \sum_{n=-\infty}^{\infty} \exp\left[-i(n+Q_0)\omega_0 t + \varphi_0\right]$

Looking at positive frequencies only:

$f_{n,\pm} = (n \pm Q_0) f_0$ (betatron sidebands)



Only positive lines can be seen by the spectrum analyzer.

Single particle power spectrum

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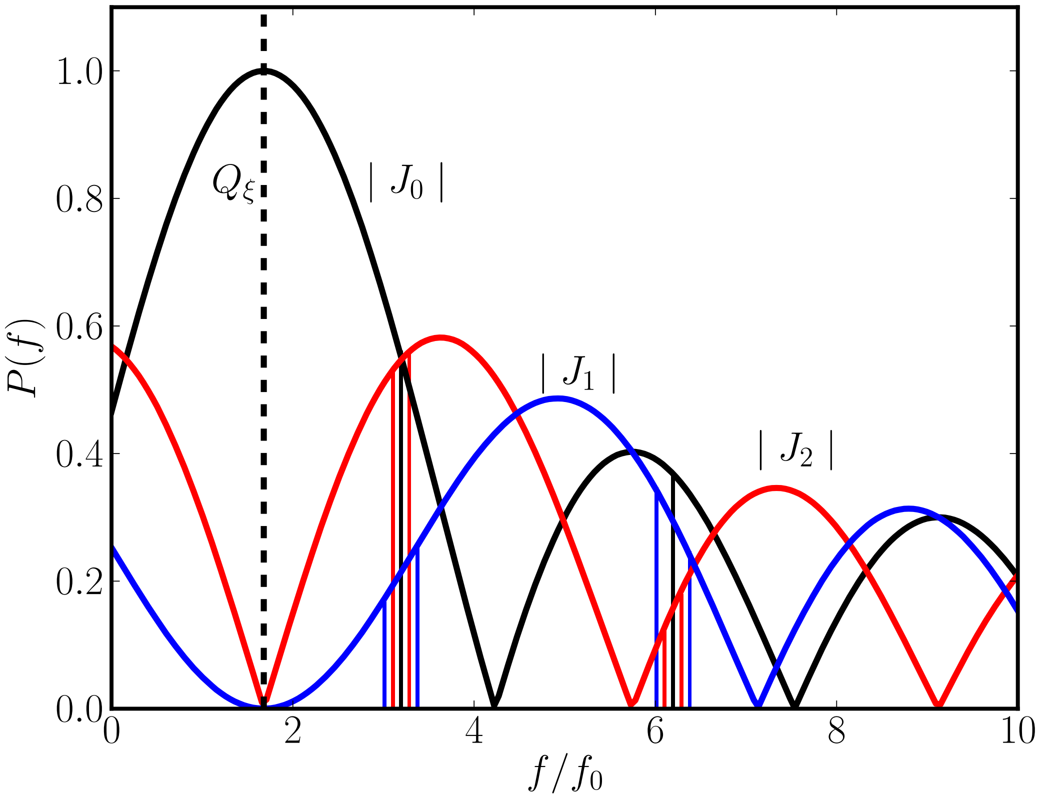
$$h(f) = |\Delta_k|^2 \sim \left| J_k \left[((n \pm Q_0) - Q_\xi)\omega_0 \hat{\tau} \right] \right|$$



Chromatic tune shift:

$$Q_x = x / h_0$$

Chromaticity:

$$\xi = \frac{\Delta Q}{\Delta p / p_0}$$