INTRABEAM SCATTERING

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Prologue

Intrabeam Scattering (IBS) is a multiple Coulomb scattering of charged particle beams (alternatively IBS is a diffusion process in all 3 transverse & longitudinal beam dimensions)

- IBS in charged particle beams causes *small* changes of the colliding *particles momenta* by addition of *multiple* random *small-angle scattering* events leading to :
  1. A *relaxation* to a thermal (energy) *equilibrium* via reallocation of the whole beam phase volume between the 3 transverse and longitudinal beam phase volumes (emittances).
  2. A continuous *diffusion growth* of the global beam phase volume *without equilibrium,* and reduction of the *beam lifetime* when the particles hit the *aperture*.
- *Touschek effect* is the particle *losses* due to *single* collision *events* at large scattering angles for which only the energy transfer from transverse to longitudinal planes is examined.
- IBS simulation consists to compute the particle *momentum variation* by *coulomb scattering* with the other particles of the beam and get the *growth rates* for the 3 degrees of freedom.
- *IBS* theory was later extended to include :
  - Amplitude & dispersion *derivatives* and lattice parameter *variations* around the *lattice*
  - Horizontal-vertical betatron *linear coupling*.

Prologue

IBS in week focusing or smooth ring lattices can be related with scattering of *gas molecules* in a *closed box* where the walls mimics the quadrupole focusing forces and the RF voltage keep the particles together. The *scattering* of the molecules leads to the *Maxwell-Boltzmann distribution* of the 3 velocity components (v_x, v_y, v_s) in which m is the molecule mass, T the temperature, k the Boltzmann's constant ($\int f\, d\boldsymbol{v}$ is normalized to 1) :

$$f(\boldsymbol{v}_x, \boldsymbol{v}_y, \boldsymbol{v}_s) = \frac{1}{(2\pi kT/m)^{3/2}} e^{-m(\boldsymbol{v}_x{}^2 + \boldsymbol{v}_y{}^2 + \boldsymbol{v}_s{}^2)/(2kT)}$$

The difference between *IBS* and *gaz molecule* scattering in a box is due to the ring orbit curvature :

- *Curvature* yields a *dispersion* so that a sudden change of *energy* will change the *betatron* amplitudes and initiate a *synchro-betatron* oscillation *coupling*.
- *Curvature* also leads to the *negative mass instability* i.e. if a particle accelerates above *transition* it becomes slower and behaves as a particle with negative mass and thus an *equilibrium* of particles *above transition energy* can't exist (transition energy $\gamma_t mc^2$ is got once $\gamma^2 = \gamma_t^2 = \frac{1}{\alpha_p} = \frac{dp/p}{dR/R}$ or $\frac{df/f}{dp/p} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_t^2} = 0$).
- *Above transition* the IBS effect is to increase the three bunch dimensions.
- *Below transition* an *equilibrium* particle distribution can *exists* (week focusing/smooth lattices).

The Intrabeam scattering effect

- Small angle **multiple Coulomb scattering** effect
  - Redistribution of beam momenta
  - Beam diffusion with impact on the beam quality (Brightness , luminosity, etc)

- **Different approaches** for the probability of scattering
  - Classical Rutherford cross section
  - Quantum approach
    - Relativistic "Golden Rule" for the 2-body scattering process

- **Several theoretical models** and their **approximations** developed over the years
  - Classical models of Piwinski (**P**) and Bjorken-Mtingwa (**BM**)
  - High energy approximations **Bane, CIMP, etc**
    - Integrals with analytic solutions

Courtesy F. Antoniou, Y. Papaphilippou, CERN

Lagrangian and Hamiltonian (briefly)

- We restrict to systems of N particles with $3N$ degrees of freedom described via Cartesian coordinates $\boldsymbol{r} = (\boldsymbol{r}_1 \cdots \boldsymbol{r}_N)$, $\boldsymbol{r}_i = (x, y, z)_i$, and $\boldsymbol{v} \equiv \dot{\boldsymbol{r}} = (\dot{\boldsymbol{r}}_1 \cdots \dot{\boldsymbol{r}}_N)$, $\dot{\boldsymbol{r}}_i = (\dot{x}, \dot{y}, \dot{z})_i$
- Assume the system exists in a ***conservative force field*** $\boldsymbol{F}^c(\boldsymbol{r})$ with *kinetic* energy $T(\boldsymbol{r}, \dot{\boldsymbol{r}})$ and *potential* $V(\boldsymbol{r})$ such as $\boldsymbol{F}^c(\boldsymbol{r}) = -\boldsymbol{\nabla}_r V(\boldsymbol{r}) \equiv -\partial V(\boldsymbol{r})/\partial \boldsymbol{r}$. The *Lagrangian* is defined as :

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \stackrel{\text{def}}{=} T(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) - V(\boldsymbol{r})$$

Lagrange's equations stem from the *variational principle*:

$$\delta I = \int_{t1}^{t2} L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t)\, dt = 0 \quad \Longrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{r}}} - \frac{\partial L}{\partial \boldsymbol{r}} = 0$$

L is then recast in an *Hamiltonian* form H

$$H(\boldsymbol{r}, \boldsymbol{p}, t) \stackrel{\text{def}}{=} \dot{\boldsymbol{r}} \cdot \boldsymbol{p} - L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \qquad \boldsymbol{p} \stackrel{\text{def}}{=} \partial L/\partial \dot{\boldsymbol{r}}$$

$\boldsymbol{p}$: *conjugate momentum to* $\boldsymbol{r}$

e.g. $L(r, \dot{r}, \ddot{r}, t) = \ddot{r}^2 - 2f(t)r \Longrightarrow \frac{d^4 r}{dt^4} = f(t)$
each Lagrangian defines a theory (realistic?)

From which *Hamilton's equations* are derived :

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}}$$

$\dot{H} = 0$ if $H = H(\boldsymbol{r}, \boldsymbol{p}) \longrightarrow$ $H = T + V = E =$ constant energy

Lagrangian and Hamiltonian (briefly)

If the total force $\boldsymbol{F}$ acting on a system contains a ***conservative*** (*Hamiltonian*) part $\boldsymbol{F}^c(\boldsymbol{r})$ and a ***non-conservative*** (*non-strictly-Hamiltonian*) part $\boldsymbol{F}^{nc}(\boldsymbol{r},\dot{\boldsymbol{r}},t)$ representing *friction*, *inelastic processes*… ($\boldsymbol{F} = -\boldsymbol{\nabla}_r V(\boldsymbol{r}) + \boldsymbol{F}^{nc}$). The *Lagrangian* of the system is then written as :

$$L(\boldsymbol{r},\dot{\boldsymbol{r}},t) \stackrel{\text{def}}{=} T(\boldsymbol{r},\dot{\boldsymbol{r}},t) - V(\boldsymbol{r}) \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{r}}} - \frac{\partial L}{\partial \boldsymbol{r}} = \boldsymbol{F}^{nc}$$

since $\boldsymbol{F}^{nc} \neq -\dfrac{\partial \tilde{V}}{\partial \boldsymbol{r}} \equiv -\boldsymbol{\nabla}_{\boldsymbol{r}}\tilde{V}$

From $H(\boldsymbol{r},\boldsymbol{p},t) = \dot{\boldsymbol{r}}\cdot\boldsymbol{p} - L(\boldsymbol{r},\dot{\boldsymbol{r}},t)$ the *(non-Hamiltonian) equations* follow :

$$1)\ \frac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{r}} - \frac{\partial L}{\partial \boldsymbol{p}} = \dot{\boldsymbol{r}}$$

$$2)\ \frac{\partial H}{\partial \boldsymbol{r}} = -\frac{\partial L}{\partial \boldsymbol{r}} = \boldsymbol{F}^{nc} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{r}}} = \boldsymbol{F}^{nc} - \frac{d\boldsymbol{p}}{dt}$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}} + \boldsymbol{F}^{nc}$$

Liouville equation

- **Γ−space :** *6N*−dim phase space coordinates, a single point (microstate) represents *N* particles labelled by *3N* positions $\boldsymbol{r} = (\boldsymbol{r}_1 \cdots \boldsymbol{r}_N)$ and momenta $\boldsymbol{p} = (\boldsymbol{p}_1 \cdots \boldsymbol{p}_N)$ with $\boldsymbol{r}_i = (x, y, z)_i$ and $\boldsymbol{p}_i = (p_x, p_y, p_z)_i$
- **Ensemble :** $\mathcal{N}$ copies of a specific microstate (N particles) each copy described by a different representative point in Γ−space $(\mathcal{N} \neq N)$
- $\boldsymbol{d\mathcal{N}(r, p, t)}$ **:** number of microstates in the volume element $d\Gamma = \prod_{i=1}^{N} d\boldsymbol{r}_i d\boldsymbol{p}_i$ about any coordinate values $(\boldsymbol{r}, \boldsymbol{p})$ at time t
- $\boldsymbol{\rho(r, p, t)}$ **:** density of representative microstates ("coarse-graining" density $\rho(\boldsymbol{r},\boldsymbol{p},t)$ is obtained by disregarding variation of ρ below small resolution in Γ-space)

Formal *density* definition

$$\rho(\boldsymbol{r}, \boldsymbol{p}, t)\boldsymbol{d\Gamma} = \lim_{\mathcal{N}\to\infty} \frac{d\mathcal{N}(\boldsymbol{r}, \boldsymbol{p}, t)}{\mathcal{N}}$$

Coarse-graining density

$$\rho(\boldsymbol{r}, \boldsymbol{p}, t)\Delta\boldsymbol{\Gamma} = \frac{\Delta\mathcal{N}(\boldsymbol{r}, \boldsymbol{p}, t)}{\mathcal{N}}$$

Liouville equation

- A microstate of *N* particles with coordinates $(\boldsymbol{r},\boldsymbol{p}) = (\boldsymbol{r}_i,\boldsymbol{p}_i)_{i=1\cdots N}$ at time t will be found at $t+\delta t$ with new coordinates $(\boldsymbol{r}'_i,\boldsymbol{p}'_i)_{i=1\cdots N} = (\boldsymbol{r}_i + \dot{\boldsymbol{r}}_i\,\delta t, \boldsymbol{p}_i + \dot{\boldsymbol{p}}_i\,\delta t + \mathcal{O}(\delta t^2))$
- The microstate density $\rho(\boldsymbol{r},\boldsymbol{p},t)$ at time t will become $\rho(\mathbf{r}',\mathbf{p}',t+\delta t)$ at $t+\delta t$
- The phase space volume $d\Gamma(t)$ at t will change into $d\Gamma(t+\delta t)$ at $t+\delta t$
- $d\mathcal{N}(\boldsymbol{r}',\boldsymbol{p}',t+\delta t) = d\mathcal{N}(\boldsymbol{r},\boldsymbol{p},t)$ because $(\boldsymbol{r}(t),\boldsymbol{p}(t))$ follow *Hamilton's* equations for (***conservative forces***) and thus *no trajectories cross* (do not escape the *6N-1 dim* surface $C(t)$ enclosing the microstates, $C(t)$ being itself a microstate !)

$$\Delta\mathcal{N}(\boldsymbol{r},\boldsymbol{p},t+\delta t) = \Delta\mathcal{N}(\boldsymbol{r},\boldsymbol{p},t)$$

$$\Rightarrow \quad \rho(\mathbf{r}',\mathbf{p}',t+\delta t)\int_{\text{in } C(t+\delta t)} d\Gamma(t+\delta t) = \rho(\boldsymbol{r},\boldsymbol{p},t)\int_{\text{in } C(t)} d\Gamma(t)$$

The relation between $d\Gamma' \stackrel{\text{def}}{=} d\Gamma(t+\delta t)$ with border $C' \stackrel{\text{def}}{=} C(t+\delta t)$ and $d\Gamma \stackrel{\text{def}}{=} d\Gamma(t)$, border $C \stackrel{\text{def}}{=} C(t)$ is

$$\int_{\text{in } C'} d\Gamma' = |J|\int_{\text{in } C} d\Gamma$$

$$J = \frac{\partial(\boldsymbol{r}'_i,\boldsymbol{p}'_i)}{\partial(\boldsymbol{r}_i,\boldsymbol{p}_i)} \quad (3N\times 3N \textit{ Jacobian}) \qquad \left.\begin{matrix}\boldsymbol{r}_i = (x,y,z)_i \\ \boldsymbol{p}_i = (p_x,p_y,p_z)_i\end{matrix}\right\} i = 1\cdots N$$

Liouville equation

Using $(\boldsymbol{r}'_i, \boldsymbol{p}'_i) = (\boldsymbol{r}_i + \dot{\boldsymbol{r}}_i\,\delta t, \boldsymbol{p}_i + \dot{\boldsymbol{p}}_i\,\delta t)$ and the *Hamilton's equations* the determinant $|J|$ of the *Jacobian* matrix writes (1st order)

$$|J| = \begin{vmatrix} \frac{\partial r_1}{\partial r_1} + \frac{\partial \dot{r}_1}{\partial r_1}\delta t & \cdots & \frac{\partial p_N}{\partial r_1} + \frac{\partial \dot{p}_N}{\partial r_1}\delta t \\ \vdots & \ddots & \vdots \\ \frac{\partial r_1}{\partial p_N} + \frac{\partial \dot{r}_1}{\partial p_N}\delta t & \cdots & \frac{\partial p_{3N}}{\partial p_N} + \frac{\partial \dot{p}_{3N}}{\partial p_N}\delta t \end{vmatrix} = \begin{vmatrix} 1 + \frac{\partial \dot{r}_1}{\partial r_1}\delta t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 + \frac{\partial \dot{p}_N}{\partial p_N}\delta t \end{vmatrix}$$

$$= 1 + \sum_{i=1}^{N}\left(\frac{\partial \dot{r}_i}{\partial r_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right)\delta t + \mathcal{O}(\delta t^2)$$

→ $|J| = 1 + \mathcal{O}(\delta t^2)$ →

$$\int_{\text{in } C(t+\delta t)} d\Gamma(t+\delta t) = \int_{\text{in } C(t)} d\Gamma(t)$$

→ *Liouville's theorem* stems from the conservation of the phase space volume in Γ−space

Liouville equation

Liouville's theorem

The microstate density $\rho(\boldsymbol{r}, \boldsymbol{p}, t)$ in Γ−space behaves like an incompressible fluid

$$\rho(\boldsymbol{r}', \boldsymbol{p}', t + \delta t) = \rho(\boldsymbol{r}, \boldsymbol{p}, t) \quad \Leftrightarrow \quad \frac{d\rho(\boldsymbol{r}, \boldsymbol{p}, t)}{dt} = 0$$

Equivalently ρ writes in differential form using the *Hamilton's equations* and *Poisson bracket* :

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{3N} \left(\frac{\partial \rho}{\partial r_i} \dot{r}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0 \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} + \dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_r \rho + \dot{\boldsymbol{p}} \cdot \boldsymbol{\nabla}_p \rho = \boldsymbol{0}$$

$$\{\rho, H\} \stackrel{\text{def}}{=} \sum_{i=1}^{3N} \left(\frac{\partial \rho}{\partial r_i} \frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial H}{\partial r_i} \right) \quad \Rightarrow \quad \frac{d\rho}{dt} + \{\rho, H\} = 0$$

Liouville's formula

Liouville equation

Consider the (*non-strictly-Hamiltonian*) equations of motion for ***non-conservative forces*** $\boldsymbol{F}^{nc}$:

$$\dot{r}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial r_i} + F_i^{nc} \;\Rightarrow\; \frac{\partial \dot{r}_i}{\partial r_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \frac{\partial F_i^{nc}}{\partial p_i} \;\Rightarrow\; |J| = 1 + \sum_{i=1}^{N} \frac{\partial F_i^{nc}}{\partial p_i} \delta t$$

$$\Rightarrow \int_{\text{in } C(t+\delta t)} d\Gamma(t+\delta t) = \left(1 + \delta t \sum_{i=1}^{N} \frac{\partial F_i^{nc}}{\partial p_i}\right) \int_{\text{in } C(t)} d\Gamma(t)$$

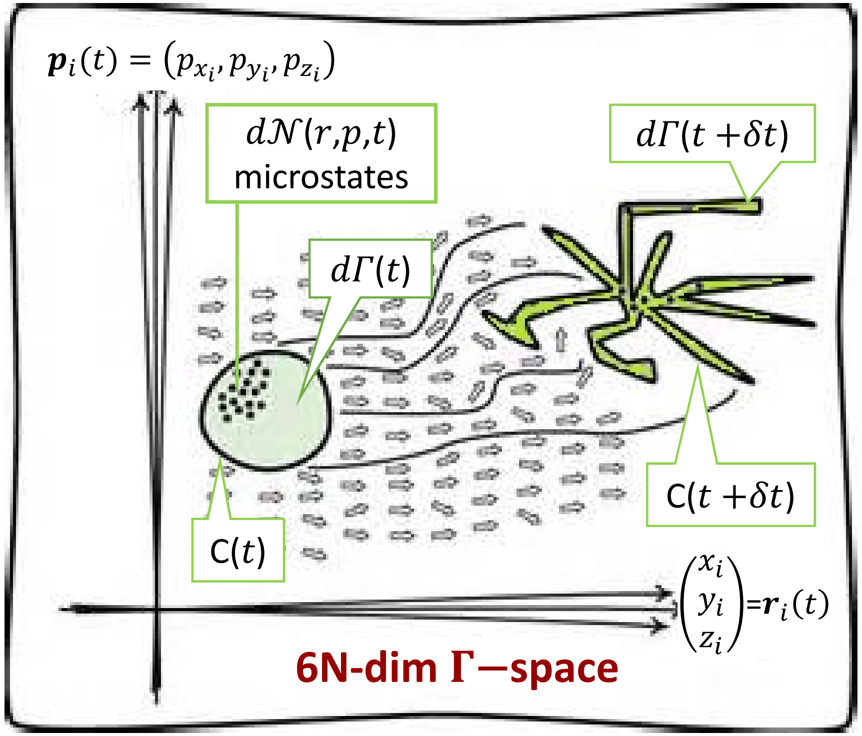
Liouville's theorem "violated" !? : incompressibility condition of $\rho(\boldsymbol{r},\boldsymbol{p},t)$ not satisfied i.e.

$$\rho(\boldsymbol{r}',\, \boldsymbol{p}',\, t+\delta t) = \left(1 + \delta t\, \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc}\right)\rho(\boldsymbol{r},\boldsymbol{p},\, t) \;\Rightarrow\; \frac{\rho(\boldsymbol{r}',\, \boldsymbol{p}',\, t+\delta t) - \rho(\boldsymbol{r},\boldsymbol{p},\, t)}{\delta t} = \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc}$$

written in differential form this lead to the equivalent results:

$$\frac{d\rho}{dt} = \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc} \;\Leftrightarrow\; \frac{\partial \rho}{\partial t} + \{\rho, H\} = \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc} \;\Leftrightarrow\; \frac{\partial \rho}{\partial t} + \dot{\boldsymbol{r}} \cdot \boldsymbol{\nabla}_r \rho + \dot{\boldsymbol{p}} \cdot \boldsymbol{\nabla}_p \rho = \boldsymbol{\nabla}_p \cdot \boldsymbol{F}^{nc}$$

Liouville equation



Microstate subset $d\mathcal{N}(r,p,t)$ inside the *6N*−dim volume $d\Gamma(t)$ of border $C(t)$ at t in Γ−space will occupy a distorted volume $d\Gamma(t+\delta t)$ of border $C(t+\delta t)$ at $t+\delta t$

Liouville (also called *collisionless Boltzmann*) *equation*

- Detailed account of the density $\rho(\boldsymbol{r(t)}, \boldsymbol{p(t)}, t)$ would require knowledge of $6N$ particle trajectories with initial conditions for all microstates of the sub-ensemble $d\mathcal{N}$ ($\sim 10^{23}$?!) in the (Γ−space) volume element $d\Gamma$.
- Practically it would be more suitable to place the phase trajectories of the *N* particles in the same *6*−dim phase space (μ−space) : a single point represents one particle labelled by *3* positions $\boldsymbol{r} = (x, y, z)$ and 3 momenta $\boldsymbol{p} = (p_x, p_y, p_z)$.
- To reach this objective the $6N$−dim *microstate* density $\rho(\boldsymbol{r}_1, \cdots \boldsymbol{r}_N, \boldsymbol{p}_1, \cdots \boldsymbol{p}_N, t)$ must be reduced a *6*−dim *particle* density $f_1(\boldsymbol{r}, \boldsymbol{p}, t)$ in (μ−space).
- This should be done via the *BBGKY hierarchy* framework to go from the *N-particles* (in Γ−space) to the *N-times 1-particle* (in μ−space) description.

Liouville equation

- The full phase space density $\rho(\boldsymbol{r}, \boldsymbol{p}, t)$ contains too much information than needed to describe the equilibrium properties of particles (e.g. 1-particle density is enough to compute a gas pressure).
- The *N*-particle density $\rho(\boldsymbol{r}_1, \boldsymbol{p}_1, \cdots \boldsymbol{r}_N, \boldsymbol{p}_N, t)$ in 6*N*–dim Γ–space is to be reduced to a *single particle density* $f_1(\boldsymbol{r}, \boldsymbol{p}, t)$ in 6–dim μ–space : the *state* of *each particle* being represented by a *single point*.
- $f_1(\boldsymbol{r}, \boldsymbol{p}, t)/N$ refers to the expectancy of finding any one of the *N* particles at time t with location $\boldsymbol{r}$ and momentum $\boldsymbol{p}$, computed from $\rho(\boldsymbol{r}_1, \boldsymbol{p}_1, \cdots \boldsymbol{r}_N, \boldsymbol{p}_N, t)$ by means of the formulae :

$$f_1(\boldsymbol{r}, \boldsymbol{p}, t) = \left\langle \sum_{i=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_i)\delta(\boldsymbol{p} - \boldsymbol{p}_i) \right\rangle \equiv \int d\Gamma \rho(\boldsymbol{r}, \boldsymbol{p}, t) \sum_{i=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_i)\delta(\boldsymbol{p} - \boldsymbol{p}_i)$$

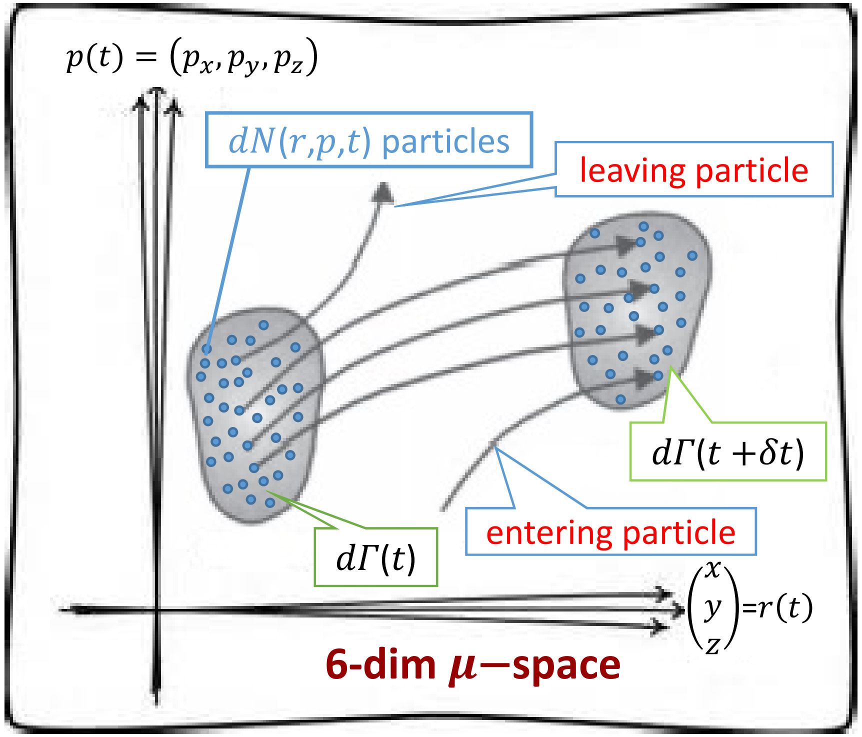
with for any function $\mathcal{O}(\boldsymbol{r}, \boldsymbol{p})$: $\langle \mathcal{O} \rangle = \int d\Gamma\, \rho(\boldsymbol{r}, \boldsymbol{p}, t)\mathcal{O}(\boldsymbol{r}, \boldsymbol{p})$. Using the first pair of delta functions to compute one set of integrals we get, assuming a symmetric density when permuting particles :

$$f_1(\boldsymbol{r}, \boldsymbol{p}, t) = N \int \prod_{i=2}^{N} d\boldsymbol{r}_i d\boldsymbol{p}_i\, \rho(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{r}_2, \boldsymbol{p}_2 \cdots, \boldsymbol{r}_N, \boldsymbol{p}_N, t)$$

$$f_1(\boldsymbol{x}) = \iint d\boldsymbol{x}_1 d\boldsymbol{x}_2 \rho(\boldsymbol{x}_1, \boldsymbol{x}_2)\{\delta(\boldsymbol{x} - \boldsymbol{x}_1) + \delta(\boldsymbol{x} - \boldsymbol{x}_2)\}$$
$$= \int d\boldsymbol{x}_2 \rho(\boldsymbol{x}_1 = \boldsymbol{x}, \boldsymbol{x}_2) + \int d\boldsymbol{x}_1 \rho(\boldsymbol{x}_1, \boldsymbol{x}_2 = \boldsymbol{x})$$

f_1 is *normalized* to N and ρ to 1

Boltzmann collision equation



Particle subset $dN(r,p,t)$ inside μ−space at t + δt due to collisions in the time δt

Liouville formula needs then to be adapted to *Boltzmann collision equation* when considering *particle interactions*

- As a result of collisions during the time interval δt particles that were inside the volume $d\Gamma = d\boldsymbol{r}d\boldsymbol{p}$ in the 6-dim μ − space may be removed from it and particles outside $d\Gamma$ may end up inside it.
- The net *gain* or *loss* of particles as a result of *collisions* during δt inside $d\Gamma$ is denoted :

$$\left[\frac{\delta f_1(\boldsymbol{r}_1, \boldsymbol{p}_1, t)}{\delta t}\right]_{\text{coll}} d\boldsymbol{r}d\boldsymbol{p}\delta t$$

where $(\delta f_1/\delta t)_{\text{coll}}$ means the rate of change of f_1. Hence the *Liouville equation* turns into the *collision Boltzmann equation*

$$\frac{\partial\rho}{\partial t} + \dot{\boldsymbol{r}} \cdot \nabla_r \rho + \dot{\boldsymbol{p}} \cdot \nabla_p \rho = \left(\frac{\delta f_1}{\delta t}\right)_{\text{coll}} \equiv \nabla_p \cdot \boldsymbol{F}^{nc} = \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{F}^{nc}$$

non conservative force field

Boltzmann collision equation

Heuristic assumptions are made to « derive » the *Boltzmann collision* equation :

- f_1 does not vary visibly over the *distance* of *interparticle force range* and over the *time scale* of the *interaction*.
- Disregard *external force* effects on the *collision cross-section* size.
- Consider only *binary collisions*.
- "*Molecular chaos*" assumption : the interacting particle momenta (velocities), before collision, are assumed to be uncorrelated, i.e.
  - the joint probability of having, at position $\boldsymbol{r}$ and time t, particles 1 & 2 of momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is equal to $f_1(\boldsymbol{r},\boldsymbol{p}_1,t)f_1(\boldsymbol{r},\boldsymbol{p}_2,t)$ (supposing that collisions are local in space so that the 2 particles sit at the same point).
- Generally the joint probability density would be equal to $f_1(\boldsymbol{r},\boldsymbol{p}_1,t)f_1(\boldsymbol{r},\boldsymbol{p}_2,t)[1+\mathrm{K}_2(\boldsymbol{r},\boldsymbol{p}_1,\boldsymbol{p}_2,t)]$ where $\mathrm{K}_2(\boldsymbol{r},\boldsymbol{p}_1,\boldsymbol{p}_2,t)$ is a *correlation* function.
- To by-pass the *molecular chaos* approximation the alternative is to work with the equations of the BBGKY hierarchy (Bogoliubov, Born, Green, Kirkwood, Yvon).

Boltzmann collision equation

Let's start with an Hamiltonian $H(\boldsymbol{r},\boldsymbol{p})$ with no interacting collision potential between particle pairs (e.g. Coulomb scattering potential). This Hamiltonian will just contain :

- *Particle kinetic energy* (for non relativistic charged particles)
- *External potential* $\Phi(\boldsymbol{r})$ (e.g. electromagnetic field for charged particle beam)

$$H(\boldsymbol{r},\boldsymbol{p}) = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + \Phi(\boldsymbol{r}_i)\right]$$

From Liouville's formula in terms of Poisson bracket and replacing the $6N$−dim density ρ in Γ−space by the 6−dim density f_1 in μ−space we get :

$$\{H, f_1\} = \frac{\partial H}{\partial \boldsymbol{r}_1}\frac{\partial f_1}{\partial \boldsymbol{p}_1} - \frac{\partial H}{\partial \boldsymbol{p}_1}\frac{\partial f_1}{\partial \boldsymbol{r}_1} = \frac{\partial \Phi}{\partial \boldsymbol{r}_1}\frac{\partial f_1}{\partial \boldsymbol{p}_1} - \frac{\boldsymbol{p}_1}{m}\frac{\partial f_1}{\partial \boldsymbol{r}_1}$$

$$\frac{\partial f_1}{\partial t} + \{f_1, H\} = \boldsymbol{0}$$

⇒

$$\frac{\partial f_1}{\partial t} - \frac{\partial \Phi}{\partial \boldsymbol{r}_1}\frac{\partial f_1}{\partial \boldsymbol{p}_1} + \frac{\boldsymbol{p}}{m}\frac{\partial f_1}{\partial \boldsymbol{r}_1} = 0$$

collisionless Boltzmann equation

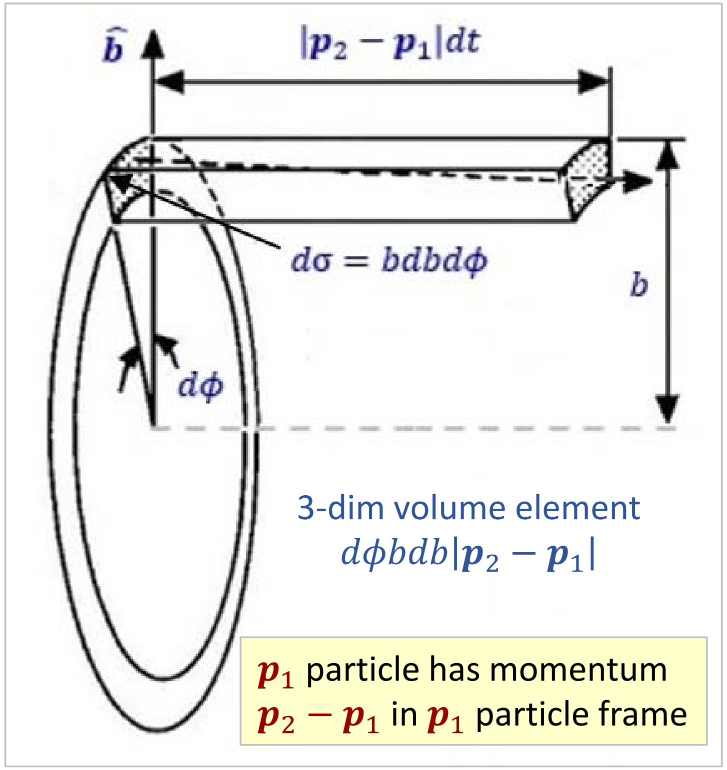
The external force $\boldsymbol{F} = m\boldsymbol{a}$ (e.g. in a plasma) includes the Lorentz force $\dot{\boldsymbol{p}} = e\,(\boldsymbol{E} + \dot{\boldsymbol{r}} \times \boldsymbol{B})$ due to externally applied fields.

Boltzmann collision equation

Collision terms :

The interaction result is characterized by the *net rate* at which *collisions increase* or *decrease* the particle number entering the *6-dim phase-space* slice $\partial \boldsymbol{r} \partial \boldsymbol{p}$ in time δt (named $\delta\mathcal{R}$) defined as : $\delta\mathcal{R}=\delta\mathcal{R}_{+}-\delta R_{-}$ where $\delta\mathcal{R}_{\pm}$ are the particle number *injected*/*ejected* in $\partial \boldsymbol{r} \partial \boldsymbol{p}$ by collisions in δt

- For $\delta\mathcal{R}_{-}$: particles are shared in 2 groups, the 1st of momenta in the interval $\partial \boldsymbol{p}$ about $\boldsymbol{p}_1$ and the 2nd of all other momenta denoted $\boldsymbol{p}_2$, the particles ejected from $\partial \boldsymbol{r} \partial \boldsymbol{p}$ are the number of collisions that the $\boldsymbol{p}_1$'s have with all other $\boldsymbol{p}_2$'s in δt. To compute $\delta\mathcal{R}_{-}$ all collisions between pairs of particles that eject one of them out of the interval $\partial \boldsymbol{p}$ about $\boldsymbol{p}_1$ are considered, i.e.
  - One particle is in $\partial \boldsymbol{r} \partial \boldsymbol{p}$ near $(\boldsymbol{r}_1, \boldsymbol{p}_1)$ the other in $\partial \boldsymbol{r}_2 \partial \boldsymbol{p}_2$ near $(\boldsymbol{r}_2, \boldsymbol{p}_2)$
  - The $\boldsymbol{p}_2$'s in $\partial \boldsymbol{r}_2$ suffer a collision with the $\boldsymbol{p}_1$'s in $\partial \boldsymbol{r}$ in time δt.
- For $\delta\mathcal{R}_{+}$: consider all pair-particle collisions that send one particle into the momentum interval $\partial \boldsymbol{p}$ about $\boldsymbol{p}_1$ in time δt which is the inverse of the original collision $(\boldsymbol{p}'_1, \boldsymbol{p}'_2) \rightleftarrows (\boldsymbol{p}_1, \boldsymbol{p}_2)$



Boltzmann collision equation

The number of particles *injected*/*ejected* into $\partial\boldsymbol{r}\partial\boldsymbol{p}$ by collisions in time δt are :

$$\delta\mathcal{R}_- = \int_{(\boldsymbol{r}_2,\boldsymbol{p}_2)} f_1(\boldsymbol{r},\boldsymbol{p}_1,t) f_1(\boldsymbol{r},\boldsymbol{p}_2,t)\, d\boldsymbol{r} d\boldsymbol{p}_1 d\boldsymbol{p}_2 \quad \delta\mathcal{R}_+ = \int_{(\boldsymbol{r}'_2,\boldsymbol{r}'_2)} f_1(\boldsymbol{r}',\boldsymbol{p}'_1,t) f_1(\boldsymbol{r}',\boldsymbol{p}'_2,t)\, d\boldsymbol{r}' d\boldsymbol{p}'_1 d\boldsymbol{p}'_2$$

All $\boldsymbol{p}_2$ particles shown (see fig. above) in the cylinder of height $|\boldsymbol{p}_2 - \boldsymbol{p}_1|\delta t$ and base area $bdbd\phi$ suffer a collision with the $\boldsymbol{p}_1$ particle in time δt ⇒ $d\boldsymbol{r} = |\boldsymbol{p}_2 - \boldsymbol{p}_1|\delta t bdbd\phi$ (idem for $\boldsymbol{p}'_1, \boldsymbol{p}'_2$).
Also since $d^3\boldsymbol{r}d^3\boldsymbol{p} = d^3\boldsymbol{r}'d^3\boldsymbol{p}'$ ⇓

⇒ $$\delta\mathcal{R}_- = \left(\int f_1 d\boldsymbol{p}_2 \,|\boldsymbol{p}_2 - \boldsymbol{p}_1| bdbd\phi\right) d^3\boldsymbol{r} d^3\boldsymbol{p}_1 \delta t \quad \delta\mathcal{R}_+ = \left(\int f_1 d\boldsymbol{p}_2 \,|\boldsymbol{p}_2 - \boldsymbol{p}_1| bdbd\phi\right) d^3\boldsymbol{r} d^3\boldsymbol{p}_1 \delta t$$

⇒ $$\delta\mathcal{R} = \int [f_1(\boldsymbol{r}',\boldsymbol{p}'_1,t) f_1(\boldsymbol{r}',\boldsymbol{p}'_2,t) - f_1(\boldsymbol{r},\boldsymbol{p}_1,t) f_1(\boldsymbol{r},\boldsymbol{p}_2,t)] |\boldsymbol{p}_2 - \boldsymbol{p}_1| d\boldsymbol{r} d\boldsymbol{p}_1 bdbd\phi$$

From Liouville equation the net number of particles that enter the 6-dim phase element $d\boldsymbol{r}d\boldsymbol{p}$ keeping on a particle trajectory during δt is zero. Likewise the collisionless Boltzmann equation writes :

$$\delta\mathcal{R}_{\text{Liouville}} \equiv d\boldsymbol{r}d\boldsymbol{p}\,\delta t \left[\frac{\partial f_1}{\partial t} - \frac{\partial\Phi}{\partial\boldsymbol{r}_1}\frac{\partial f_1}{\partial\boldsymbol{p}_1} + \frac{\boldsymbol{p}}{m}\frac{\partial f_1}{\partial\boldsymbol{r}_1}\right] = 0$$

Boltzmann collision equation

Hence the above term $\delta\mathcal{R}$ can be cast into the form :

$$\frac{\delta\mathcal{R}}{d\boldsymbol{r}d\boldsymbol{p}_1\delta t} = \int [f_1(\boldsymbol{r},\boldsymbol{p}_1',t)f_1(\boldsymbol{r},\boldsymbol{p}_2',t) - f_1(\boldsymbol{r},\boldsymbol{p}_1,t)f_1(\boldsymbol{r},\boldsymbol{p}_2,t)]|\boldsymbol{p}_2 - \boldsymbol{p}_1|d\boldsymbol{p}_2 b db d\phi \equiv \left[\frac{\delta f_1(\boldsymbol{r},\boldsymbol{p}_1,t)}{\delta t}\right]_{\text{coll}}$$

- The quantity $bd\phi db \equiv d\sigma$ having dimensions of area can be written as $d\sigma = (d\sigma/d\Omega)d\Omega$ in which $|d\sigma/d\Omega|$ is the *differential cross-section* (see below).
- Replacing $|\boldsymbol{p}_2 - \boldsymbol{p}_1|/m$ by the *velocity* $|\boldsymbol{v}_2 - \boldsymbol{v}_1|$ (*non relativistic* particles) the collision term writes:

$$\left(\frac{\delta f_1}{\delta t}\right)_{\text{coll}} = \int d\boldsymbol{v}_2 \int d\Omega \left|\frac{d\sigma}{d\Omega}\right| |\boldsymbol{v}_2 - \boldsymbol{v}_1| \, [f_1(\boldsymbol{r},\boldsymbol{p}_1',t)f_1(\boldsymbol{r},\boldsymbol{p}_2',t) - f_1(\boldsymbol{r},\boldsymbol{p}_1,t)f_1(\boldsymbol{r},\boldsymbol{p}_2,t)]$$

Putting $(\delta f_1/\delta t)_{\text{coll}}$ in the collisionless Boltzmann equation yields the *Boltzmann collision equation* :

$$\left(\frac{\partial}{\partial t} - \frac{\partial\Phi}{\partial\boldsymbol{r}_1}\frac{\partial}{\partial\boldsymbol{p}_1} + \frac{\boldsymbol{p}_1}{m}\frac{\partial}{\partial\boldsymbol{r}_1}\right)f_1 = \int d\boldsymbol{v}_2 \int d\Omega \left|\frac{d\sigma}{d\Omega}\right| |\boldsymbol{v}_2 - \boldsymbol{v}_1| \, [f_1(\boldsymbol{r},\boldsymbol{p}_1',t)f_1(\boldsymbol{r},\boldsymbol{p}_2',t) - f_1(\boldsymbol{r},\boldsymbol{p}_1,t)f_1(\boldsymbol{r},\boldsymbol{p}_2,t)]$$

Particle interactions modify the Liouvillian flow

Boltzmann collision equation

Kinematics of collisions :

- A cylindrical polar coordinates is taken to do the above integral : the *scattering* angle θ refers to the x-axis *parallel* to $\boldsymbol{p}_2 - \boldsymbol{p}_1$ (before x_1), the perpendicular plane is parametrized by the y-axis parallel to the *impact parameter* $\widehat{\boldsymbol{b}}$ (unit vector) and by the angle ϕ, r_m is the *distance of closest approach*.

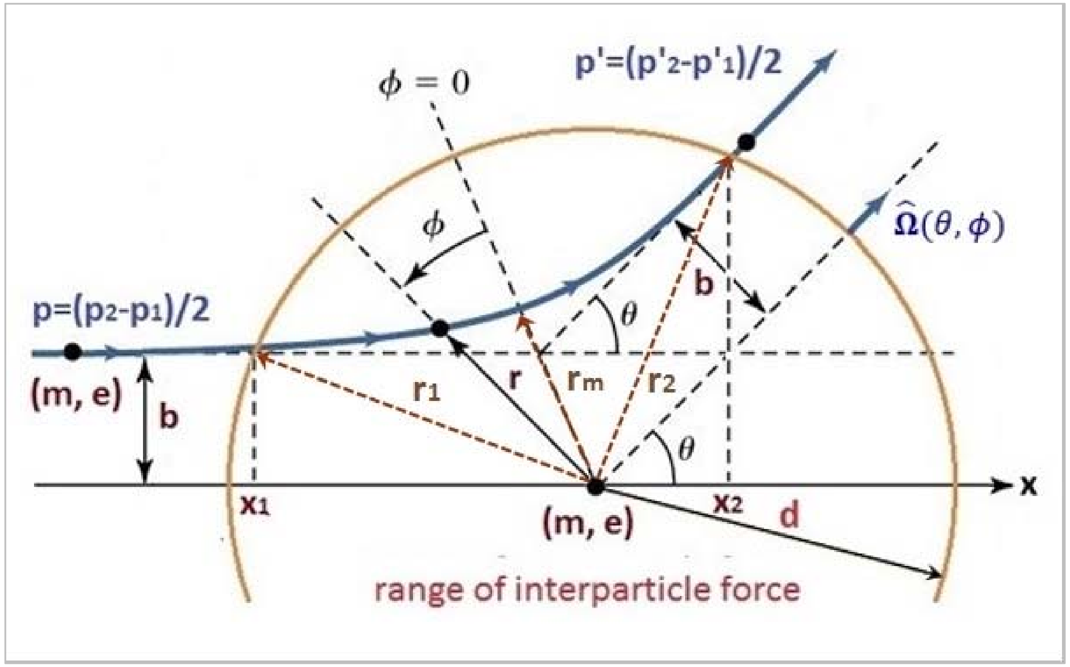
- *Non-relativistic* collision of 2 particles of mass m and momenta $\boldsymbol{p}_{1,2} = m\boldsymbol{v}_{1,2}$ seen from a frame in which one particle is at rest at $x = 0$.
- The out-going momenta $\boldsymbol{p}'_{1,2}$ are given from the conditions :
  1. Conserved momentum : $\boldsymbol{p}'_2 + \boldsymbol{p}'_1 = \boldsymbol{p}_2 + \boldsymbol{p}_1$
  2. Conserved energy :

     $|\boldsymbol{p}'_2|^2 + |\boldsymbol{p}'_1|^2 = |\boldsymbol{p}_2|^2 + |\boldsymbol{p}_1|^2$ ⇔

     $\boldsymbol{p}'_2 - \boldsymbol{p}'_1 = |\boldsymbol{p}_2 - \boldsymbol{p}_1|\widehat{\Omega}(\theta, \phi)$ ⇒

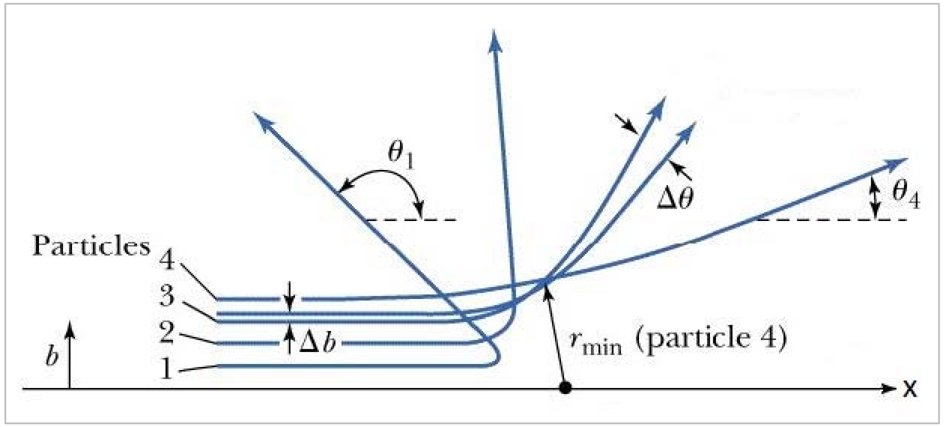
     $|\boldsymbol{p}'_2 - \boldsymbol{p}'_1| \equiv |\boldsymbol{p}_2 - \boldsymbol{p}_1|$ (constant modulus)

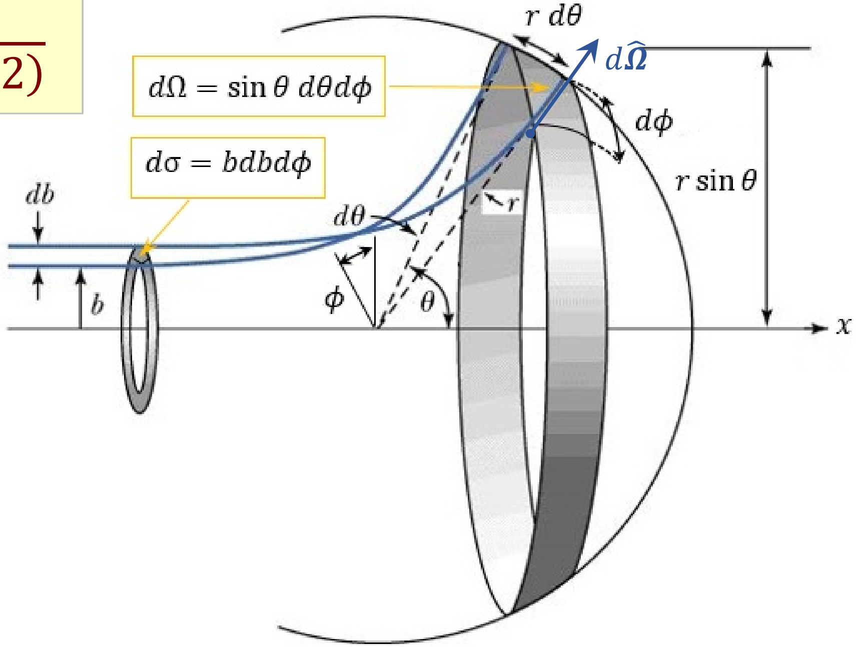
     where $\widehat{\boldsymbol{\Omega}}(\theta, \phi)$ is a *solid angle* unit vector



Boltzmann collision equation

- Differential cross-section : $|d\sigma/d\Omega| = |db/d\theta|$ [m^2]
  - This is the number of particles *scattered* per *unit time*, unit *incident flux* and oriented *solid angle* $\hat{\boldsymbol{\Omega}}(\theta,\phi)$ (the absolute value $|\cdots|$ comes because θ usually decreases when b increases)
  - Geometrically the next figures show a scattering process with $d\Omega = \sin\theta\, d\theta d\phi$ and $d\sigma = b\, dbd\phi$ where θ depends on the *interparticle force* law, the *relative momentum* $|\boldsymbol{p}_2 - \boldsymbol{p}_1|$ and *impact parameter* b
- Rutherford scattering : $\left|\frac{d\sigma}{d\Omega}\right| = \left(\frac{me^2}{4\pi\epsilon_0|\boldsymbol{p}_2 - \boldsymbol{p}_1|^2}\right)^2 \frac{1}{\sin^4(\theta/2)}$
  - Small θ yield large b ($\theta_{min} = 0 \rightarrow b_{max} = \infty$)





Equilibrium particle density

- **Equilibrium** : At equilibrium the 1−particle density $f_1(\boldsymbol{r},\boldsymbol{p})$ has no explicit time dependence :

  $\partial f_1/\partial t = 0 \longrightarrow \{H_1, f_1\} = 0 \longrightarrow f_1 = f_1(H_1)$ with $H_1(\boldsymbol{r},\boldsymbol{p}) = \boldsymbol{p}^2/2m + \Phi(\boldsymbol{r})$

- **Maxwell-Boltzmann distribution**: Similarly at equilibrium the collision integral vanishes :

$$f_1(\boldsymbol{r},\boldsymbol{p}_1)f_1(\boldsymbol{r},\boldsymbol{p}_2) = f_1(\boldsymbol{r},\boldsymbol{p}'_1)f_1(\boldsymbol{r},\boldsymbol{p}'_2) \qquad \ln f_1(\boldsymbol{r},\boldsymbol{p}_1) + \ln f_1(\boldsymbol{r},\boldsymbol{p}_2) = \ln f_1(\boldsymbol{r},\boldsymbol{p}'_1) + \ln f_1(\boldsymbol{r},\boldsymbol{p}'_2)$$

  where the l.h.s. refers to momenta before collision the r.h.s. to the those after collision.
  The equality is satisfied by any additive *invariant* quantities during the collision, e.g.

$$\ln f_1(\boldsymbol{r},\boldsymbol{p}) = -\beta[\boldsymbol{p}^2/2m + \Phi(\boldsymbol{r})] \Rightarrow f_1(\boldsymbol{r},\boldsymbol{p}) = \alpha e^{-\beta[\boldsymbol{p}^2/2m+\Phi(\boldsymbol{r})]}$$

  α and β are constants, from which the *Maxwell-Boltzmann velocity density* (for $\Phi(\boldsymbol{r}) = 0$) follows :

For a gaz of N particles in a box volume V for $\boldsymbol{p} = m\boldsymbol{v}$, $\boldsymbol{u}$ an overall *drift*, k the *Boltzmann* constant (the integral of f_1 over the ***3-dim*** box volume V is equal to N since $f_1 d\boldsymbol{p}$ must be normalized to N) :

$$f_1(\boldsymbol{v}) = \frac{N}{V}\left(\frac{\beta m}{2\pi}\right)^{3/2} e^{-\beta m(\boldsymbol{v}-\boldsymbol{u})^2/2} \quad \overset{\beta=1/kT}{\Longleftrightarrow} \quad f_1(\boldsymbol{v}) = \frac{N}{V}\frac{1}{(2\pi kT/m)^{3/2}} e^{-m(\boldsymbol{v}-\boldsymbol{u})^2/(2kT)}$$

INTRABEAM SCATTERING

- Part 2 : Intrabeam scattering
  - Core IBS model
  - IBS analytical model
  - Original Piwinski model
  - Bjorken-Mtingwa model

- Part 3 : Applications
  - IBS & LHC (7 TeV)
  - IBS & ELENA (100 keV)
  - Epilogue

The Intrabeam scattering effect

- Theoretical models calculate the **IBS growth rates**:

$$\frac{1}{T_i} \propto \frac{N}{\gamma \varepsilon_{xn} \varepsilon_{yn} \varepsilon_{sn}} f(optics, \gamma, \varepsilon_{xn}, \varepsilon_{yn}, \varepsilon_{sn})$$

- Complicated integrals averaged around the rings
  - Depend on optics and beam properties

✓ They have been well benchmarked for hadron machines

➢ For lepton machines the work is in progress
  - ➢ Need to benchmark the IBS effect in the presence of SR and QE
  - ➢ Studies and publications from: ATF(2001), CesrTA, SLS, SPEAR3

➢ Main drawbacks:
  - Gaussian beams assumed
  - Betatron coupling not trivial to be included
  - Impact on damping process (especially in strong IBS regimes)?

➢ Tracking codes **SIRE** (A. Vivoli) and **CMAD-IBStrack** (M. Pivi, T. Demma)
  - Based on the classical Rutherford cross section

Courtesy F. Antoniou, Y. Papaphilippou, CERN

Core IBS model

Continuation… from Part 1

Transverse & longitudinal beam growth rate estimate : A strategy in 7 steps

- Following Piwinski's calculations of beam size *growth*/*decrease rates* due to IBS effect are sketched.
- The presented kinematics & dynamics of charge particle pair collisions refer to Piwinski 1974 & 1986.

1. Transform the momenta of the colliding particles from the *LAB* to the centre of mass (*CM*) frame
2. Calculate the changes in momenta due to an *elastic collision*.
3. Transform of the momenta back to the *LAB* frame.
4. Relate the changes in *momenta* to changes in *transverse* & *longitudinal emittances*.
5. Average over the *scattering angle* distribution using the classical *Rutherford* cross-section.
6. Average over the *particle momentum* & *position* distributions in a bunch.
7. The *averaged emittances* allow to work out the *growth* or *decrease rates* of the bunch sizes.

Strategy step 1-3: momenta kinematics

Core IBS model

In line with Piwinski the relative longitudinal and transverse *momentum changes* after a collision between two particles (labelled 1, 2) can be cast (after some hard-working task) into the form :

$$\delta \boldsymbol{p}_{1,2} = \boldsymbol{p}'_{1,2} - \boldsymbol{p}_{1,2} =$$

defining

$$\theta = \frac{p_{x1} - p_{x2}}{p} \equiv x'_1 - x'_2 \qquad \zeta = \frac{p_{z1} - p_{z2}}{p} \equiv z'_1 - z'_2$$

$$\gamma\xi = \frac{p_1 - p_2}{p} \qquad 2\alpha \equiv \alpha_1 + \alpha_2 = \sqrt{\theta^2 + \varsigma^2}$$

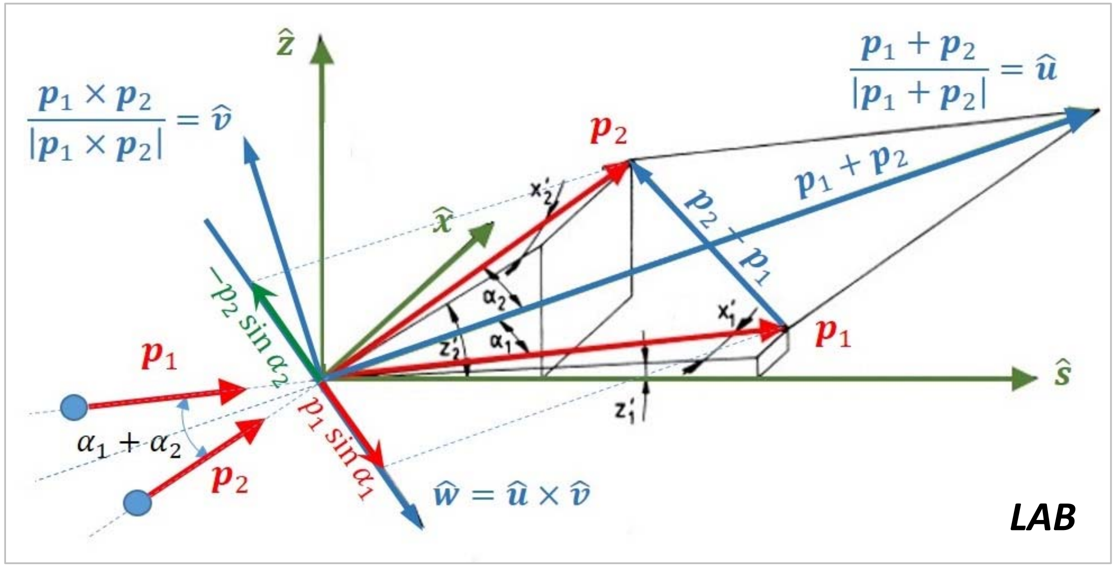
$$\frac{\delta p_s}{p} \approx \frac{\delta p}{p} = \frac{\gamma}{2}\left[2\alpha \sin\bar{\phi} \sin\bar{\psi} + \xi(\cos\bar{\psi} - 1)\right]$$

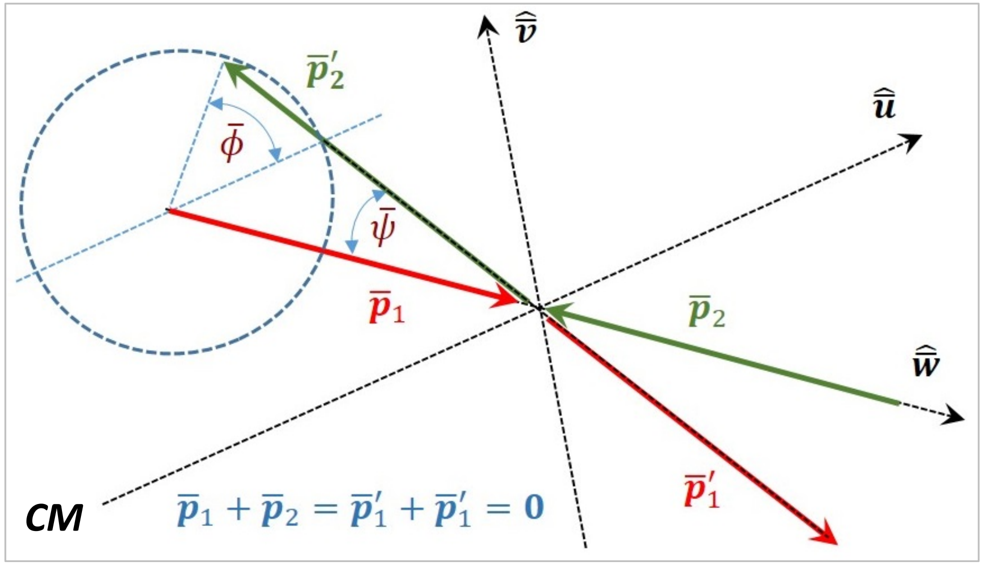
$$2\frac{\delta p_x}{p} = \left[\xi\sqrt{1 + \frac{\xi^2}{4\alpha^2}}\cos\bar{\phi} - \frac{\xi\theta}{2\alpha}\sin\bar{\phi}\right]\sin\bar{\psi} + \theta(\cos\bar{\psi} - 1)$$

$$2\frac{\delta p_z}{p} = \left[\xi\sqrt{1 + \frac{\xi^2}{4\alpha^2}}\cos\bar{\phi} - \frac{\xi\zeta}{2\alpha}\sin\bar{\phi}\right]\sin\bar{\psi} + \theta(\cos\bar{\psi} - 1)$$

- $\delta \boldsymbol{p}_{1,2}$ is the back momenta *Lorentz transform* from momenta in ad-hoc *CM* frame $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})$-axes to the *LAB* frame $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$-axes ($p_{1,2} = |\boldsymbol{p}_{1,2}|$, $\boldsymbol{p}$ is the mean particle momentum, $\hat{\boldsymbol{s}}$ =unit vector, γ the *Lorentz* factor, $\bar{\psi}$ & $\bar{\phi}$ the axial & azimuthal collision angles in *CM*, $2\alpha \equiv \alpha_1 + \alpha_2$ is the angle between particle momenta in *LAB*)
- $\boldsymbol{p}'_{1,2}$ are the rotated momenta after collision with angles $\bar{\psi}$ & $\bar{\phi}$ (expressed in *LAB* frame).
- $\boldsymbol{p}_{1,2}$ are the momenta before collision written as $\boldsymbol{p}_{1,2} = p_{s1,2}(1, x'_{1,2}, z'_{1,2})$ via $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{z}})$-coordinates in *LAB* frame and $\boldsymbol{p}_{1,2} = p_{1,2}(\cos\alpha_{1,2}, 0, \pm\sin\alpha_{1,2})$ via $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})$-coordinates in *CM* frame (cf. next Fig.)

Core IBS model





- Particle momenta $p_{1,2}$ *before* collision in ***LAB*** frames ($\hat{s}, \hat{x}, \hat{z}$)
- Relation between initial $p_{1,2}$ and final $p'_{1,2}$ is quite complex
- The overlaid ($\hat{u}, \hat{v}, \hat{w}$) frame is aligned on ***CM*** particle motion

Particle momenta *before* collision ($\bar{p}_1$,$\bar{p}_2$) and *after* ($\bar{p}'_1$, $\bar{p}'_2$) in the *CM* frame ($\hat{u}, \hat{v}, \hat{w}$) ($\hat{u}$ being the Lorentz-transformed longitudinal axis from *LAB* to *CM* frame)

The *change of particle momentum* after collision leads to a parallel change of the particle *invariants* (i.e. longitudinal & transverse *emittances*) which result supposing that transverse particle positions are not altered during the interaction time (assumed to be short enough).

Strategy step 1-3: momenta kinematics

Strategy step 4: emittance changes

Core IBS model

- The radial particle movement from the closed orbit is the sum of betatron & momentum deviation.
- The invariants are the beam emittances $\varepsilon_{x,z}$ & H (for *bunched beams*) in which $\alpha_{x,z}, \beta_{x,z}, \gamma_{x,z}$ are the Twiss parameters, with $\beta_{x,z}\gamma_{x,z}-\alpha_{x,z}^2=1$, $2\alpha_{x,z}=-\beta'_{x,z}$, Ω is the synchrotron frequency :

$$x = x_\beta + D_x \Delta p/p \qquad z = z_\beta$$
$$x' \equiv p_x/p = x'_\beta - D'_x \Delta p/p \qquad z' \equiv p_z/p = z'_\beta$$

$$\varepsilon_x = \gamma_x x_\beta^2 + 2\alpha_x x_\beta x'_\beta + \beta_x x'^2_\beta$$
$$H = (\Delta p/p)^2 + \Omega^{-2}\left[\tfrac{d}{dt}(\Delta p/p)\right]^2$$

The change $\delta\varepsilon_{x,z}$ of $\varepsilon_{x,z}$ works out as (swap x with z for $\delta\varepsilon_z$) :

$$\delta\varepsilon_x = \gamma_x(2x_\beta\delta x_\beta + \delta x_\beta^2) + 2\alpha_x\left(x'_\beta\delta x_\beta + x_\beta\delta x'_\beta + \delta x_\beta\delta x'_\beta\right) + \beta_x\left(2x'_\beta\delta x'_\beta + \delta x'^2_\beta\right)$$

Assuming there is no vertical dispersion i.e. $D_z=D'_z=0$ and that $x_{1,2}$ & $z_{1,2}$ stay constant during the short collision time so that only $x'_{1,2}$ & $z'_{1,2}$ vary with the momentum change. Since $\delta(\Delta p/p)=\delta p/p$ as the mean momentum $p=|\mathbf{p}|$ is constant without acceleration, the variations δx_β, $\delta x'_\beta$, $\delta z'_\beta$ can be written in term of betatron amplitudes as follows :
(e.g. $\delta x=\delta x_\beta+D_x\Delta p/p=\delta x_\beta+D_x\delta(\Delta p/p)=\delta x_\beta+D_x\delta p/p \equiv 0 \Longrightarrow \delta x_\beta=-D_x\delta p/p$)

Strategy step 5: scattering angle averages

Core IBS model

$$\delta x_\beta = -D_x \delta p/p \quad \delta x'_\beta = \delta p_x/p - D'_x \delta p/p \quad \delta z'_\beta = \delta p_z/p$$

The changes $\delta\varepsilon_{x,z}$ & δH of $\varepsilon_{x,z}$ & H *after collision* can be rewritten (in which $\widetilde{D}_x = \alpha_x D_x + \beta_x D'_x$ and by disregarding the time variation of Ω during the collision) as :

$$\frac{\delta\varepsilon_x}{\beta_x} = -\frac{2}{\beta_x}\left[\gamma_x x_\beta D_x + \alpha_x D'_x + x'_\beta \widetilde{D}_x\right]\frac{\delta p}{p} + \frac{D_x^2 + \widetilde{D}_x^2}{\beta_x^2}\left(\frac{\delta p}{p}\right)^2 + 2\left(x'_\beta + \frac{\alpha_x}{\beta_x} x_\beta\right)\frac{\delta p_x}{p} + \left(\frac{\delta p_x}{p}\right)^2 - \frac{2\widetilde{D}_x}{\beta_x}\frac{\delta p}{p}\frac{\delta p_x}{p}$$

$$\frac{\delta\varepsilon_z}{\beta_z} = 2\left(z'_\beta + \frac{\alpha_z}{\beta_z} z_\beta\right)\frac{\delta p_z}{p} + \left(\frac{\delta p_z}{p}\right)^2$$

$$\delta H = 2\frac{\Delta p}{p}\frac{\delta p}{p} + \left(\frac{\delta p}{p}\right)^2$$

The beam phase space volume change can be found by averaging the particle invariant variation over the collisions.

- For a scattering process Piwinski introduced the derivative $d\langle\varepsilon_{x,z}\rangle/d\bar{t}$ i.e. the mean emittance change of a 1st particle by averaging with all betatron angles (or momentum spread) of a 2nd particle.
- Further averages over positions, betatron angles (or momentum deviations) of the 1st particle must be done to get the total mean emittance change of all particles : i.e. integrate over the phase space with the probability density law P ($\bar{P}$) in the *LAB* & *CM* frames. In formula this writes as follows :

Strategy step 5:
scattering angle averages

Core IBS model

Physics of collisions

$$\left\langle \frac{d}{d\bar{t}} \frac{\langle \varepsilon_x \rangle}{\beta_x} \right\rangle = \int 2c\bar{\beta}\bar{P} \int_{\bar{\psi}_{\min}}^{\pi} d\,\bar{\psi} \int_0^{2\pi} d\bar{\phi}\, \frac{d\bar{\sigma}}{d\bar{\Omega}} \frac{\delta\varepsilon_x}{\beta_x} \sin\bar{\psi}\, d\bar{\tau}$$

The outer $\langle \cdots \rangle$ denotes an average round the optics parameters, $d\bar{\sigma}/d\bar{\Omega}$ is the *Rutherford* differential cross-section for the scattering into a solid angle element $d\bar{\Omega}(\bar{\phi},\bar{\psi})$ in the *CM* frame. The *proper time* intervals in *CM* & *LAB* frames are $d\bar{t}$ & dt with $dt{=}\gamma d\bar{t}$, $2c\bar{\beta}$ is the relative velocity of two colliding particles with $\bar{\boldsymbol{v}}_1{+}\bar{\boldsymbol{v}}_2{=}0$ in *CM* frame. P is defined as a probability density product using 12 variables and can be expresses in *LAB* into the form (defining for short $\eta_{1,2}{\equiv}\Delta p_{1,2}/p_{1,2}$) :

$$P = P_{\eta s}(\eta_1, s_1)\, P_{\eta s}(\eta_2, s_2)\, P_{x_\beta x'_\beta}(x_{\beta_1}, x'_{\beta_1}) P_{x_\beta x'_\beta}(x_{\beta_2}, x'_{\beta_2}) P_{z_\beta z'_\beta}(z_{\beta_1}, z'_{\beta_1}) P_{z_\beta z'_\beta}(z_{\beta_2}, z'_{\beta_2})$$

Among the 12 P variables 3 are dependent since during the short collision time the 2 particle positions are assumed not to change i.e. :

$$s_1{=}s_2{=}s \qquad x_1{=}x_{\beta_1}{+}D_x\eta_1{\equiv}x_2{=}x_{\beta_2}{+}D_x\eta_2 \qquad z_1{=}z_{\beta_1}{\equiv}z_2{=}z_{\beta_2}$$

The distribution P will be examined in more details later.

Strategy step 5:
scattering angle averages

Core IBS model

The scattering angle distribution is now considered. The *Rutherford* differential cross-section for a *non-relativistic* Coulomb collision of 2 *ions* of charge Z and atomic mass A in a *CM* frame (i.e. $\bar{\beta} \ll 1$) is :

$$\frac{d\bar{\sigma}(\bar{\psi})}{d\bar{\Omega}} = \left(\frac{AmZ^2e^2}{4\pi\varepsilon_0|\bar{\boldsymbol{p}}_2 - \bar{\boldsymbol{p}}_1|^2}\right)^2 \frac{1}{\sin^4(\bar{\psi}/2)} = \left(\frac{Z^2r_0mc^2}{2\bar{T}}\right)^2 \frac{1}{\sin^4(\bar{\psi}/2)} = \left(\frac{Z^2}{A}\frac{r_0}{4\bar{\beta}^2}\right)^2 \frac{1}{\sin^4(\bar{\psi}/2)}$$

with $\bar{T}=|\bar{\boldsymbol{p}}_2 - \bar{\boldsymbol{p}}_1|^2/2Am=2Am\bar{\beta}^2c^2$ is the ion *kinetic energy*, $2Am\bar{\beta}c$ is the *relative momentum* between the hitting ions for which $\bar{\boldsymbol{p}}_1 + \bar{\boldsymbol{p}}_2=0$ in *CM*, $r_0=e^2/4\pi\varepsilon_0mc^2$ is the classical proton radius.

To evaluate $\bar{\beta}$ the above expression $|\bar{\boldsymbol{p}}_2 - \bar{\boldsymbol{p}}_1|=2m\bar{\beta}c$ in the *CM* frame must be Lorentz transformed back to the *LAB* frame to link $\bar{\beta}c$ with βc. All calculations done it is found in first approximation :

$$\bar{\beta} \approx \frac{\beta\gamma}{2}\left[\frac{1}{\gamma^2}\frac{(p_1 - p_2)^2}{p^2} + (x_1' - x_2')^2 + (z_1' - z_2')^2\right]^{\frac{1}{2}} = \frac{\beta\gamma}{2}\sqrt{(\xi^2 + \theta^2 + \zeta^2}$$

in which βc is the average particle velocity in the *LAB* frame. The two integrals over $\bar{\psi}$ & $\bar{\phi}$ needed to evaluate part of the average time-derivative of $\langle\varepsilon\rangle/\beta_x$ are computed replacing $\delta p/p$ & $\delta p_{x,z}/p$ by their expressions given in terms of parameters α, ξ, θ, ζ, $\bar{\phi}$, $\bar{\psi}$ yielding after rather lengthy calculations :

Strategy step 5: scattering angle averages

Core IBS model

$$I_{x,\bar{\phi},\bar{\psi}} \equiv \int_{\bar{\psi}_{\min}}^{\pi} d\,\bar{\psi} \int_0^{2\pi} d\bar{\phi}\, \frac{d\bar{\sigma}}{d\bar{\Omega}} \frac{\delta\varepsilon_x}{\beta_x} \sin\bar{\psi}$$

$$= \frac{Z^4}{A^2}\frac{\pi r_0^2}{8\bar{\beta}^4}\int_{\bar{\psi}_m}^{\pi} d\,\bar{\psi}\,\frac{\sin\bar{\psi}}{\sin^4(\bar{\psi}/2)}\Bigg\{(1-\cos\bar{\psi})\left[\frac{2x_\beta}{\beta_x}\big(\gamma_x D_x\gamma\xi + \alpha_x(D_x'\gamma\xi - \theta)\big) + 2x_\beta'\left(\frac{\widetilde{D}_x\gamma\xi}{\beta_x} - \theta\right)\right.$$

$$\left. + \frac{{D_x}^2 + {\widetilde{D}_x}^2}{\beta_x^2}\gamma^2\xi^2 + \theta^2 - \frac{2\widetilde{D}_x}{\beta_x}\gamma\xi\theta\right]$$

$$+ \sin^2\bar{\psi}\left(\frac{{D_x}^2 + {\widetilde{D}_x}^2}{4\beta_x^2}\gamma^2(\theta^2 + \zeta^2 - 2\xi^2) + \frac{1}{4}(\xi^2 + \zeta^2 - 2\theta^2) + \frac{3\widetilde{D}_x}{2\beta_x}\gamma\xi\theta\right)\Bigg\}$$

The smallest angle $\bar{\psi}_{\min}$ is defined by the maximum *impact parameter* $\bar{b}_{\max}$ fixed by the beam height :

$$\tan\left(\frac{\bar{\psi}_{\min}}{2}\right) = \frac{Z^2}{A}\frac{r_0}{2\bar{\beta}^2\bar{b}_{\max}}$$

To calculate analytically these 2 integrals it is assumed that $4A^2\bar{\beta}^4\bar{b}_{\max}^2/Z^4r_0^2 \gg 1$ (i.e. $\bar{\psi}_{\min} \ll 1$) then :

$$\int_{\bar{\psi}_{\min}}^{\pi} \frac{\sin\bar{\psi}\,(1-\cos\bar{\psi})}{\sin^4(\bar{\psi}/2)}\,d\,\bar{\psi} = -8\ln\left[\sin\frac{\bar{\psi}_{\min}}{2}\right] \approx 8\ln\left[\frac{2}{\bar{\psi}_{\min}}\right] = 4\ln\left[\frac{4}{\bar{\psi}_{\min}^2}\right] = 4\ln\left[\frac{4A^2\bar{\beta}^4\bar{b}_{\max}^2}{Z^4r_0^2}\right]$$

Strategy step 5: scattering angle averages

Core IBS model

$$\int_{\bar{\psi}_{\min}}^{\pi} \frac{\sin^3\bar{\psi}}{\sin^4(\bar{\psi}/2)} d\bar{\psi} = -4(1+\cos\bar{\psi}_{\min}) - 16\ln\left[\sin\frac{\bar{\psi}_{\min}}{2}\right] \approx 16\ln\left[\frac{2}{\bar{\psi}_{\min}}\right] = 8\ln\left[\frac{4A^2\bar{\beta}^4\bar{b}_{\max}^2}{Z^4 r_0^2}\right]$$

After reorganizing the integrals it follows :

$$I_{x,\bar{\phi},\bar{\psi}} = \int_{\bar{\psi}_{\min}}^{\pi} d\bar{\psi} \int_0^{2\pi} d\bar{\phi} \frac{d\bar{\sigma}}{d\bar{\Omega}} \frac{\delta\varepsilon_x}{\beta_x} \sin\bar{\psi}$$

$$= \frac{Z^4}{A^2}\frac{\pi {r_0}^2}{4\bar{\beta}^4} \left\{ \frac{4x_\beta}{\beta_x}\left(\gamma_x D_x \gamma\xi + \alpha_x (D_x'\gamma\xi - \theta)\right) + 4x_\beta'\left(\frac{\tilde{D}_x\xi}{\beta_x} - \theta\right) + \xi^2 + \zeta^2 + \frac{{D_x}^2 + {\tilde{D}_x}^2}{\beta_x^2}\gamma^2(\xi^2 + \theta^2) + \frac{2\tilde{D}_x}{\beta_x}\gamma\xi\theta \right\} \ln\left[\frac{A^2}{Z^4}\frac{4\bar{\beta}^4\bar{b}_{\max}^2}{{r_0}^2}\right]$$

$$\ln\left[\frac{A^2}{Z^4}\frac{4\bar{\beta}^4\bar{b}_{\max}^2}{{r_0}^2}\right] \approx 2\ln\left[\frac{2}{\bar{\psi}_{\min}}\right] = 2\bar{C}_{\log}$$

$C_{\log}$ is the *Coulomb logarithm* in *LAB* frame (cf Bjorken Mtingwa) but $\bar{C}_{\log}$ is in *CM*. Its logarithmic dependence makes it slowly change over a big range of the elements involved in its definition.

$$C_{\log} \equiv \ln\left[\frac{A2\beta^2 b_{\max}}{Z^2 r_0}\right]$$

Similarly the integrals $I_{z,\bar{\phi},\bar{\psi}}$ and $I_{s,\bar{\phi},\bar{\psi}}$ for the vertical and longitudinal momenta can be worked out assuming no vertical dispersion and then put together, yielding the transverse and longitudinal scattering integrals :

Strategy step 5: scattering angle averages

Core IBS model

$$\begin{pmatrix} I_{s,\bar{\phi},\bar{\psi}} \\ I_{x,\bar{\phi},\bar{\psi}} \\ I_{z,\bar{\phi},\bar{\psi}} \end{pmatrix} \equiv \int_{\bar{\psi}_{\min}}^{\pi} d\,\bar{\psi} \int_0^{2\pi} d\bar{\phi} \sin\bar{\psi} \frac{d\bar{\sigma}}{d\bar{\Omega}} \begin{pmatrix} \dfrac{\delta H}{\gamma^2} \\ \dfrac{\delta\varepsilon_x}{\beta_x} \\ \dfrac{\delta\varepsilon_z}{\beta_z} \end{pmatrix} = \frac{Z^4}{A^2}\frac{{r_0}^2}{4\bar{\beta}^4} \ln\left[\frac{A^2}{Z^4}\frac{4\bar{\beta}^4\bar{b}_{\max}^2}{{r_0}^2}\right] \times$$

$$\left\{ \begin{array}{c} -\dfrac{4\Delta p}{\gamma p}\xi + \theta^2 + \zeta^2 \\ \dfrac{4x_\beta}{\beta_x}\left(\gamma_x D_x \xi + \alpha_x (D'_x \gamma\xi - \theta)\right) + 4x'_\beta\left(\dfrac{\widetilde{D}_x \xi}{\beta_x} - \theta\right) + \xi^2 + \zeta^2 + \dfrac{{D_x}^2 + {\widetilde{D}_x}^2}{\beta_x^2}\gamma^2(\xi^2 + \theta^2) + \dfrac{2\widetilde{D}_x}{\beta_x}\gamma\xi\theta \\ -\dfrac{4\alpha_z z_\beta}{\beta_z}\zeta - 4z'_\beta\theta + \xi^2 + \zeta^2 \end{array} \right\}$$

The computation of the mean change of the invariants $\varepsilon_{x,z}$ & H of all particles due to the multiple particle collisions requires to average the above three integrals of the two colliding particles over the 12 variables, reduced to 9 as $(s_{1,2}, x_{1,2}, z_{1,2})$ are dependent, via the probability densities P $(\bar{P})$.

Strategy step 6: particle beam averages

Core IBS model

Changing the 9 variables of the joint probability law P into the new ones $\eta, s, \xi, x_\beta, x'_\beta, \theta, z, z', \zeta$:

$$P(\eta_1, \eta_2, s_1, x_{\beta_1}, x'_{\beta_1}, x'_{\beta_2}, z_1, z'_1, z'_2) \longmapsto P(\eta, s, \xi, x_\beta, x'_\beta, \theta, z, z', \zeta)$$

by means of the substitutions we get, via the Jacobian of the transformation the volume element $d\tau$ (the «variables» s_2, x_{β_2}, z_2 disappear as they depend upon s_1, x_{β_1}, z_1 via their tight constraints) :

$$x_{\beta_{1,2}} = x_\beta \mp D_x \gamma \xi/2 \quad x'_{\beta_{1,2}} = x'_\beta \pm (\theta - D'_x \gamma \xi)/2$$

$$z'_{1,2} = z' \pm \zeta/2 \quad \eta_{1,2} = \eta \pm \gamma \xi/2$$

$$d\tau = ds\,d\eta\,d\xi\,dx_\beta\,dx'_\beta\,d\theta\,dz\,dz'\,d\zeta = |J|\,ds\,d\eta_1\,d\eta_2\,dx_{\beta_1}\,dx'_{\beta_1}\,dx'_{\beta_2}\,dz_1\,dz'_1, dz'_2 \quad \text{with} \quad |J| = \gamma$$

Hence the formal expression for the mean change of the invariants $\varepsilon_{x,z}$ & H can be cast into the form where $\langle \varepsilon_u \rangle / \beta_u$ stands for and $\langle H \rangle / \gamma^2$:

$$\left\langle \frac{d}{d\bar{t}} \frac{\langle \varepsilon_u \rangle}{\beta_u} \right\rangle = \int 2c\bar{\beta}\bar{P} \begin{pmatrix} I_{s,\bar{\phi},\bar{\psi}} \\ I_{x,\bar{\phi},\bar{\psi}} \\ I_{z,\bar{\phi},\bar{\psi}} \end{pmatrix} d\bar{\tau}$$

P being now symmetrical with respect to ξ, θ, ζ it follows that the integrals cancel for the linear terms in ξ, θ, ζ of the integrand.

Strategy step 6: particle beam averages

Core IBS model

$$\frac{d}{d\bar{t}}\begin{pmatrix}\frac{\langle H\rangle}{\gamma^2}\\ \frac{\langle\varepsilon_x\rangle}{\beta_x}\\ \frac{\langle\varepsilon_z\rangle}{\beta_z}\end{pmatrix} = \frac{Z^4}{A^2}\frac{\pi c r_0^2}{2}\times$$

$$\int \bar{P}\ln\left[\frac{A^2}{Z^4}\frac{4\bar{\beta}^4\bar{b}_{\max}^2}{{r_0}^2}\right]\begin{Bmatrix}\theta^2+\zeta^2-2\xi^2\\ \xi^2+\zeta^2-2\theta^2+\frac{{D_x}^2+{\tilde{D}_x}^2}{\beta_x^2}\gamma^2(\zeta^2+\theta^2-2\xi^2)+\frac{6\tilde{D}_x}{\beta_x}\gamma\xi\theta\\ \xi^2+\theta^2-2\zeta^2\end{Bmatrix}\frac{d\bar{\tau}}{\bar{\beta}^3}$$

- This formula for the mean change of the invariants $\varepsilon_{x,z}$ & H makes no assumption about the density *distribution* $\bar{P}$.
- To derive IBS analytical models (not in closed form!) it is usually assumed that betatron amplitudes, angles, momentum deviations and synchrotron coordinates are *Gaussian* distributed for *bunched* beams, the synchrotron coordinate being uniformly distributed for *unbunched* beams.

Strategy step 6: particle beam averages

IBS analytical model

Let's define Gaussian *distributions* $P_{x_\beta x'_\beta}$ & $P_{z_\beta z'_\beta}$ (with $z_\beta \equiv z$ & $z'_\beta \equiv z'$ and assuming $D_z = D'_z = 0$) for the betatron amplitudes & angles and $P_{\eta s}$ for momentum and bunch length deviations (*bunched* beams) :

$$P_{x_\beta x'_\beta} = \frac{\sqrt{1+\alpha_x^2}}{2\pi\sigma_{x_{\beta_{1,2}}}\sigma_{x'_{\beta_{1,2}}}} \exp\left[-Q\left(x_{\beta_{1,2}}, x'_{\beta_{1,2}}\right)\right] \quad Q\left(x_\beta, x'_\beta\right) = \frac{1+\alpha_x^2}{2}\left(\frac{x_\beta^2}{\sigma_{x_\beta}^2} + \frac{2x_\beta x'_\beta \alpha_x}{\sigma_{x_\beta}\sigma_{x'_\beta}\sqrt{1+4\sigma_{x_\beta}^2}} + \frac{{x'_\beta}^2}{\sigma_{x'_\beta}^2}\right)$$

$$P_{\eta s} = \frac{1}{2\pi\sigma_{\eta_{1,2}}\sigma_{s_{1,2}}} \exp\left[-\frac{\eta_{1,2}^2}{2\sigma_{\eta_{1,2}}^2} - \frac{(s-s_0)^2}{2\sigma_{s_{1,2}}^2}\right]$$

σ_{x_β}, $\sigma_{x'_\beta}$, σ_η are rms values of the related variables, σ_s the rms bunch length, $\Delta s = s - s_0$ the synchrotron coordinate, i.e. the position relative to the synchronous particle.

Ditto in vertical plane $P_{z_\beta z'_\beta}$.

Q=constant is a tilted ellipse with is correlation coefficient $\rho_x = \alpha_x/\sqrt{1+\alpha_x^2}$. The probability distribution P must be well-matched to the *Courant-Snyder invariant* $\varepsilon_x = \gamma_x x_\beta^2 + 2\alpha_x x_\beta x'_\beta + \beta_x x'^2_\beta$ which is the phase space area divided by π i.e. ε_x=area/π.

Strategy step 6: particle beam averages

IBS analytical model

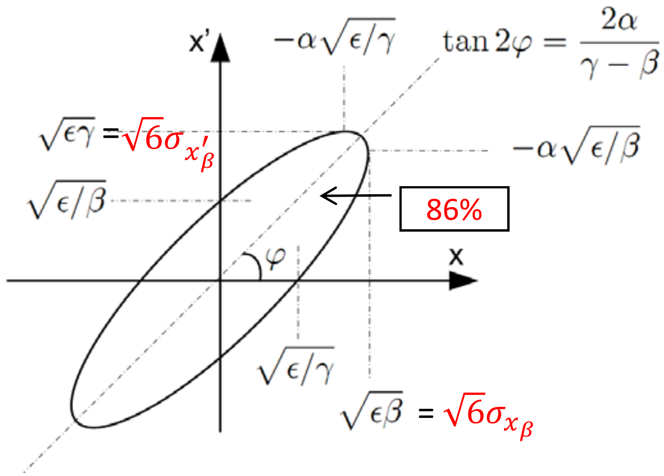
The emittance describes the phase space area usedby the beam, i.e. for a phase space area covering a fraction F of a Gaussian beam with rms value σ_{x_β} the emittance at F [%] of particles in phase space is :

$$\varepsilon_x = \frac{2\sigma_{x_\beta}^2}{\beta_x} \ln[1 - F]$$

e.g. the emittance at $F=(39, 86, 95)\%$ are $\varepsilon_x=(1, 4, 6)\sigma_{x_\beta}^2/\beta_x$. Notice that the beam width containing a projected beam fraction $F_{\mathrm{proj}}=95\%$ onto the betatron horizontal amplitude axis is $2\sigma_{x_\beta}<\sqrt{6}\sigma_{x_\beta}$ yielding the "projected emittance" $\tilde{\varepsilon}_x = 4\sigma_{x_\beta}^2/\beta_x$ (not the same as above !).

Using the related betatron amplitude and angle rms values σ_{x_β} and $\sigma_{x'_\beta}=\sigma_{x_\beta}\sqrt{\gamma_x/\beta_x}$ the *probability* $P_{x_\beta x'_\beta}$ can be rewritten (also $P_{z_\beta z'_\beta}$, P_η being unchanged) as :

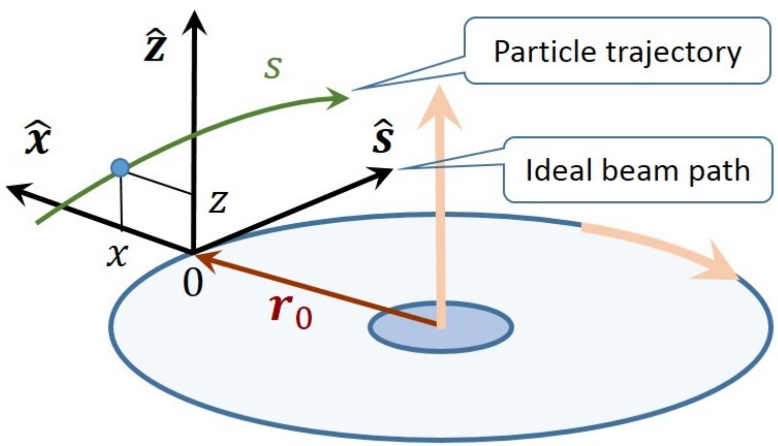
$$P_{x_\beta x'_\beta} = \frac{\beta_x}{2\pi\sigma_{x_{\beta_{1,2}}}^2} \exp\left[-\frac{\beta_x}{2\sigma_{x_{\beta_{1,2}}}^2}\left(\gamma_x x_{\beta_{1,2}}^2 + 2\alpha_x x_{\beta_{1,2}} x'_{\beta_{1,2}} + \beta_x x'^2_{\beta_{1,2}}\right)\right]$$



Strategy step 6: particle beam averages

IBS analytical model

In the *CM* frame all derivatives d/ds are reduced by γ because of the *Lorentz* contraction along s (e.g. $\bar{P}=P/\gamma$, $\bar{\sigma}_{x'_\beta}=\sigma_{x'_\beta}/\gamma$), the transverse sizes & relative momentum spread are unchanged (e.g. $\bar{\sigma}_{x_\beta}=\sigma_{x_\beta}$, $\bar{\sigma}_\eta=\sigma_\eta$, $\bar{b}_{\max}=b_{\max}$) and the bunch length turns into $\bar{\sigma}_s=\gamma\sigma_s$.



Accelerator & storage ring moving coordinates

$\boldsymbol{r}(s)=\boldsymbol{r}_0(s)+d\boldsymbol{r}(s$ $\quad d\boldsymbol{r}(s)=x(s)\hat{\mathbf{x}}+z(s)\hat{\mathbf{z}}$

The *relative velocity* between 2 scattering ions being $2\bar{\beta}c$ the probability for a collision in $\bar{\phi}$ and $\bar{\psi}$ per unit time in the *CM* frame is $2\bar{\beta}c\bar{P}[d\bar{\sigma}/d\bar{\Omega}]$. Hence the scattering probability per unit time in a storage ring is, with $dt=\gamma d\bar{t}$:

$$P_{\text{scat}}=2\bar{\beta}c\frac{P}{\gamma^2}\frac{d\bar{\sigma}}{d\bar{\Omega}}$$

Integrating in the *CM* frame the mean invariant changes over the variables $s,\eta,x_\beta,x'_\beta,z_\beta,x'_\beta$ (integrals over ξ,θ,ζ & $2\bar{\beta}\approx\beta\gamma\sqrt{\xi^2+\theta^2+\zeta^2}$ is still to be done).

Strategy step 6-7: beam averages & IBS rise times

IBS analytical model

- Now the P's must be inserted into the mean invariant changes $\varepsilon_{x,z}$ & H for further integration.
- For this replace the above 7 probability densities $P_{x_{\beta_{1,2}} x'_{\beta_{1,2}}}, P_{z_{\beta_{1,2}} z'_{\beta_{1,2}}}, P_{\eta_{1,2}}, P_s^2$ by their expressions given in terms of the 9 variables $s, \eta, \xi, x_\beta, x'_\beta, \theta, z, z', \zeta$ and use $\bar{\beta} = \beta\gamma\sqrt{(\xi^2 + \theta^2 + \zeta^2}/2$, i.e.

$$P_{x_{\beta_1} x'_{\beta_1}}\left(x_\beta - D_x\gamma\xi/2, x'_\beta + (\theta - D'_x\gamma\xi)/2\right) \quad P_{z_{\beta_1} z'_{\beta_1}}\left(z_\beta, z'_\beta + \zeta/2\right) \quad P_{\eta_1}(\eta + \gamma\xi/2)$$

- The integrations over the 6 variables $x_\beta, x'_\beta, z, z', \eta, s$ can be done with the help of the integral (in which a and b can be any of the 6 variables $x_\beta \ldots s$) :

$$\int_{-\infty}^{\infty} \exp[-ay^2 - 2by]dy = \sqrt{\pi/a}\,\exp[b^2/a]$$

- Before the integration all the variables have to be Lorentz transformed to the *CM* frame, and after integration they must be transformed back to the *LAB* frame. Considering all the beam particles, the "final" result is, after tedious manipulations (3 more integrals over ξ, θ, ζ must still be solved!) :

Strategy step 6-7: beam averages & IBS rise times

IBS analytical model

$$\frac{d}{dt}\begin{pmatrix}\frac{\langle H\rangle}{\gamma^2}\\ \frac{\langle\varepsilon_x\rangle}{\beta_x}\\ \frac{\langle\varepsilon_z\rangle}{\beta_z}\end{pmatrix} = \mathcal{A}\iiint_{-\infty}^{\infty}\exp\left[-\frac{\gamma^2\xi^2}{4\sigma_\eta^2}-\frac{1}{\sigma_{x_\beta}^2}\left(\left[D_x^2+\tilde{D}_x^2\right]\gamma^2\xi^2-\frac{\beta_x^2}{4}\theta^2+\frac{\beta_x\tilde{D}_x}{2}\gamma\xi\theta+\frac{\beta_z^2}{4}\zeta^2\right)\right]\times$$

$$\ln\left[\frac{\gamma^4}{16}\frac{A^2\beta^4 b_{\max}^2}{Z^4 r_0^2}(\xi^2+\theta^2+\zeta^2)^2\right]\times$$

$$\begin{Bmatrix}\theta^2+\zeta^2-2\xi^2\\ \xi^2+\zeta^2-2\theta^2+\frac{{D_x}^2+{\tilde{D}_x}^2}{\beta_x^2}\gamma^2(\zeta^2+\theta^2-2\xi^2)+\frac{6\tilde{D}_x}{\beta_x}\gamma\xi\theta\\ \xi^2+\theta^2-2\zeta^2\end{Bmatrix}\frac{d\xi d\theta d\zeta}{(\xi^2+\theta^2+\zeta^2)^{3/2}}$$

where for a bunched beam : $\mathcal{A}=\frac{Z^4}{A^2}\frac{c r_0^2 N_b}{32\pi^2\beta^3\gamma^4\varepsilon_x\varepsilon_z\varepsilon_s}\quad \varepsilon_s=\sigma_\eta\sigma_s$

To solve the 3 remaining integrals over ξ, θ, ζ further approximations would be required !

Strategy step 7: IBS rise times

Original Piwinski model

In his initial model (1974), besides cancelling $\beta'_{x,z}$, $\beta'_{x,z}$, D_z Piwinski makes use of the *smoothed focusing* approximation to derive IBS formulae for approximate *mean lattice parameters*. After hard-working manipulations the IBS growth rates write (for bunched beams, $\langle\cdots\rangle$ denotes a mean value) :

$$\langle\beta_x\rangle = \frac{\langle R\rangle}{Q_x} \quad \langle D_x\rangle \approx \frac{\langle\beta_x\rangle}{Q_x} = \frac{R}{Q_x^2} \quad \frac{1}{Q_x^2} \approx \alpha_p \equiv \frac{1}{\gamma_t^2}$$

$\langle R\rangle$ is the mean ring radius, Q_x the *betatron tune*, γ_t the *transition energy*, α_p the *momentum compaction factor*.

$$\sigma_{x_\beta,z_\beta} = \left(\beta_{x,z}\varepsilon_{x,z}\right)^{1/2} \quad \sigma_{x'_\beta,z'_\beta} = \left(\varepsilon_{x,z}/\beta_{x,z}\right)^{1/2} \quad \sigma_h^{-2} = \sigma_\eta^{-2} + D_x^2\sigma_{x_\beta}^{-2} \quad \alpha_p \equiv \frac{1}{2\pi\langle R\rangle}\oint\frac{D_x(s)}{R(s)}ds$$

$$\begin{pmatrix} \dfrac{1}{\tau_\eta} \\ \dfrac{1}{\tau_x} \\ \dfrac{1}{\tau_z} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2\sigma_\eta^2}\dfrac{d\sigma_\eta^2}{dt} \\ \dfrac{1}{2\sigma_{x_\beta}^2}\dfrac{d\sigma_{x_\beta}^2}{dt} \\ \dfrac{1}{2\sigma_{z_\beta}^2}\dfrac{d\sigma_{z_\beta}^2}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{\mathcal{A}}{2}\dfrac{\sigma_h^2}{\sigma_\eta^2} f(a,b,c) \\ \dfrac{\mathcal{A}}{2}\left[f\left(\dfrac{1}{a},\dfrac{b}{a},\dfrac{c}{a}\right) + \dfrac{D_x^2\sigma_\eta^2}{\sigma_{x_\beta}^2} f(a,b,c)\right] \\ \dfrac{\mathcal{A}}{2} f\left(\dfrac{1}{b},\dfrac{a}{b},\dfrac{c}{b}\right) \end{pmatrix}$$

$$f(a,b,c) = 8\pi \times \int_0^1 \left(\ln\left[\frac{c^2}{2}\left\{\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right\}\right] - C\right)\frac{1-3x^2}{\sqrt{pq}}dx$$

$$p = a^2 + (1-a^2)x^2 \quad q = b^2 + (1-b^2)x^2$$

$$a = \frac{\sigma_h}{\gamma\sigma_{x'_\beta}} \quad b = \frac{\sigma_h}{\gamma\sigma_{z'_\beta}} \quad c = \beta\sigma_h\sqrt{\frac{2Ab_{\max}}{Z^2 r_0}}$$

$$c = \sigma_h \exp\left[C_{\log}/2\right]$$

$C = 0.577$ Euler's constant, $f(a,b,c)$ is integrated numerically.

Original Piwinski model

Invariants

- *Above transition energy* the particle property is often identify by a *negative mass* comportment.
- Association with a *gas* in a *closed box* is not valid and the overall *oscillation energy* can *increase*.
- The *beam behaviour* can be described via a global *invariant* which can be cast into a form close to the sum of the mean invariant change $\langle \varepsilon_{x,z} \rangle$ & $\langle H \rangle$ over the collisions for all particles, i.e. multiplying $\langle H \rangle / \gamma^2$ by $1 - \gamma^2 D_x^2 / \beta_x^2$) in the summation yields a non invariant quantity because D_x / β_x varies.
- *Smoothed focusing* approx. for the *tune*, *momentum compaction factor* and *transition energy* yields :

$$\langle H \rangle \left(\frac{1}{\gamma^2} - \frac{D_x^2}{\beta_x^2} \right) + \frac{\langle \varepsilon_x \rangle}{\beta_x} + \frac{\langle \varepsilon_z \rangle}{\beta_z} \neq \text{constant}$$

$$\langle H \rangle \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \right) + \frac{\langle \varepsilon_x \rangle}{\beta_x} + \frac{\langle \varepsilon_z \rangle}{\beta_z} = \text{constant}$$

$$\eta_t = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$ (*slip factor*)

$$\frac{d}{d\bar{t}} \left[\langle H \rangle \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \right) + \frac{\langle \varepsilon_x \rangle}{\beta_x} + \frac{\langle \varepsilon_z \rangle}{\beta_z} \right] = 0$$

- *Below transition* ($\eta_t < 0$) the sum of the 3 (positive) invariants is limited and hence the 3 oscillation energies, so the "emittances" are redistributed in all 3 phase planes holding the whole phase space invariant, the distribution P is stable : equilibrium exist (like gaz molecules in a closed box).
- *Above transition* ($\eta_t > 0$) the overall oscillation energy can increase as $\eta_t > 0$: no possible equilibrium.

Bjorken-Mtingwa model

- ***Phase space distribution & 6D phase space volume***
- ***Emittances ($\varepsilon_x, \varepsilon_z, \varepsilon_s$) & Angles - momentum spread***

Gaussian 6-dim phase-space *distributions* for the beam are considered which can be expressed into the form :

$$P(\boldsymbol{r},\boldsymbol{p}) = \frac{N_b}{\Gamma} e^{-S(\boldsymbol{r},\boldsymbol{p})} \quad \Gamma = \int d\boldsymbol{r} d\boldsymbol{p}\, e^{-S(\boldsymbol{r},\boldsymbol{p})} \quad S(\boldsymbol{r},\boldsymbol{p}) = S^{(\mathrm{h})} + S^{(\mathrm{v})} + S^{(\mathrm{l})}$$

Γ is the 6-dim *phase-space volume*, N_b the *particle* number *per bunch*, $\boldsymbol{r}=(x,z,\eta)$ & $\boldsymbol{p}=(p_x,p_z,p_s)$ the positions & momenta of the particles within the bunch, $\sigma_{x_\beta}, \sigma_{z_\beta}, \sigma_s, \sigma_\eta$ the rms bunch *width*, *height*, *length*, *momentum spread*, ε_x, ε_z, ε_s the rms transverse & longitudinal emittances (only bunched beam are discussed). The transverse & longitudinal phase space components write ($\bar{s}$ being the mean longitudinal particle position i.e. that of the synchronous particle):

$$S^{(\mathrm{h})} = \frac{\beta_X}{2\sigma_{x_\beta}^2}\left(\gamma_x x_\beta^2 + 2\alpha_x x_\beta x'_\beta + \beta_x x'^2_\beta\right) \quad S^{(\mathrm{v})} = \frac{\beta_X}{2\sigma_{z_\beta}^2}\left(\gamma_z z_\beta^2 + 2\alpha_z z_\beta z'_\beta + \beta_z z'^2_\beta\right) \quad S^{(\mathrm{l})} = \frac{\eta^2}{2\sigma_\eta^2} + \frac{(s-\bar{s})^2}{2\sigma_s^2}$$

$$\varepsilon_x = \frac{\sigma_{x_\beta}^2}{\beta_x} \quad \varepsilon_z = \frac{\sigma_{z_\beta}^2}{\beta_z} \quad \varepsilon_s = \sigma_\eta \sigma_s$$

$$x' = \frac{\Delta p_x}{\bar{p}} \quad z' = \frac{\Delta p_z}{\bar{p}} \quad \eta = \frac{\Delta p}{\bar{p}} \;\&\; \sigma_\eta = \frac{\sigma_p}{\bar{p}}$$

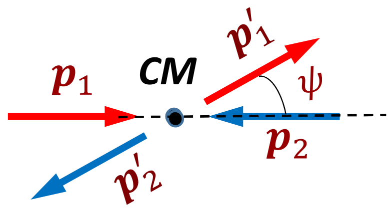
$$x_\beta = x - D_x\eta \quad x'_\beta = x' - D'_x\eta$$
$$z_\beta = z - D_z\eta \quad z'_\beta = z' - D'_z\eta$$

Bjorken-Mtingwa model

- ***Golden Rule for two-body scattering in the CM frame***

Bjorken & Mtingwa approach of IBS theory was via the *S-matrix* formalism related to quantum electrodynamics. (*QED*). Hence they use the Fermi scattering "*Golden Rule*" to compute physical parameters for interactions between particles & fields, by evaluating the relevant Feynman diagram with the *Feynman rules*.

In a 2−body scattering process particles 1 & 2 with 4-momenta $p_{1,2} \overset{\text{def}}{=} p^{\mu}_{1,2}$ (i.e. energy-momentum 4-vector $p^{\mu}_{1,2}$) interact each other to give after collision two 4-momenta $p'_{1,2} \overset{\text{def}}{=} p'^{\mu}_{1,2}$ ($\boldsymbol{p}_1 + \boldsymbol{p}_2 \rightarrow \boldsymbol{p}'_1 + \boldsymbol{p}'_2$) whose interaction rate is :



$$\frac{d\mathcal{P}}{dt} = \frac{1}{2}\int d\boldsymbol{r}\,\frac{d\boldsymbol{p}_1}{\gamma_1}\frac{d\boldsymbol{p}_2}{\gamma_2}P(\boldsymbol{r},\boldsymbol{p}_1)P(\boldsymbol{r},\boldsymbol{p}_2)|\mathcal{M}|^2\frac{dp'_1}{\gamma'_1}\frac{dp'_2}{\gamma'_2}\frac{\delta^4(p'_1+p'_2-p_1-p_2)}{(2\pi)^2}$$

$\mathcal{M}$ is the Coulomb scattering *amplitude* containing the physics of the process (*S-matrix*). Here r^{μ} denotes a contravariant vector which with the covariant vector r_{μ} make a product $r^{\mu} r_{\mu} = g_{\mu\nu} r^{\mu} r^{\nu}$ invariant thru Lorentz transform ($g_{11}=1, g_{22}=g_{33}=g_{44}=-1, g_{\mu\neq\nu}=0$) : **metric = (4−momentum)^2 = (energy)^2 − (3−momentum)^2**

$$r \overset{\text{def}}{=} r^{\mu} \equiv (ct, \boldsymbol{r}) = (ct, x, z, s) \quad p \overset{\text{def}}{=} p^{\mu} \equiv \left(\frac{E}{c}, \boldsymbol{p}\right) = \left(\frac{E}{c}, p_x, p_z, p_s\right)$$

$$p_1 \cdot p_2 \overset{\text{def}}{=} p_1^{\mu} p_{2\mu} = \frac{E_1 E_2}{c^2} - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 \quad r \cdot p \overset{\text{def}}{=} r^{\mu} p_{\mu} = ct\frac{E}{c} - \boldsymbol{r} \cdot \boldsymbol{p}$$

Bjorken-Mtingwa model

- ***Golden Rule for two-body scattering in the CM frame***

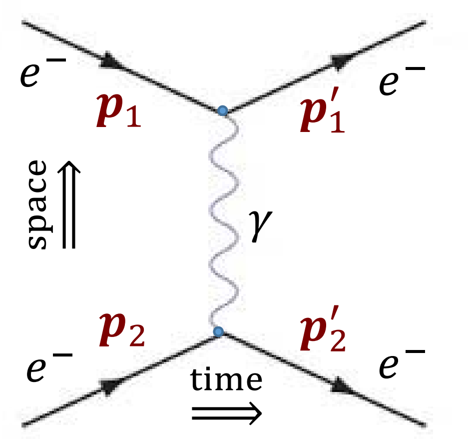
- The aim here is to determine the *amplitude* $|\mathcal{M}|^2$ for a Coulomb scattering between 2 electrons of mass m via the exchange of a virtual photon of 4-momentum q, by means of the Feynman rules.
- To simplify the computations with respect to a "real-life" *QED* 4-body process $e^- + e^- \rightarrow e^- + e^-$ (electrons spin 1, massless photon spin 0) spin 0 for both particles and boson is assumed (toy model).
  - The coupling constant g_e in *QED* which specify the interaction strength between electrons and photons is related to the *fine structure constant* α as : $g_e = \sqrt{4\pi\alpha}$.
  - A boson *propagator* $f(q)$ is associated with the wavy line in the *Feynman diagram* and represents the transfer of momentum from one e^- to the other e^- via the virtual photon γ (see e.g. Griffith's book for details) :

$$f(q) = \frac{i}{q_\mu^2} \quad i = \sqrt{-1}$$

Hence $\mathcal{M}$ for *elastic collisions* (momentum & kinetic energy are *conserved*) writes as :

$$\mathcal{M} \equiv g_e f(q) = \frac{4i\pi\alpha}{q_\mu^2} \quad q_\mu = (p_1' - p_1)_\mu \quad q_\mu^2 \equiv q^\mu q_\mu = -(p_1' - p_1)^2$$

In *SI* units $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137}$ and in *HL* Heaviside-Lorentz units ($\epsilon_0 = \hbar = c = 1$) $\alpha = \frac{e^2}{4\pi}$



Bjorken-Mtingwa model

- ***Golden Rule for two-body scattering in the CM frame***

To see the link of $\mathcal{M}$ with a collisional process rewrite explicitly q_μ^2 :

$$q_\mu^2 = -(p_1'\text{-}p_1)^2 = -\left(p_1'^2 + p_1^2 \text{-} 2p_1 \cdot p_1'\right) = -\left[\frac{E_1^2 + E_1'^2}{c^2} - |p_1^2| - |p_1'^2| - 2\left(\frac{E_1 E_1'}{c^2} - |p_1||p_1'|\cos\psi\right)\right] = 2|p_1^2|(1 - \cos\psi) = |p_1^2|\sin^2[\psi/2]$$

because of momentum & energy conservation. Finally the amplitude $\mathcal{M}$ takes the form ($\alpha = e^2/4\pi$ in *HL* units) :

$$\mathcal{M} = \frac{4i\pi\alpha}{\boldsymbol{p}_1^2 \sin^2[\psi/2]} \quad |\mathcal{M}|^2 = \left(\frac{e^2}{\boldsymbol{p}_1^2 \sin^2[\psi/2]}\right)^2 \equiv \frac{d\sigma(\boldsymbol{p}_1, \psi)}{d\Omega}$$

$|\mathcal{M}|^2$ is thus the differential cross section of the two electron collisional process. Before ending $d\mathcal{P}/dt$ is rewritten introducing the 6-dim beam distribution $P(\boldsymbol{r}, \boldsymbol{p})$ into it, yielding :

$$\frac{d\mathcal{P}}{dt} = \frac{N_b}{2\Gamma^2} \int d\boldsymbol{r} \frac{d\boldsymbol{p}_1}{\gamma_1} \frac{d\boldsymbol{p}_2}{\gamma_2} e^{-S(\boldsymbol{r},\boldsymbol{p}_1) - S(\boldsymbol{r},\boldsymbol{p}_2)} \frac{d\sigma}{d\Omega} \frac{d\boldsymbol{p}_1'}{\gamma_1'} \frac{d\boldsymbol{p}_2'}{\gamma_2'} \frac{\delta^4(p_1' + p_2' - p_1 - p_2)}{(2\pi)^2}$$

Calculations are far to be finished, moreover they are not easy (cf Bjorken-Mtingwa, 1983). So the final and well known formulae for the 3 growth-rates are just given below without proof. On the other hand their use is easy.

Bjorken-Mtingwa model

- ***Final steps of IBS theory providing quantifiable growth rates***

The *IBS growth rates* τ_u^{-1} in the 3 directions $u = x$ (horiz), z (vert) and s (long) are for *bunched* beams :

$$\frac{1}{\tau_u} = \frac{d\ln\varepsilon_u}{dt} = \frac{N_b c r_0^2 C_{\log}}{\gamma 8\pi\beta^3\gamma^3\varepsilon_x\varepsilon_z\sigma_s\sigma_\eta}\frac{Z^4}{A^2}\left\langle\int_0^\infty \frac{d\lambda}{\sqrt{\det[L+\lambda I]}}\{\mathrm{Tr}[L_u]\,\mathrm{Tr}[(L+\lambda I)^{-1}] - 3\,\mathrm{Tr}[L_u(L+\lambda I)^{-1}]\}\right\rangle$$

in which the bracket $\langle\cdots\rangle$ denotes an average over the lattice period, with $L = L_x + L_z + L_s$ and :

$$L_x = \frac{\beta_x}{\varepsilon_x}\begin{pmatrix} 1 & -\gamma\phi_x & 0 \\ -\gamma\phi_x & \gamma^2 H_x/\beta_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad L_z = \frac{\beta_z}{\varepsilon_z}\begin{pmatrix} 1 & 0 & 0 \\ -\gamma\phi_z & \gamma^2 H_z/\beta_z & 0 \\ 0 & -\gamma\phi_z & 1 \end{pmatrix} \quad L_s = \frac{\gamma^2}{\sigma_\eta^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

After the bracket expansion the growth rates are simplified via relevant approximations and tedious computations into the form ($u = x, z, s$):

$$\phi_{x,z} = \frac{D_{x,z}\alpha_{x,z} + D'_{x,z}\beta_{x,z}}{\beta_{x,z}} \quad H_{x,z} = \frac{D_{x,z}^2 + \beta_{x,z}^2\phi_{x,z}^2}{\beta_{x,z}}$$

$$\frac{1}{\tau_u} = \frac{N_b c r_0^2 C_{\log}}{\gamma 8\pi\beta^3\gamma^3\varepsilon_x\varepsilon_y\sigma_s\sigma_\eta}\frac{Z^4}{A^2}\left\langle\Delta_u\int_0^\infty d\lambda\frac{(a_u\lambda + b_u)\sqrt{\lambda}}{(\lambda^3 + a\lambda^2 + b\lambda + c)^{3/2}}\right\rangle$$

$$\Delta_x = \frac{\gamma^2 H_x}{\beta_x} \quad \Delta_z = \frac{\gamma^2 H_z}{\beta_z} \quad \Delta_s = \frac{\gamma^2}{\sigma_\eta^2}$$

The 9 coefficients $a, b, c, a_x, b_x, a_z, b_z, a_s, b_s$ (not reproduced here) depend on the lattice optics parameters.

Bjorken-Mtingwa model

- ***Final steps of IBS theory providing quantifiable growth rates***

c is the speed of light, β, γ the Lorentz factors & $\alpha_u, \beta_u, D_u, D'_u$ the optics parameters. The longitudinal emittance ε_s is defined by the product $\varepsilon_s = \sigma_s \sigma_\eta$ [m] or by the momentum p as $\varepsilon_s = \pi p \sigma_s \sigma_\eta \beta^{-1} c^{-1}$ [eVs] (*bunched* beam).

- $C_{\log}$ is defined in terms of the impact parameter $r_{\min}$ (the larger of the *classical* distance of closest approach $r_{\min}^C$ or the *quantum* mechanical diffraction limit from the nuclear radius $r_{\min}^{QM}$) and $r_{\max}$ (the smaller of the mean rms *beam size* $\sigma_x = (\langle\beta_x\rangle\varepsilon_x)^{1/2}$ or λ_D the Debye length). Here $C_{\log}$=20 $10 \lesssim C_{\log} \lesssim 20$).
- The Bjorken-Mtingwa IBS model assumes a 6-dim *Gaussian* beam *density*.

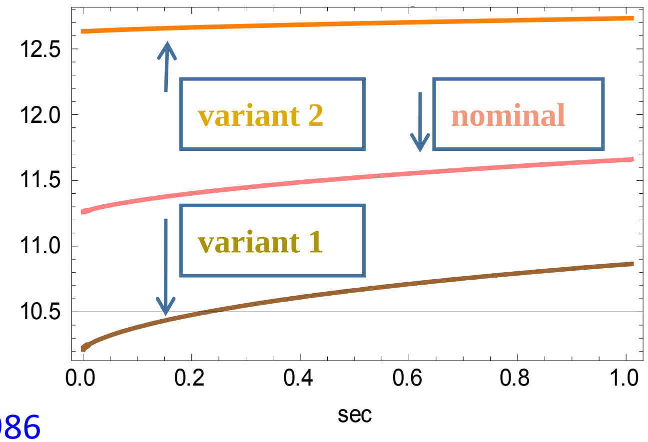
$$C_{\log} = \min\left[\frac{\ln r_{\max}}{\ln r_{\min}}\right] \quad r_{\max} = \min[\sigma_x, \lambda_D] \quad r_{\min} = \max[r_{\min}^C, r_{\min}^{QM}]$$

$$\lambda_D = \frac{7.434}{Z}\sqrt{\frac{2E_\perp}{\rho}} \quad \rho = \frac{N_b \times 10^{-6}}{\sqrt{64\pi^3 \langle\beta_x\rangle \varepsilon_x \langle\beta_y\rangle \varepsilon_y \sigma_z^2}} \quad E_\perp = \frac{(\gamma^2-1)E_0}{2}\frac{\varepsilon_x}{\langle\beta_x\rangle}$$

$$r_{\min}^C = \frac{1.44 \times 10^{-9} Z^2}{2E_\perp} \quad r_{\min}^{QM} = \frac{1.973 \times 10^{-13}}{\sqrt{8E_\perp E_0}}$$

M. Zisman, S. Chattopadhyay, J. Bisognano, "ZAP user's manual", 1986

Coulomb logarithm (ELENA ring 100keV)



INTRABEAM SCATTERING

- Part 3 : Applications
  - IBS & LHC (7 TeV)
  - IBS & ELENA (100 keV)
  - Epilogue

IBS Calculations

Continuation… from Part 2

The IBS growth rates in one turn (or one time step)

$$\frac{1}{T_i} = \langle f_i \rangle$$

Complicate integrals averaged around the ring

Horizontal, vertical and longitudinal **equilibrium states** and **damping times** due to SR damping

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}(\varepsilon_x - \varepsilon_{x0}) + \frac{2\varepsilon_x}{T_x(\varepsilon_x, \varepsilon_y, \sigma_p)}$$

$$\frac{d\varepsilon_y}{dt} = -\frac{2}{\tau_y}(\varepsilon_y - \varepsilon_{y0}) + \frac{2\varepsilon_y}{T_y(\varepsilon_x, \varepsilon_y, \sigma_p)}$$

$$\frac{d\sigma_p}{dt} = -\frac{1}{\tau_p}(\sigma_p - \sigma_{p0}) + \frac{\sigma_p}{T_p(\varepsilon_x, \varepsilon_y, \sigma_p)}$$

If = 0 → **Steady State emittances**

If ≠0

- Steady state exists if we are below transition or in the presence of SR damping
- dt should be much smaller than the IBS growth times
- Good scanning of optics is important in order not to skip large IBS kick points

Courtesy F. Antoniou, Y. Papaphilippou, CERN

IBS & LHC (7 TeV)

LHC and SLHC beam parameter with improved variants

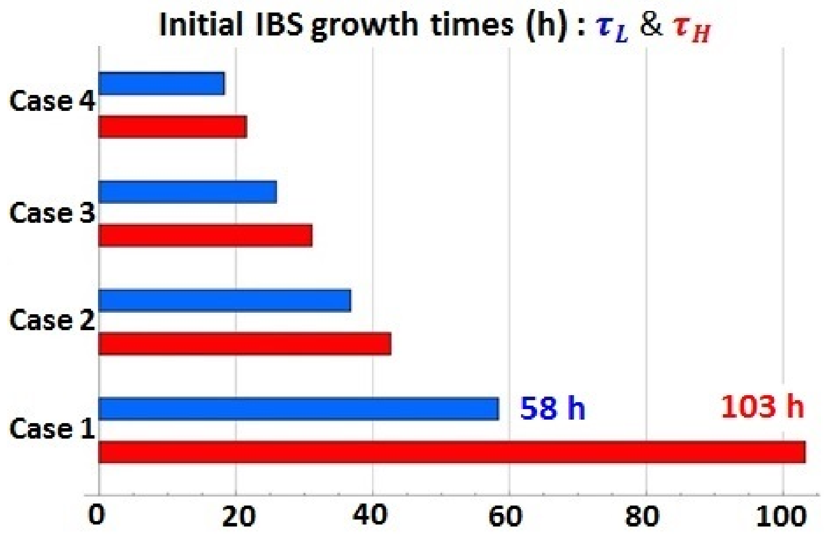
$\mathcal{L} = \frac{f_{rev} n_b N_b \gamma}{2 r_p \beta^*} \lvert \Delta Q_{bb} \rvert$	LHC Luminosity with nominal beam intensity		SLHC Luminosity	
	Case 1 Initial IR triplet	**Case 2** IR phase 1 triplet : $\beta^* = 0.30$ m reduced emittance	**Case 3** Ultimate N_b : $\beta^* = 0.25$ m reduced emittance	**Case 4** >Ultimate N_b : $\beta^* = 0.15$ m reduced emittance
N_b (10^{11})	1.15	1.15	1.70	2.36
$\varepsilon_{H,V}^n = \varepsilon^n = \gamma\varepsilon$ rms μm	3.75	2.54	2.65	2.60
β^* m	0.55	0.30	0.25	0.15
$\sigma_{H,V}^* = \sigma^*$ μm	16.58	10.11	9.40	7.21
σ_L mm	75.50	75.50	75.50	75.50
$\sigma_{\Delta p/p}$ (10^{-4})	1.13	1.13	1.13	1.13
ε_L rms eVs	0.62	0.62	0.62	0.62
Crossing angle θ μrad	285	337	355	454
ΔQ_{bb} head-on**	1.00	1.09	1.43	1.37
$\mathcal{L}$uminosity (10^{34}) $cm^{-2}s^{-1}$	1.00	2.00	4.65	10.29

** ΔQ_{bb} normalized to the value of the nominal beam

- 1st case : nominal beam and LHC parameters at top energy give the nominal luminosity of $10^{34} cm^{-2}s^{-1}$
- 2nd case: new optics will rise the crossing angle to 337 μrad and the luminosity to $2 \times 10^{34} cm^{-2}s^{-1}$
- 3rd case : will raise the head-on beam-beam tune shift to 1.43 and the luminosity to $4.65 \times 10^{34} cm^{-2}s^{-1}$
- 4th case : with an intensity of 2.36×10^{11} protons/bunch a top luminosity of $\sim 10^{35} cm^{-2}s^{-1}$ can be got

IBS & LHC (7 TeV)

IBS effects in the SLHC

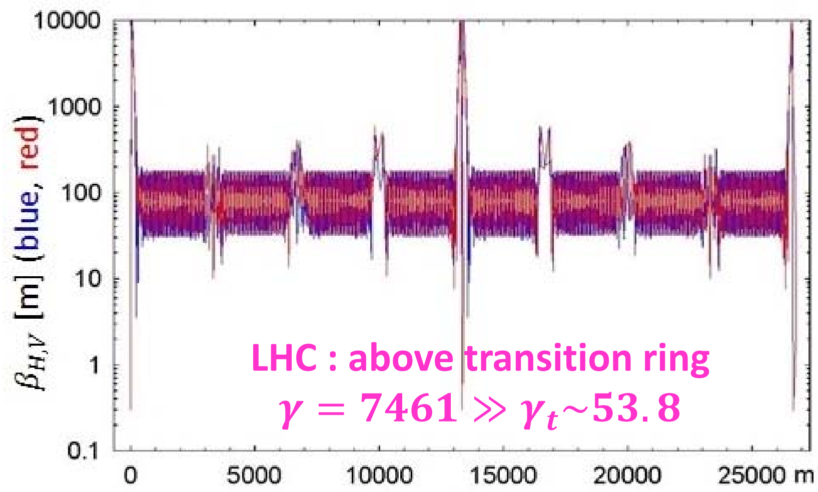


IBS (**Bjorken-Mtingwa** model) and *synchrotron radiation* calculation to estimate the *LHC* & *SLHC* beam emittances evolution during *7 TeV physics coasts* are done for the *4 nominal* & *reduced emittance beam* cases

- IBS growth rates : $$\frac{1}{\tau_{L,H,V}} = \frac{N_b c r_0^2 C_{\log}}{8\pi\beta^3\gamma^4\varepsilon_H\varepsilon_V\sigma_L\sigma_{\Delta p/p}}\langle H_{L,H,V}\rangle$$
- Longitudinal emittance : $\varepsilon_L = \pi p \sigma_L \sigma_{\Delta p/p}(\beta c)^{-1}$

		$\Delta\varepsilon_L/\varepsilon_L$	$\Delta\varepsilon_H/\varepsilon_H$	$\Delta\varepsilon_V/\varepsilon_V$
1st case	Initial IR triplet	16%	9%	-0.0001%
2nd case	IR phase 1 triplet ($\beta^* = 0.30$ m) reduced emittance	24%	21%	-0.001%
3rd case	Ultimate N_b ($\beta^* = 0.25$ m) reduced emittance	32%	27%	-0.001%
4th case	>Ultimate N_b ($\beta^* = 0.15$ m) reduced emittance	44%	37%	-0.001%

IBS emittance growth after a 10 hours beam coast



SLHC betatron functions for $\beta^* = 0.30$ m (IP1 and IP5)

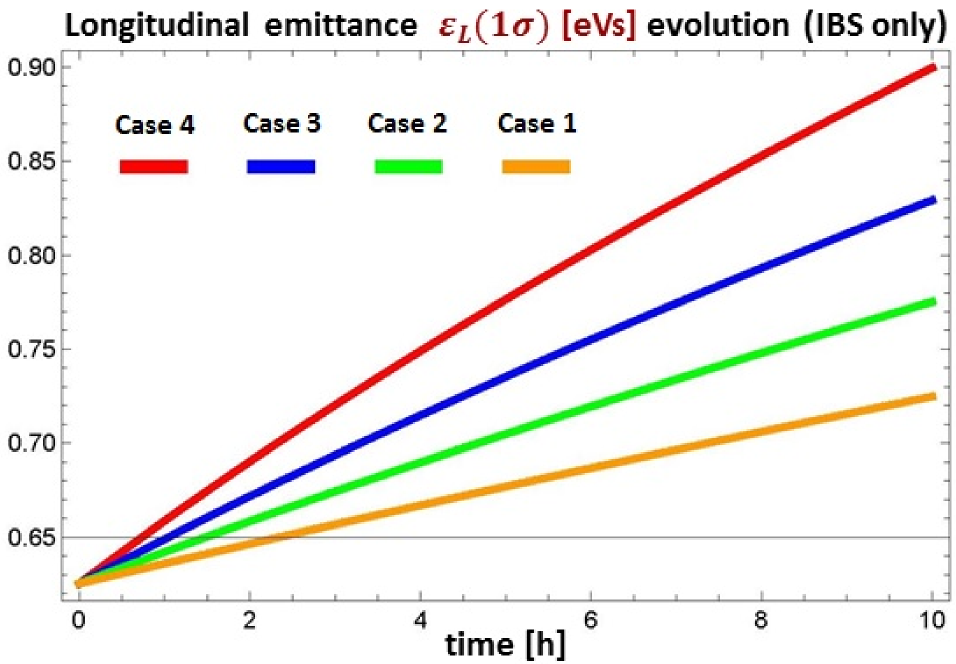
IBS & LHC (7 TeV)

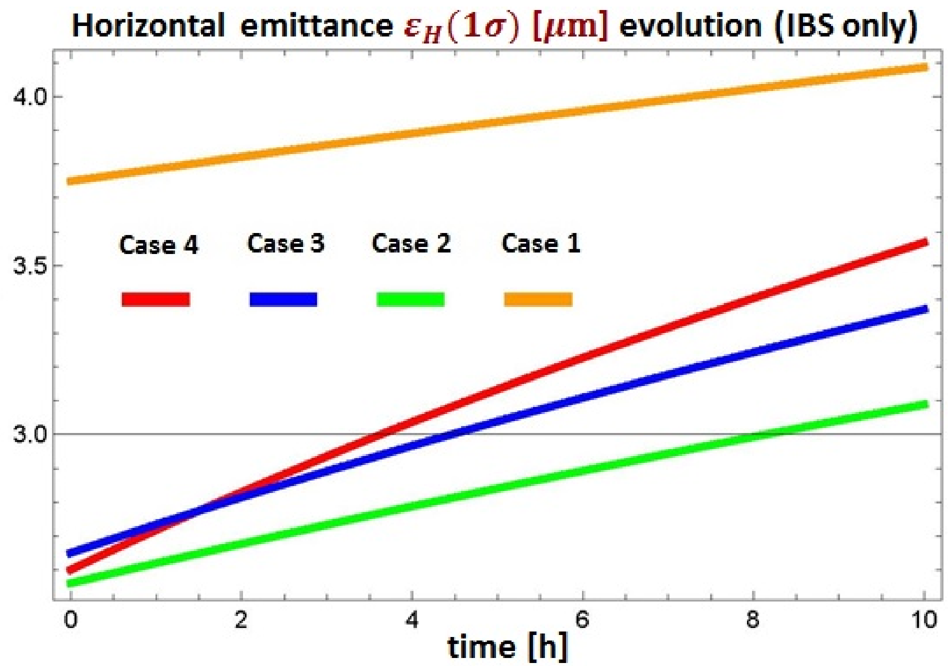
IBS effects in the SLHC

- A constant beam intensity for the duration of the beam storage period is assumed in the computations.
- The next 2 figures show the evolution of the *longitudinal* & *horizontal emittances* over a *10 hours beam coast*.
- IBS growth-rates $\tau_{L,H,V}^{-1}$ were calculated iteratively by step Δt of 5 minutes updating the emittances at each iteration i:

$$\varepsilon_{L,H,V}(i+1) = \varepsilon_{L,H,V}(i)e^{\Delta t/\tau_{L,H,V}(i)}$$

$$i = i + 1 \quad \tau_{L,H,V}^{-1}(i+1) = d \ln \varepsilon_{L,H,V}(i)/dt$$

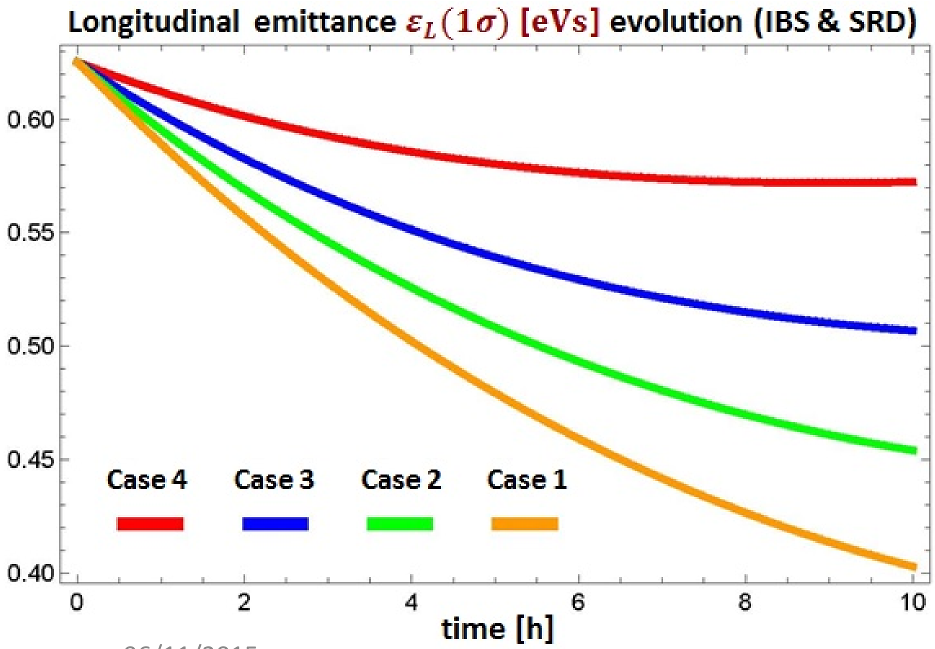


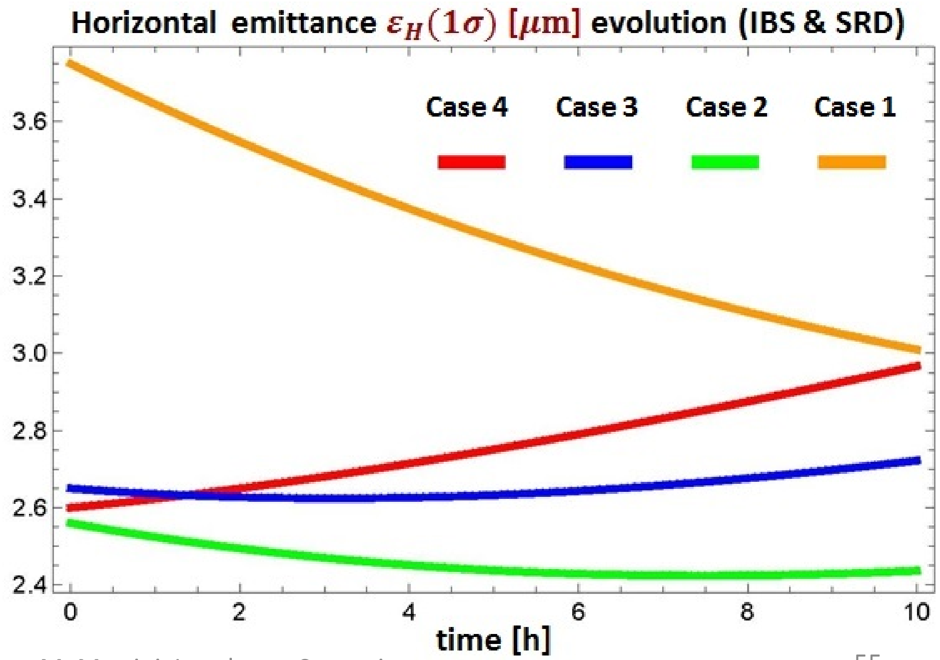


IBS & LHC (7 TeV)

IBS & synchrotron radiation damping effects in the SLHC

- The synchrotron radiation turns into a visible effect for the LHC/SLHC proton beams at 7 TeV collision energy. *Emittances shrink* with *damping* times of : **12.9 h** in the *longitudinal* and **26.0** h in the 2 *transverse* planes.
- Synchrotron radiation damping (*SRD*) is modelled substituting in the previous formula $\tau_{L,H,V}(i)$ by $\left(\tau_{L,H,V}^{-1}(i) - \tau_{\mathrm{srd}_{L,H,V}}^{-1}\right)^{-1}$
- The next 3 figures show the evolution of the *longitudinal* & *transverse emittances* over a *10 hours beam coast*.
- *SRD* dominates the *IBS growth* in the *longitudinal* & *vertical* planes for the 4 cases, in *horizontal* the emittance damps over the all coast only for *case 1* while, for *cases 2-4* it grows at some point in time during the coast.

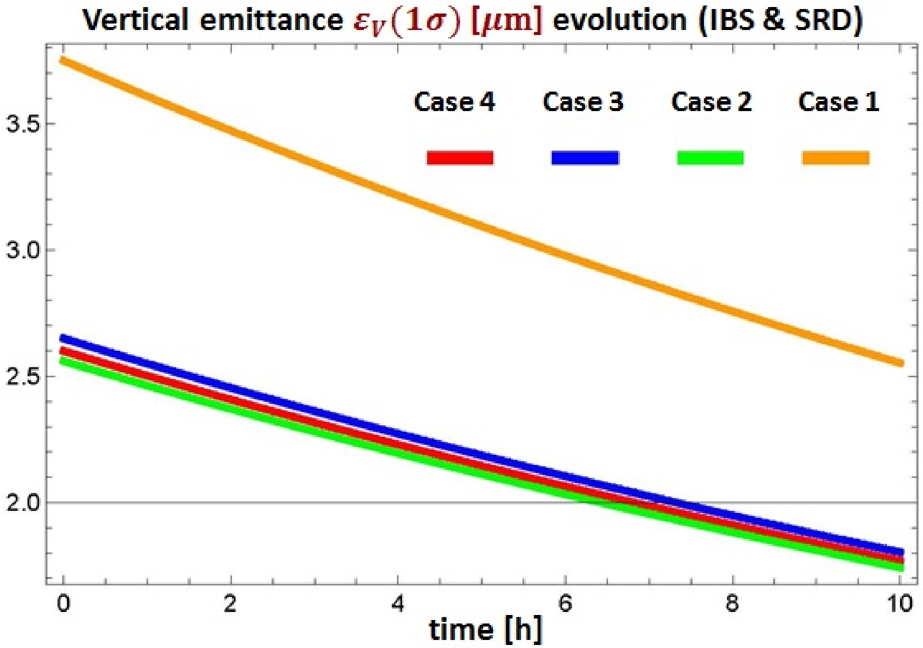




IBS & LHC (7 TeV)

IBS & synchrotron radiation damping effects in the SLHC

Table : *Emittance changes* after a 10 hours beam coast resulting from the effects of IBS and synchrotron radiation damping



		$\Delta\varepsilon_L/\varepsilon_L$	$\Delta\varepsilon_H/\varepsilon_H$	$\Delta\varepsilon_V/\varepsilon_V$
1st case	Initial IR triplet	-36%	-20%	-32%
2nd case	IR phase 1 triplet ($\beta^* = 0.30$ m) reduced emittance	-27%	-5%	-32%
3rd case	Ultimate N_b ($\beta^* = 0.25$ m) reduced emittance	-19%	3%	-32%
4th case	>Ultimate N_b ($\beta^* = 0.15$ m) reduced emittance	-8%	14%	-32%

IBS emittance changes after a 10 hours beam coast

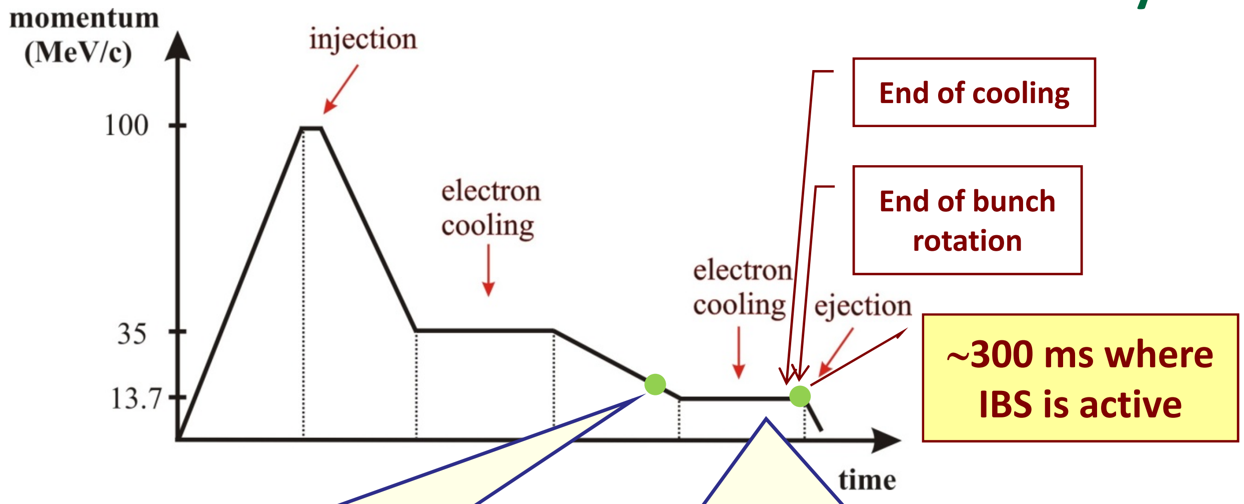
Conclusion

- **Longitudinal** & **vertical** : cases 1-4: *emittances* of all the *luminosity* scenarios are kept within target specifications.
- **Horizontal** : *emittances* stay in requirements cases 1-2: (*nominal* 10^{34} & *first IR upgrade* 2×10^{34} cm^{-2}s^{-1} luminosities, case 3: $\sim$3% *blow-up* expected (*ultimate* intensity $N_b = 2.36 \times 10^{11}$) & case 4: $\sim$14% ($\sim 10^{35}$ cm^{-2}s^{-1} *peak* luminosity).

Globally for most scenarios the evolution of *emittances* during the 10 hours coast is kept inside the design values

IBS & ELENA (100 keV)

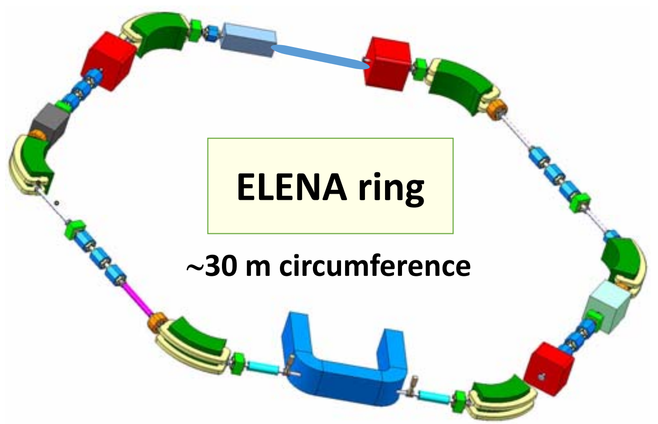
ELENA deceleration cycle



Momentum	~ 13.7 MeV/c
Beam intensity	2.5 10^7 (1 bunch)
Physical $\varepsilon_{H,V}$ (95%)	5 mm.mrad
$\Delta p/p$ (95%)	3 10^{-4}
Bunch length (95%)	10.1 m (circumf/3)

Momentum (energy)	13.7 MeV/c (100 keV)
Bunch intensity	6.25 10^6 (4 bunches)
Physical $\varepsilon_{H,V}$ (95%)	4 mm.mrad
$\Delta p/p$ (95%)	3 10^{-4}
Bunch length (95%)	1.3 m

ELENA (Extra Low Energy Antiproton) is a compact ring for *cooling* and more *deceleration* of **5.3 MeV** *antiprotons* sent by the Antiproton Decelerator to give dense beams at **100 keV** energies



ELENA : below transition ring
$\gamma = 1.0001 < \gamma_t \sim 1.9$

- 1st plateau : 4 bunches injection at 100 MeV/c from AD followed by beam cooling.
- 2nd plateau : Deceleration down to 35 MeV/c and cooling again.
- 3rd plateau : Last deceleration down to 13.7 MeV/c, beam cooled down to emittances needed for ELENA experiments.

IBS & ELENA (100 keV)

Nominal beam parameter and variant study

Ejection momentum/energy	13.7MeV/c	100 keV
Injected/ejected beam intensity	3 10^7	2.5 10^7
Number of extracted bunches	4	
Extracted bunch intensity	6.25 10^6	

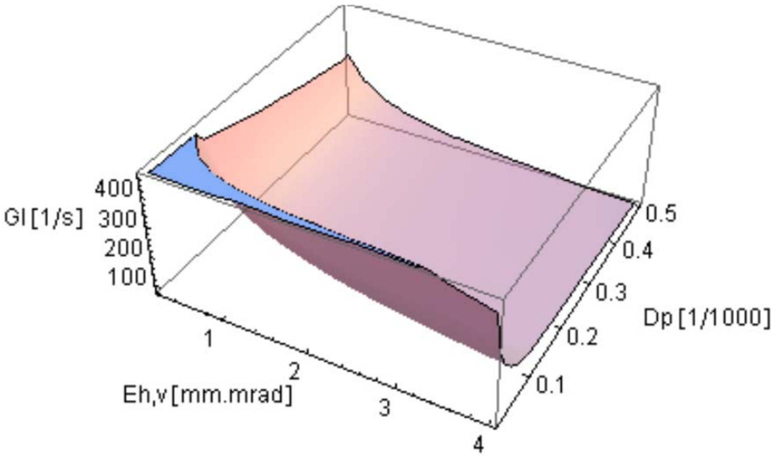
$\varepsilon_{H,V}^{rms} = 1\,\mu m, \sigma_{\Delta p/p} = 0.325\,m\,(75\,ns), \sigma_{\Delta p/p} = 0.075\,‰\,(7.510^{-5})$ $\varepsilon_{H,V}^{rms} = \pi p \sigma_{BL} \sigma_{BL} (\beta c)^{-1}$

	σ_{BL} m	$BL^{95\%}$ m	$\sigma_{\Delta p/p}$ ‰	$\Delta p/p^{95\%}$ ‰	$\varepsilon_L{}^{rms}$ eVs	$\varepsilon_L{}^{95\%}$ eVs	$\varepsilon_{H,V}^{rms}$ μm	$\varepsilon_{H,V}^{95\%}$ μm
Nominal beam	0.325	1.3	0.075	0.3	2.4 10^{-4}	9.6 10^{-4}	1.0	4.0
Variant 1	0.325	1.3	0.025	0.1	0.8 10^{-4}	3.2 10^{-4}	0.5	2.0
Variant 2	0.325	1.3	0.125	0.5	4.0 10^{-4}	16 10^{-4}	2.5	10.0

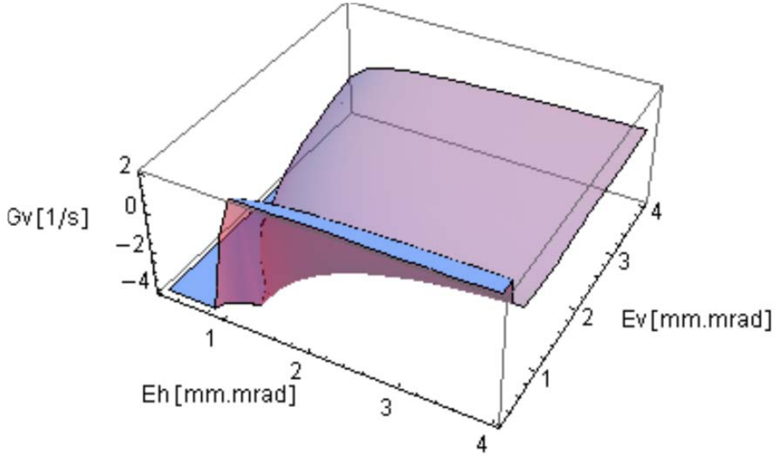
Initial nominal beam emittances with variants on the 100 keV plateau

IBS & ELENA (100 keV)

Longitudinal IBS



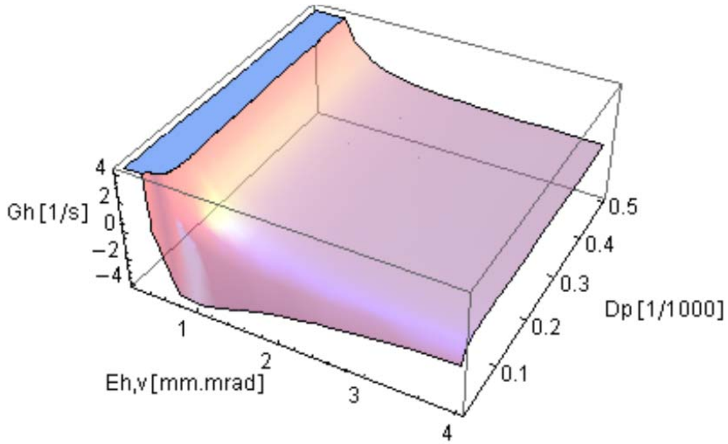
Growth-rate $1/\tau_L$ vs ($\sigma_{\Delta p/p}$, ε_H) for ε_H = ε_V & σ_{BL}= 0.325 m



Growth-rate $1/\tau_V$ vs (ε_H, ε_V) for $\sigma_{\Delta p/p}$ =0.075 ‰ & σ_{BL}= 0.325 m

Vertical IBS

Horizontal IBS



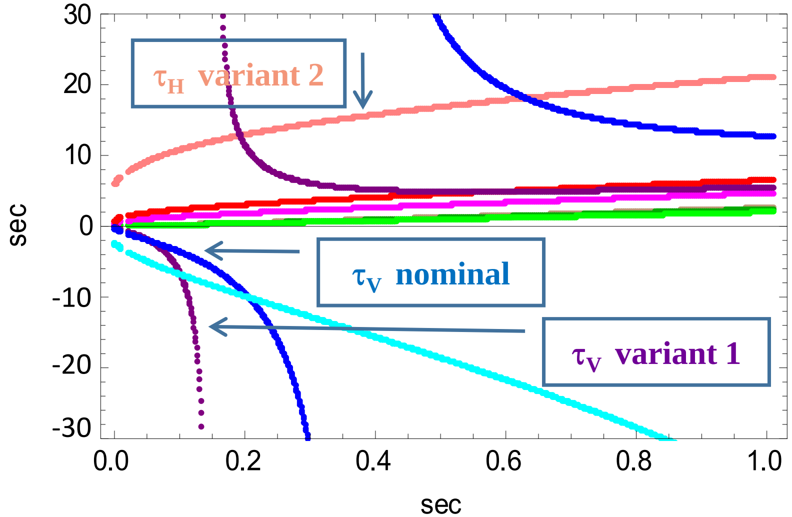
Growth-rate $1/\tau_H$ vs (ε_H, ε_V) for $\sigma_{\Delta p/p}$ =0.075 ‰ & σ_{BL}=0.325 m

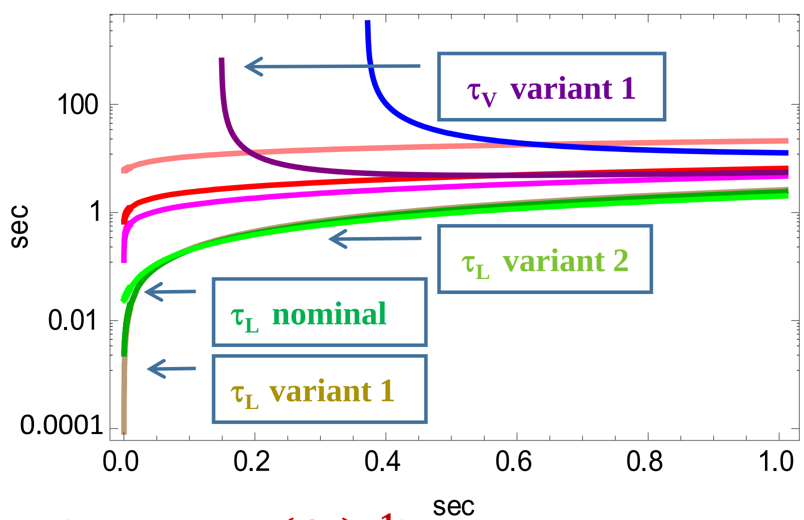
Bjorken-Mtingwa

IBS calculation model

IBS & ELENA (100 keV)

IBS growth times evolution



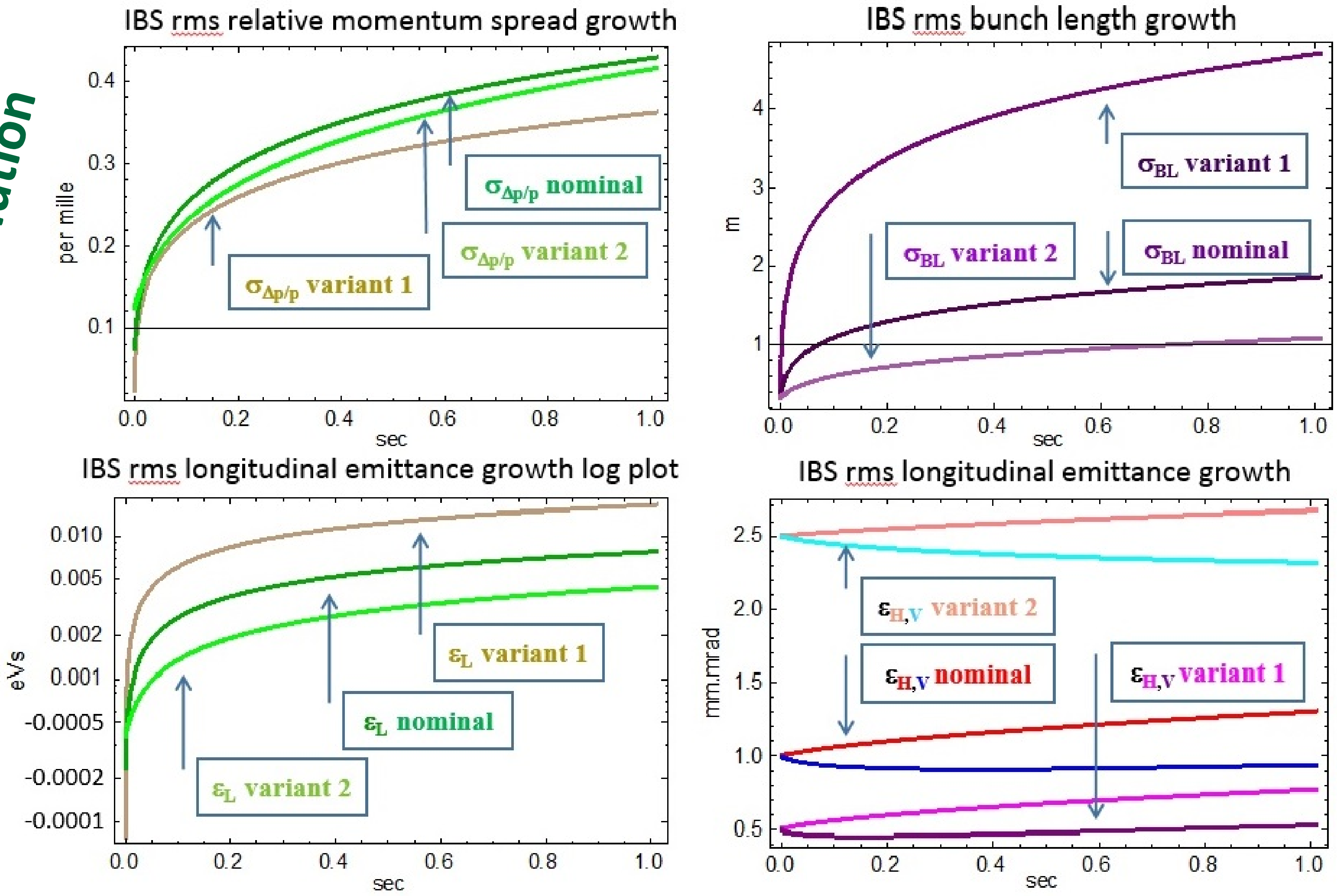


IBS growth-times $\tau_{L,H,V}$ evolution $(\varepsilon_L = \pi p \sigma_{BL} (\beta c)^{-1})$

ELENA initial rms beam emittances and IBS growth times at 100 keV ejection								
	σ_{BL} m	$\sigma_{\Delta p/p}$ ‰	ε_L eVs	ε_H µm	ε_V µm	τ_L ms	τ_H s	τ_V s
Nominal beam	0.325	0.075	$2.4\ 10^{-4}$	1.0	1.0	2.40	0.67	-0.27
Variant 1	0.325	0.025	$0.8\ 10^{-4}$	0.5	0.5	0.09	0.13	-0.04
Variant 2	0.325	0.125	$4.0\ 10^{-4}$	2.5	2.5	24.0	5.92	-2.44

IBS & ELENA (100 keV)

IBS beam parameter evolution



IBS & ELENA (100 keV)

Comments on variant performance & study extra variants

Assuming one or several bunches circulate for ~1 s on the 100 keV *plateau* : the above plots show that *none* of the *3 scenarios* are fully *satisfactory* because the *bunch length* and *momentum spread* will suffer too much *blow-up* due to *IBS* :

Nominal : bunch length and momentum spread growth after 1 s on the 100 keV plateau is **Big !**

σ_{BL}(1s) =1.9 m , $\sigma_{\Delta p/p}$(1s) =0.4 ‰ (95% bunch length=7.4 m instead of 1.3 m !)

Variant 1 : bunch length and momentum spread increases after 1 s on the 100 keV plateau is **Huge !**

σ_{BL}(1s) =4.7 m, $\sigma_{\Delta p/p}$(1s) =0.4 ‰ (95% bunch length=18.8 m !)

Variant 2 : bunch length and momentum spread increases after 1 s on the 100 keV plateau is still too **Large !**

σ_{BL}(1s) =1.1 m, $\sigma_{\Delta p/p}$(1s) =0.4 ‰ (95% bunch length=4.3 m !)

	σ_{BL} m	$BL^{95\%}$ m	$\sigma_{\Delta p/p}$ ‰	$\Delta p/p^{95\%}$ ‰	ε_L^{rms} eVs	$\varepsilon_L^{95\%}$ eVs	$\varepsilon_{H,V}^{rms}$ μm	$\varepsilon_{H,V}^{95\%}$ μm
variant 3	**0.325**	1.3	**0.250**	1	**8 10^{-4}**	32 10^{-4}	**1.0**	4.0
variant 4	**0.325**	1.3	**0.375**	1.5	**12 10^{-4}**	48 10^{-4}	**1.0**	4.0
variant 5	**0.325**	1.3	**0.500**	2	**16 10^{-4}**	60 10^{-4}	**1.0**	4.0

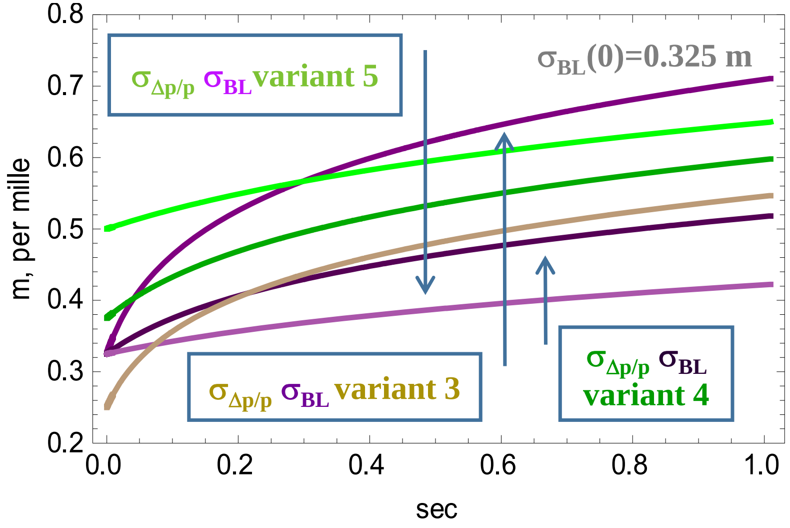
Three more variant scenarios with higher relative momentum spreads

IBS & ELENA (100 keV)

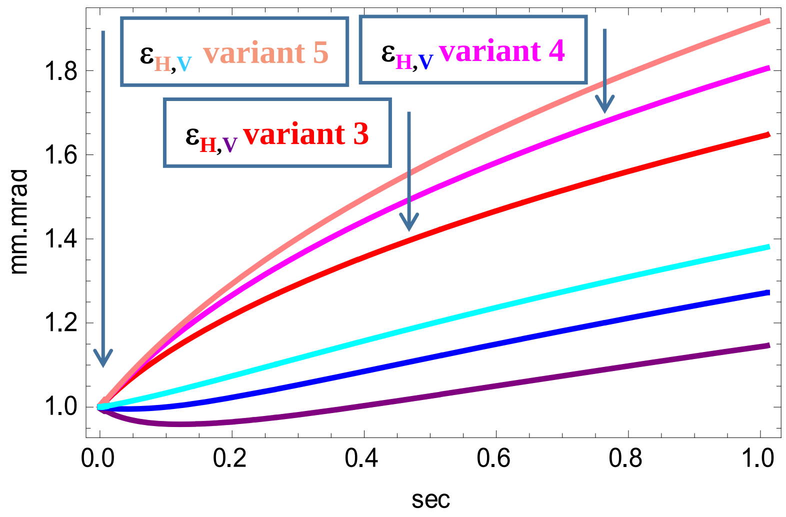
Additional IBS variant beam study

Plots of the beam parameter evolution for the three new variant scenarios

IBS rms bunch length and momentum spread growth



IBS rms physical transverse emittance growth



Evolution of the momentum spread and bunch length (left) and transverse emittances (right)

Summary of the IBS variant beam performance

The table shows that among the *3 new scenarios* investigated the *variant 5* is the *best* because the **bunch length** and **momentum spread** will suffer only **30% blow-up due to IBS after 1s** on the 100 keV plateau (**13% blow-up after 0.3s**)

Nominal : the bunch length and momentum spread growth after 1 [s] on the 100 keV plateau is **Big !**

σ_{BL}(1s) =1.9 m , $\sigma_{\Delta p/p}$(1s) =0.4 ‰ (95% bunch length=7.4 m instead of 1.3 m at t=0 !)

Variant 5 : the bunch length and momentum spread growth after 1 [s] looks **Fine**

σ_{BL}(1s) =0.4 m, $\sigma_{\Delta p/p}$(1s) =0.6 ‰ (95% bunch length=1.7m !)

	$\sigma_{BL}(t)/\sigma_{BL}(0)$		$\sigma_{\Delta p/p}(t)/\sigma_{\Delta p/p}(0)$		$\varepsilon_L(t)/\varepsilon_L(0)$		$\varepsilon_H(t)/\varepsilon_H(0)$		$\varepsilon_V(t)/\varepsilon_V(0)$	
Growth factor at t=	1 s	0.3 s	1 s	0.3 s	1 s	0.3 s	1s	0.3 s	1 s	0.3 s
Nominal beam	5.7	4.4	5.7	4.4	32.5	19.0	1.31	1.13	0.94	0.91
variant 1	14.5	11.3	14.5	11.3	205.0	125.3	1.25	1.54	1.05	0.92
variant 2	3.3	2.4	3.3	2.4	11.0	5.9	1.07	1.03	0.93	0.96
variant 3	2.19	1.75	2.19	1.75	4.78	3.04	1.65	1.29	1.15	0.98
variant 4	1.59	1.32	1.59	1.32	2.54	1.75	1.81	1.36	1.27	1.05
variant 5	1.30	1.13	1.30	1.13	1.69	1.29	1.92	1.40	1.38	1.12

IBS beam growth factor : beam parameter at time *t* over the initial one at *t*=0 along the 100 keV plateau

Epilogue

- Exchange of energies between *horizontal & vertical* $\beta-$oscillation *& synchrotron* oscillation due to IBS was first studied by Piwinski (1974).
- The derivatives of the amplitude function & dispersion β_x' & D_x' were implemented into a CERN code by Piwinsky & Sacherer (1977) and used for *rise-time* calculations in diverse proton storage rings.
- Likewise IBS *rise-times* were also derived by Bjorken-Mtingwa (1983) using a quantum field theory approach giving a broad description of IBS theory.
- Between 2005 & 2012 the derivatives of vertical $\beta-$function & dispersion β_z' & D_z' were implemented by Bjorken-Mtingwa, Carli, Piwinski, Zimmermann. *Mathematica Notebooks* were written.
- IBS theory was extended to *horizontal & vertical* $\beta-$oscillation *linear coupling* (skew quadrupoles or solenoids) by Piwinski (1990). The process is applied to the generalized emittances specified thru the $\beta-$oscillation eigenvectors (e.g. as calculated by *MADX*). IBS with coupling was fully implemented into a *Mathematica Notebook* by Carli (2012) and used for *ELENA* antiproton IBS studies at 100 keV energy.

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