Wakefields and Impedances I

An Introduction

R. Wanzenberg
M. Dohlus

The CERN Accelerator School

CAS
Nov. 2, 2015

HELMHOLTZ ASSOCIATION

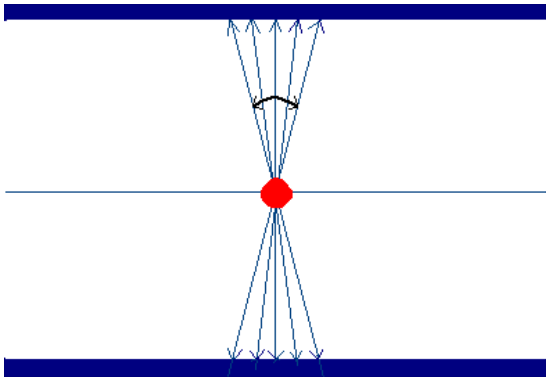
DESY

Introduction

- Wake Field = the track left by a moving body (as a ship) in a fluid (as water); broadly : a track or path left
- Impedance = Fourier Transform (Wake Field)

Electric Field of a Bunch

Point charge in a beam pipe



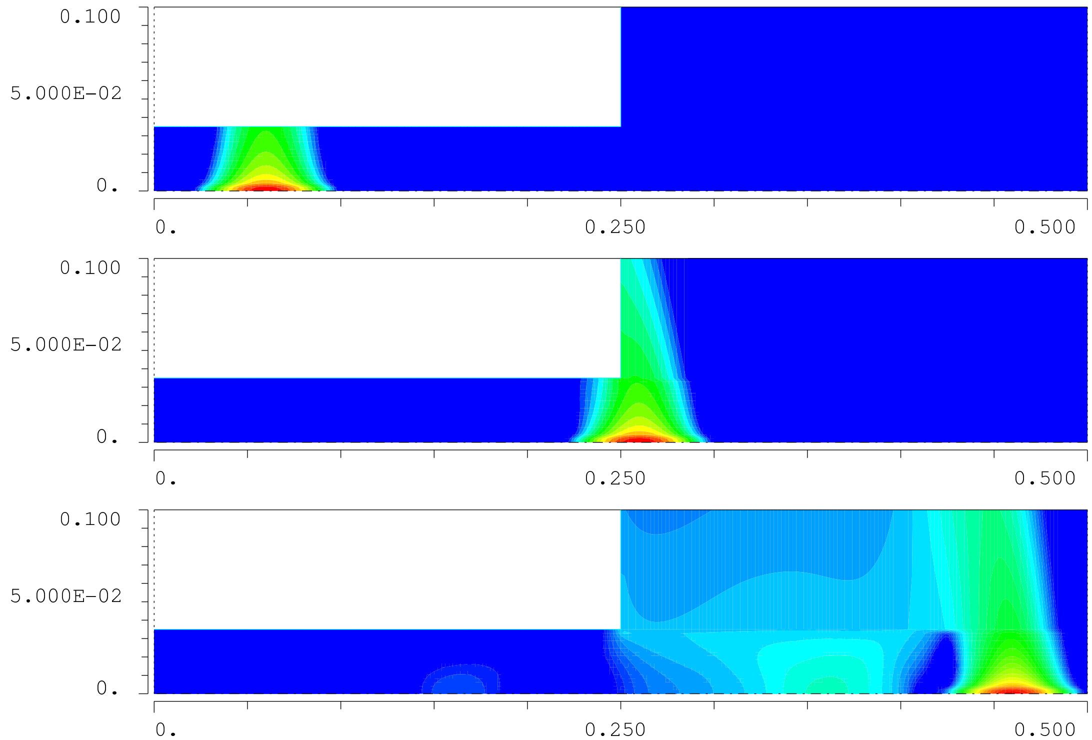
Opening angle

$$\phi = \frac{0.511\,\mathrm{MeV}}{E}$$

$$E = 10\,\mathrm{MeV}, =>$$

$$\phi = 50\,\mathrm{mrad} = 2.89^\circ$$

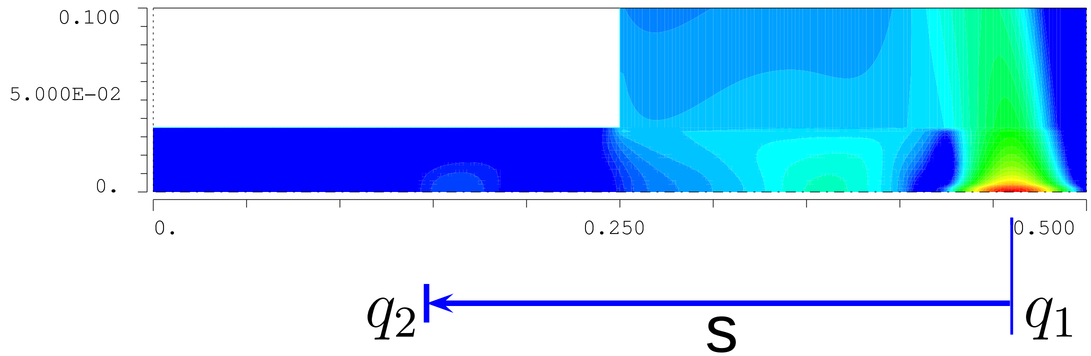
Gaussian Bunch, Step out transition



Effects on a test charge

Change of energy $\Delta\mathcal{E}(s) = q_2 \int dz\, E_z(r, z, t = (s+z)/c)$

Bunch charge q_1
Test charge q_2



Equation of motion

$$q_1: \quad z = ct$$
$$q_2: \quad z = ct - s$$

$E_z(r, z, t)$ depends on the total bunch charge

Wakepotential

$$\mathcal{W}_{||}(s) = \frac{1}{q_1} \int dz\, E_z(r, z, t(s,z))$$

$$t(s,z) = \frac{1}{c}\,(s+z)$$

$$\Delta\mathcal{E}(s) = q_2\, q_1\, \mathcal{W}(s)$$

Bunch charge q_1

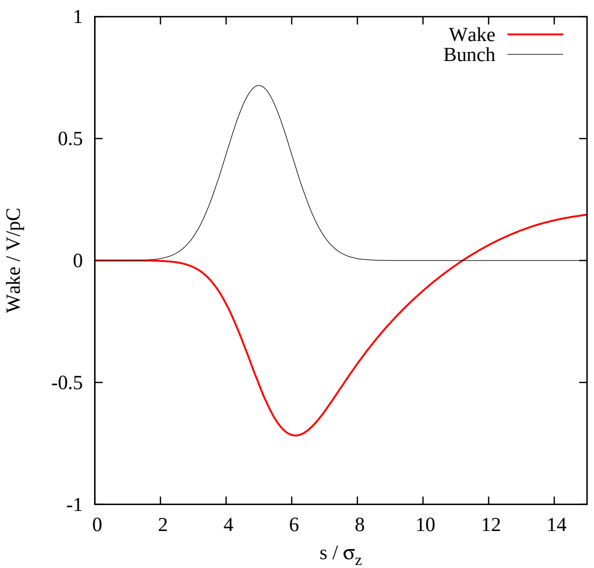
Test charge q_2

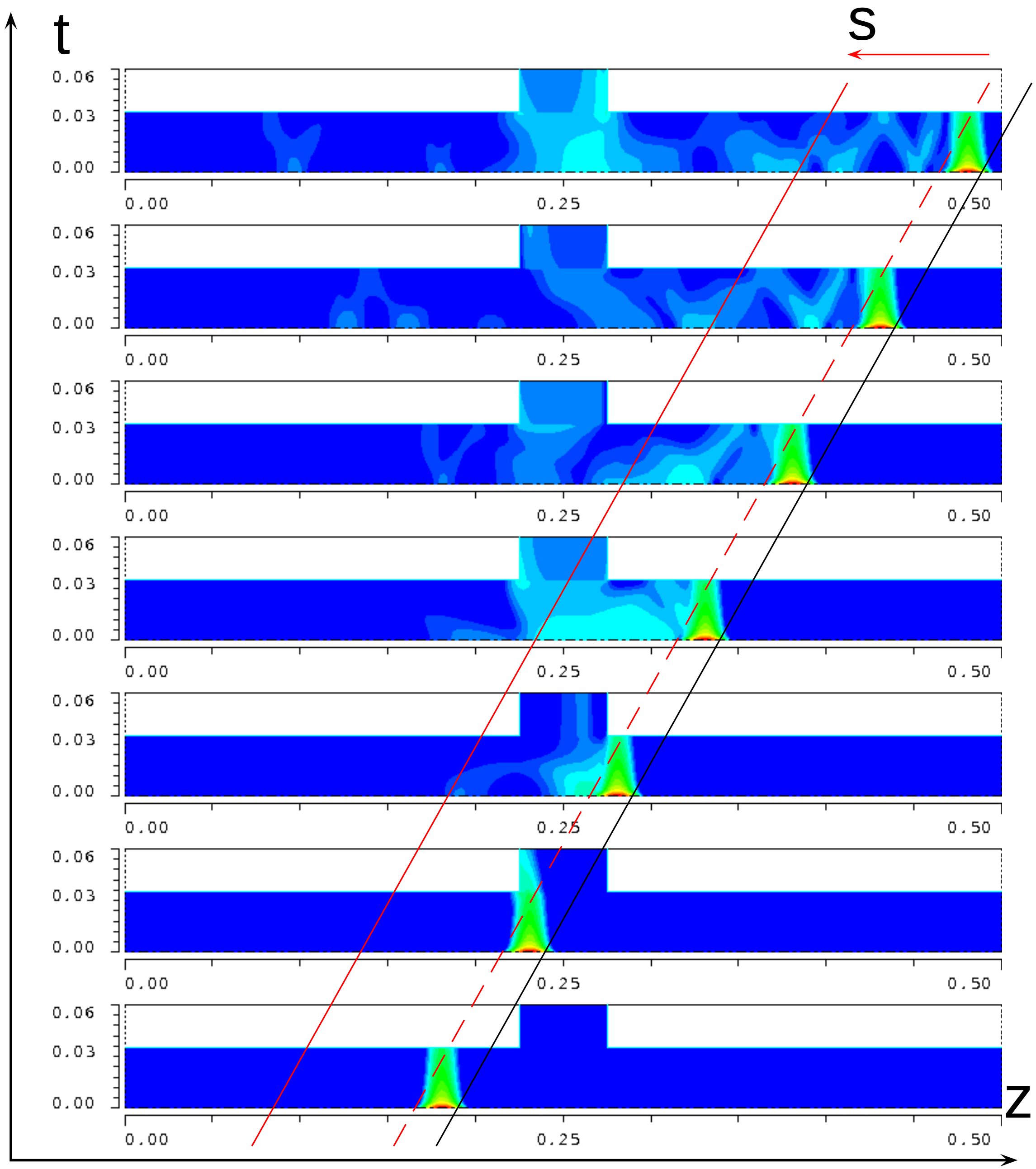
Equation of motion:

$$q_1 : \quad z = \quad c\,t$$

$$q_2 : \quad z = \quad c\,t - s$$

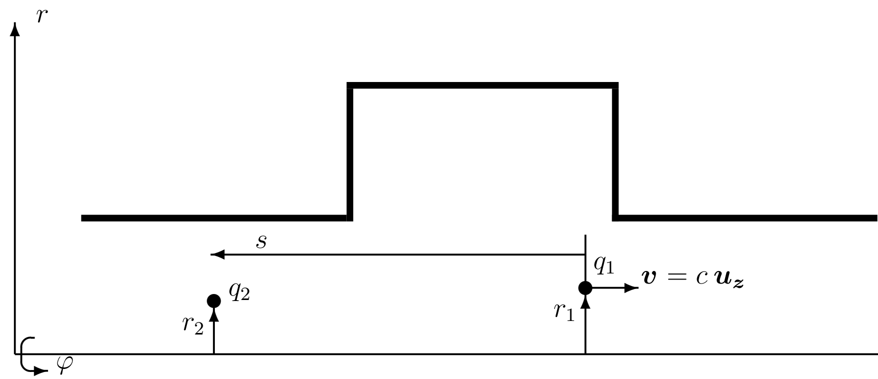
Longitudinal Wakepotential





Wake: point charge versus bunch

Longitudinal Wakepotential

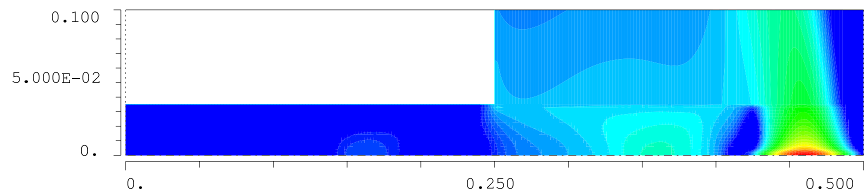


Point charge

$$W_{||}(s) = \frac{1}{q_1} \int dz\, E_{z,p.c.}(r, z, t(s,z))$$

Bunch

$$\mathcal{W}_{||}(s) = \frac{1}{q_1} \int dz\, E_z(r, z, t(s,z))$$



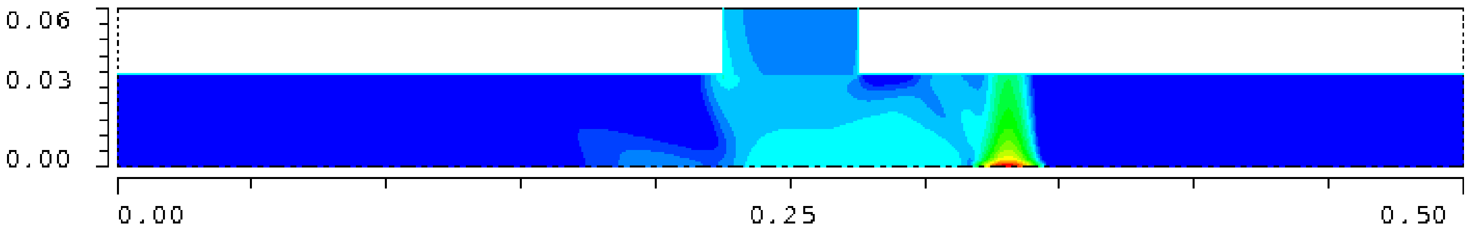
$$\mathcal{W}_{||}(r_{2\perp}, s) = \int_0^\infty ds'\, \lambda(s - s')\, W_{||}(r_{2\perp}, s')$$

The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are generated by the charge distribution ρ and the current density $\boldsymbol{j}$.
They are solutions of the Maxwell equations and have to obey several boundary conditions.

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2} \frac{\partial}{\partial t} \boldsymbol{E} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \boldsymbol{j} = c\, \boldsymbol{u}_z\, \rho$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial}{\partial t} \boldsymbol{B} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} \rho \qquad \rho = q_1\, \lambda_\perp\, \lambda \qquad \lambda(z) = \frac{1}{\sigma\, \sqrt{2\,\pi}} \exp\left(-\frac{z^2}{2\sigma^2}\right)$$

Approximations



Wake Field
Rigid beam approximation

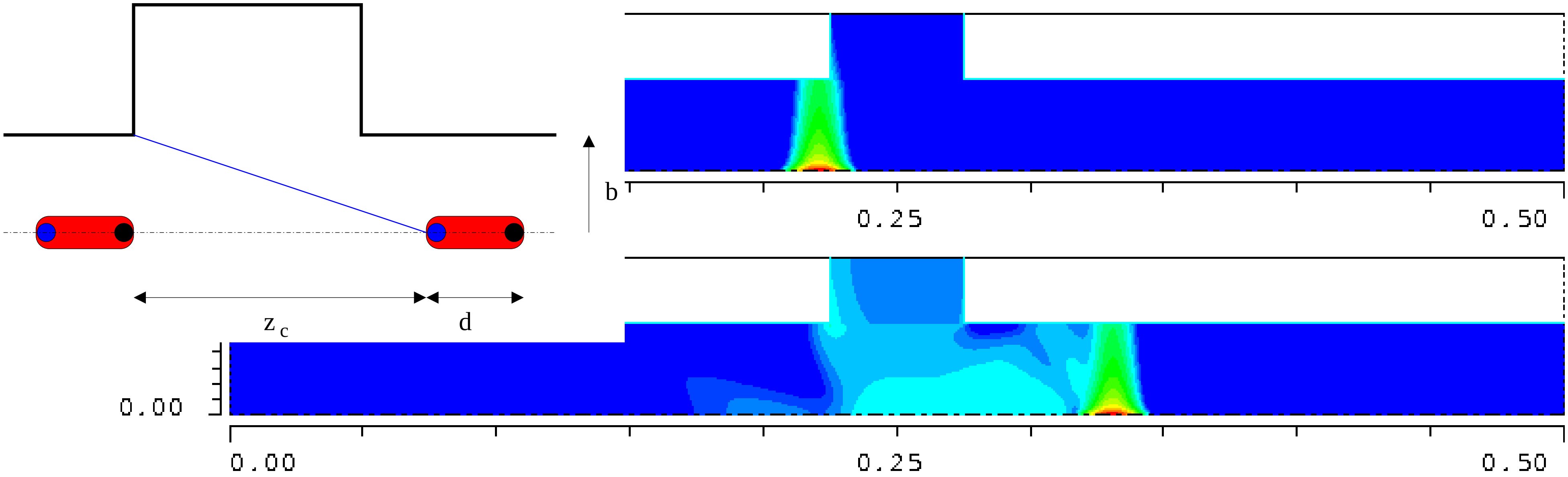
- The wake field does not affect the motion of the beam
- The wake field does not affect the motion of the test charge (only the energy or momentum change is calculated)

The interaction of the beam with the environment is *not* self consistent.

Nevertheless, one can use results from wake field calculations for

a turn by turn tracking code - sometimes cutting a beam into slices.

Catch-up distance



Catch-up distance z_c

$$\frac{z_c + d}{c} = t = \frac{\sqrt{{z_c}^2 + b^2}}{c} \qquad z_c = \frac{b^2 - d^2}{2\,d} = \frac{b^2}{2\,d} - \frac{d}{2}$$

Example:

$z_c \approx 75\,\mathrm{mm}$

$b = 35\,\mathrm{mm},\ d \approx 2\,\sigma_z = 8\,\mathrm{mm}$

$R = 60\,\mathrm{mm},\ L = 50\,\mathrm{mm}$

$$\lim_{d \to \mathrm{small}} z_c = \mathrm{LARGE}$$

Numerical Calculations

There exist several numerical codes to calculate wakefields. Examples are:

Non commercial codes (2D, r-z geometry)

- ABCI, Yong Ho Chin, KEK

  http://abci.kek.jp/abci.htm

- Echo 2D, Igor Zagorodnov, DESY

  http://www.desy.de/˜zagor/WakefieldCode_ECHOz/

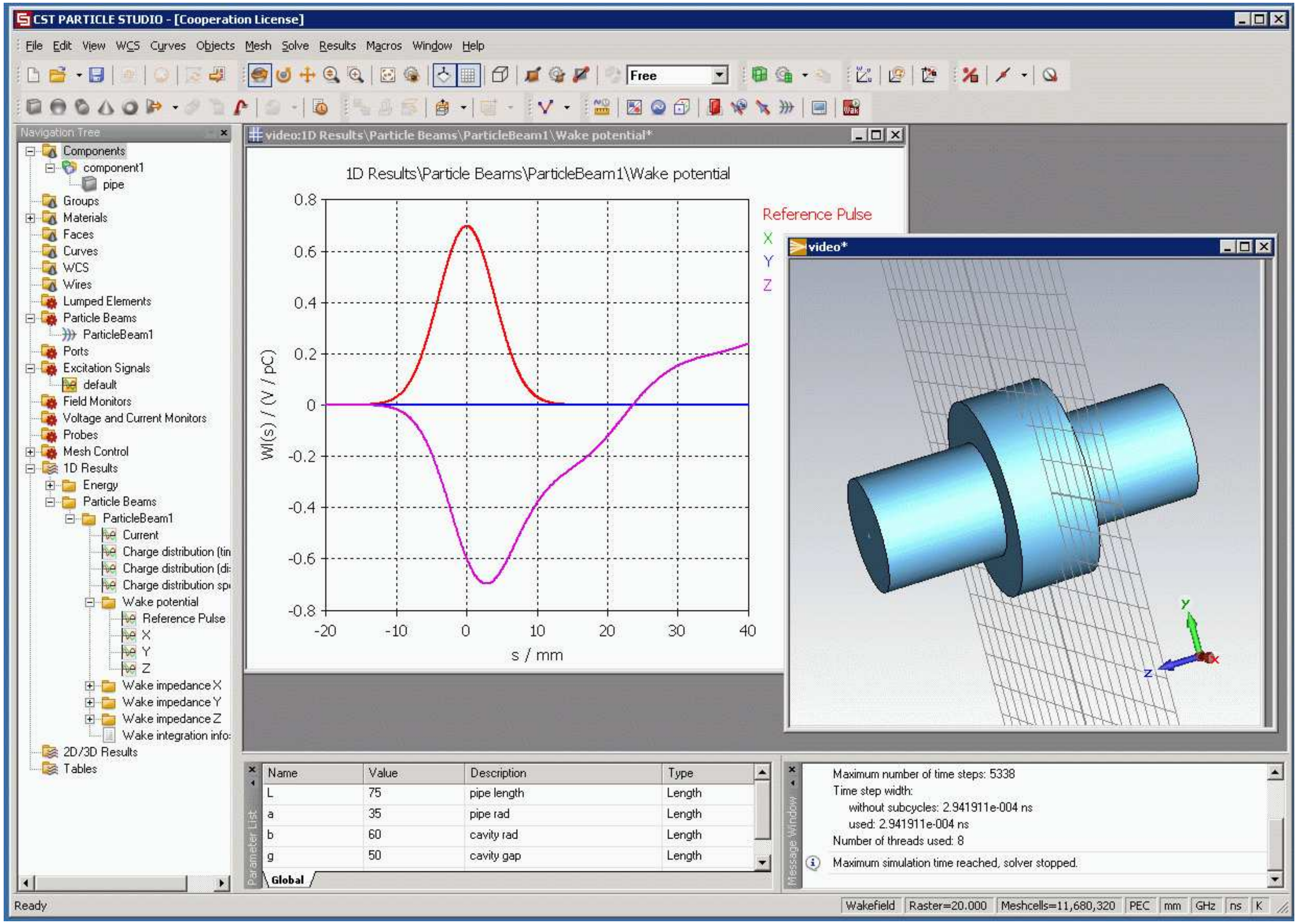
Commercial codes (3D)

- GdfidL
- CST (Particle Studio, Microwave Studio)

The Maxwell equations are solved on a grid

Example: wakepotential of a cavity

Numerical calculation of the wakepotential with CST Particle Studio:



Loss Parameter

Wakepotential

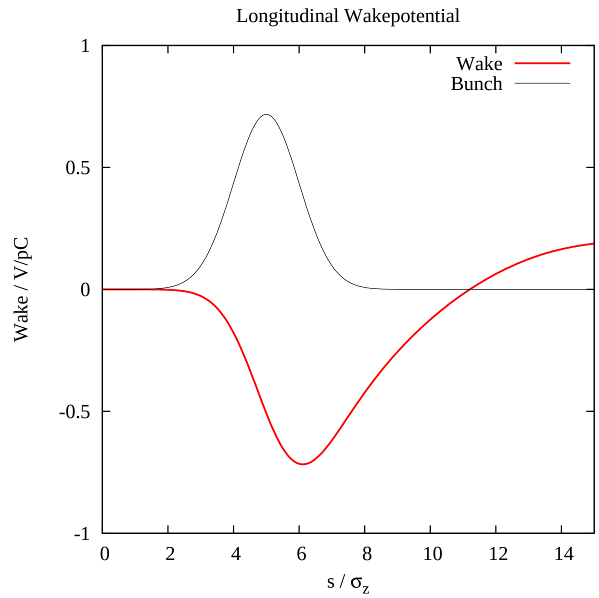
$$\mathcal{W}(s) = \frac{1}{q_1} \int dz\, E_z(r, z, (s+z)/c)$$

Line charge density

$$\lambda(s) = \frac{1}{\sigma_z \sqrt{2\,\pi}} \ \exp\left(-\frac{1}{2}\,\frac{s^2}{{\sigma_z}^2}\right)$$

Total loss parameter

$$k_{tot} = \int ds\, \lambda(s)\, \mathcal{W}(s)$$

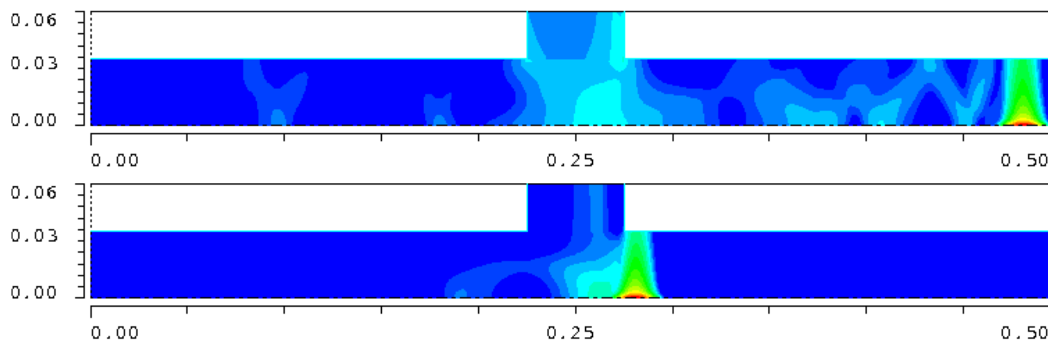


$k_{tot} = 0.46\, \frac{\mathrm{V}}{\mathrm{pC}}$, σ_z = 4 mm, Cavity: $L = 5$ cm, $R = 6$ cm, pipe radius 3.5 cm

Energy loss

$$\Delta\mathcal{E} = P = {q_{bunch}}^2\, k_{tot}$$

$$\Delta\mathcal{E} = (16\,\mathrm{nC})^2\, 0.46\, \frac{\mathrm{V}}{\mathrm{pC}} = 0.118\ \mathrm{mV\,C}$$



Bunch population $N = 10 \cdot 10^{10}$,

PETRA $C = 2304\,\mathrm{m}, T = 7.68\mu\mathrm{s}$

$P = \frac{1}{T}\,\Delta\mathcal{E} = 15.2\,\mathrm{W}$

Transverse Wake Fields

Lorentz Force

$$\boldsymbol{F}(\boldsymbol{r_2}, t) = q_2\,(\boldsymbol{E}(\boldsymbol{r_2}, t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r_2}, t))$$

Longitudinal Wakepotential

$$\mathcal{W}_{||}(\boldsymbol{r_2}_\perp, s) = \frac{1}{q_1} \int dz\, E_z(\boldsymbol{r_2}_\perp, z, (s+z)/c)$$

Wakepotential

$$\boldsymbol{\mathcal{W}}(\boldsymbol{r_2}_\perp, s) = \frac{1}{q_1} \int_{-\infty}^{\infty} dz [\boldsymbol{E}(\boldsymbol{r_2}_\perp, z, t) + c\,\boldsymbol{e}_z \times \boldsymbol{B}(\boldsymbol{r_2}_\perp, z, t)]_{t=(s+z)/c}$$

Change of momentum of a test charge q_2

$$\Delta \boldsymbol{p}(\boldsymbol{r_2}_\perp, s) = \frac{1}{c}\, q_2\, q_1\, \boldsymbol{\mathcal{W}}(\boldsymbol{r_2}_\perp, s)$$

kick on an electron (q_2) due to the bunch charge q_1

$$\boldsymbol{\theta}(\boldsymbol{r_2}_\perp, s) = \frac{e}{E}\, q_1\, \boldsymbol{\mathcal{W}}_\perp(\boldsymbol{r_2}_\perp, s)$$

Transverse Wakepotential

$$\mathcal{W}_\perp(\boldsymbol{r}_{2\perp}, s) = \frac{1}{q_1}\int_{-\infty}^{\infty} dz[\boldsymbol{E}_\perp(\boldsymbol{r}_{2\perp}, z, t) + c\boldsymbol{e}_z \times \boldsymbol{B}_\perp(\boldsymbol{r}_{2\perp}, z, t)]_{t=(s+z)/c}$$

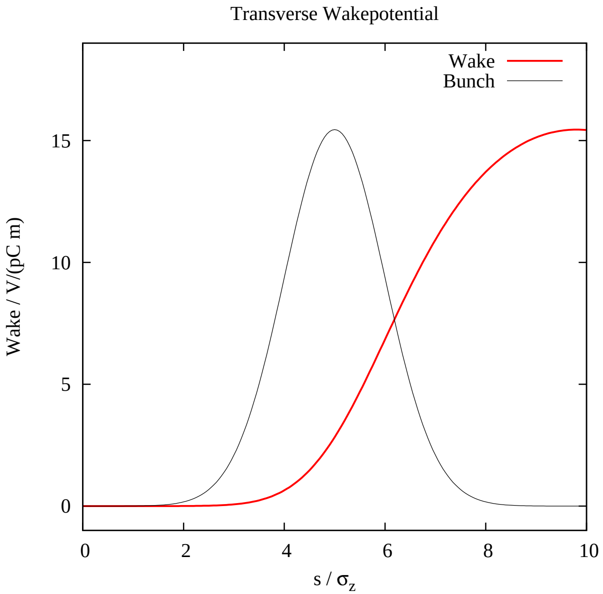
Bunch charge q_1

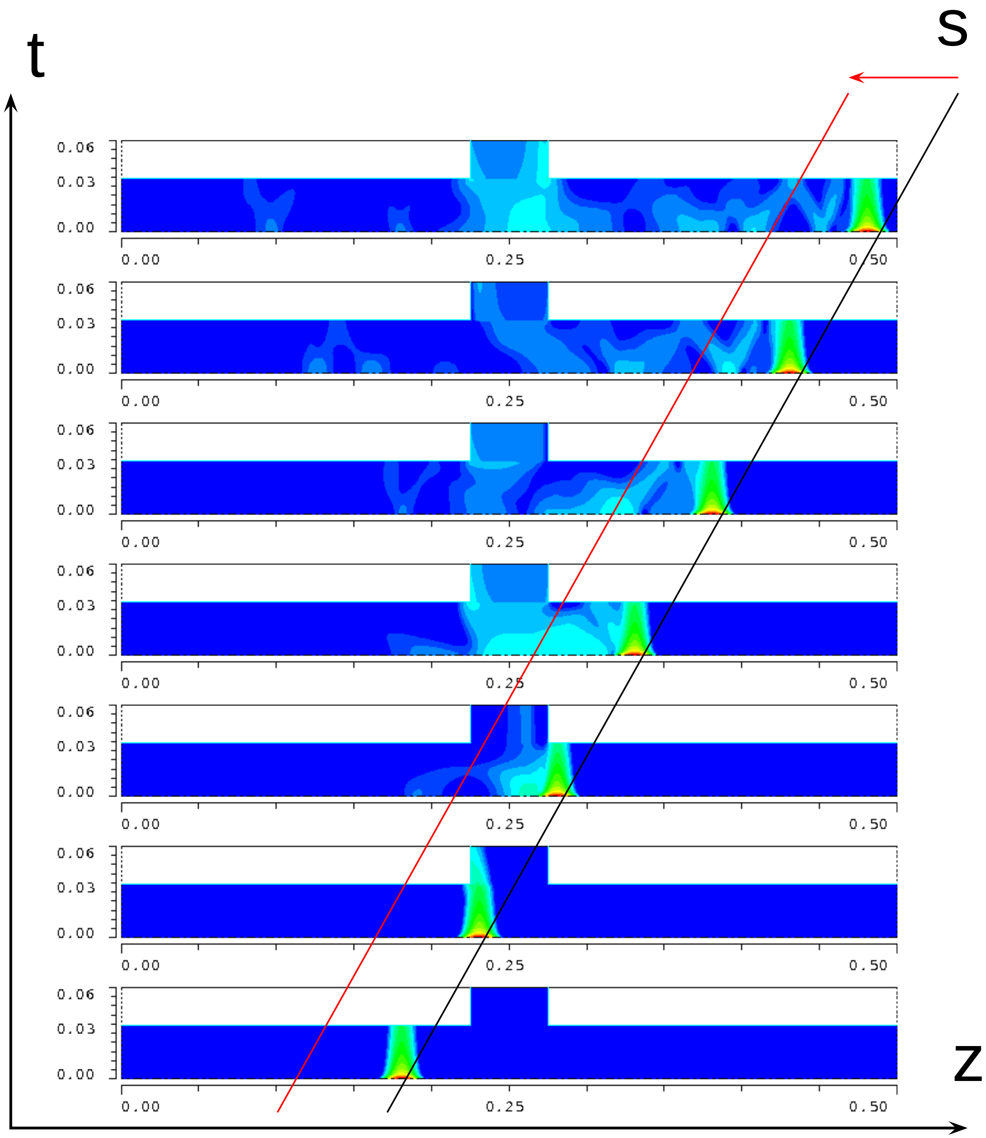
Test charge q_2

Equation of motion:

$$q_1: \quad z = \quad ct$$

$$q_2: \quad z = \quad ct - s$$

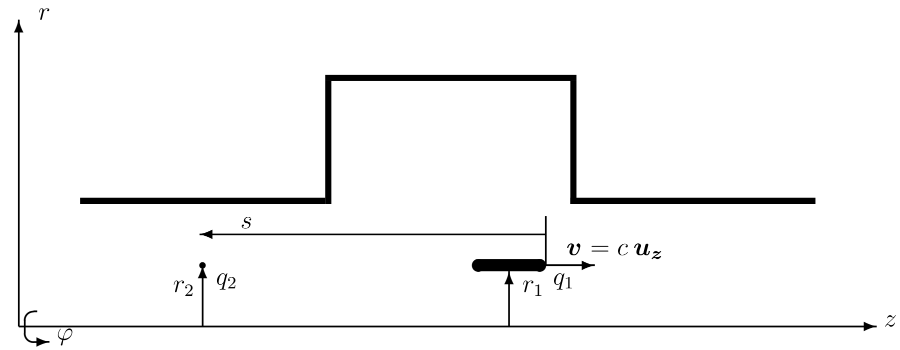




Kick Parameter

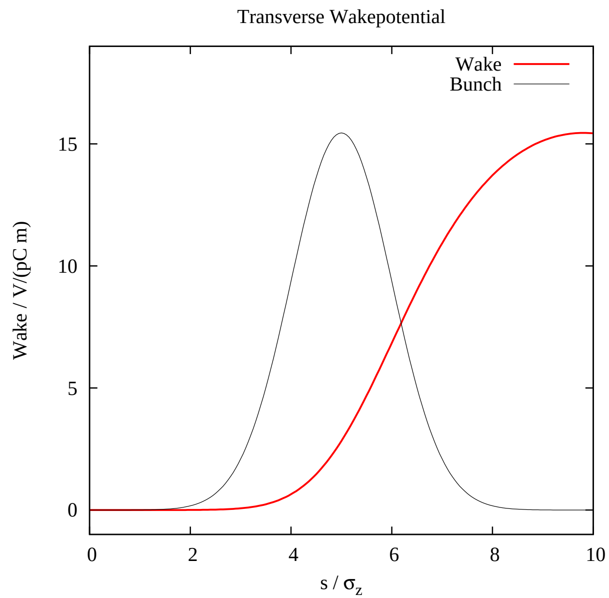
Wakepotential

$$\boldsymbol{\mathcal{W}}(r_2, s) = \frac{1}{q_1}\int_{-\infty}^{\infty} dz[\boldsymbol{E}(r_2, z, t) + c\,\boldsymbol{e}_z \times \boldsymbol{B}(r_2, z, t)]_{t=(s+z)/c}$$



Line charge density

$$\lambda(s) = \frac{1}{\sigma_z\sqrt{2\,\pi}}\;\exp\left(-\frac{1}{2}\,\frac{s^2}{{\sigma_z}^2}\right)$$



Kick parameter ($\mathrm{V/(pC\,m)}$)

$$k_\perp = \int ds\,\lambda(s)\,\frac{1}{r_2}\mathcal{W}_\perp(r_2, s)$$

$$k_\perp = 3.6\,\frac{\mathrm{V}}{\mathrm{pC\,m}}$$

$\mathcal{W}_\perp$ radial, vertical or horizontal component of the wake potential

Betatron Tune shift in a storage ring

$$\Delta Q_\beta = \frac{I_{bunch}\,T_0}{4\pi\,E/e}\,\beta\,k_\perp, \quad (\beta\ \text{function, optics})$$

From Wakes to Impedance

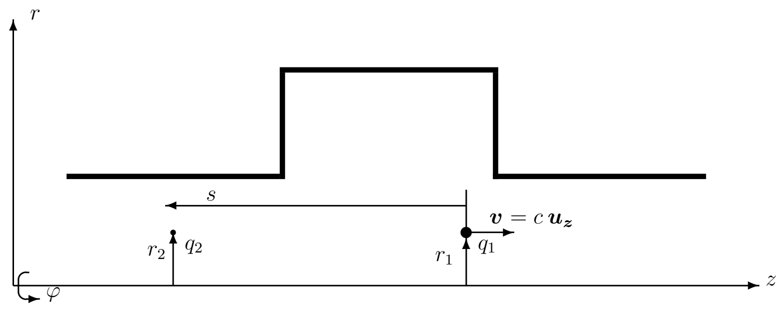
- Wake Field = the track left by a moving ship in water
- Impedance = Fourier Transform (Wake Field)

Warning: Wakefields have a bad reputation and the generation of wakes may be even illegal in some communities:
NO WAKE please on Chicago river (PAC 2001)

Impedance

Longitudinal impedance

$$Z_{||}(r_{2\perp},\omega) = \frac{1}{c}\int_{-\infty}^{\infty} ds\, W_{||}(r_{2\perp},s)\,\exp(-i\frac{\omega}{c}\,s)$$



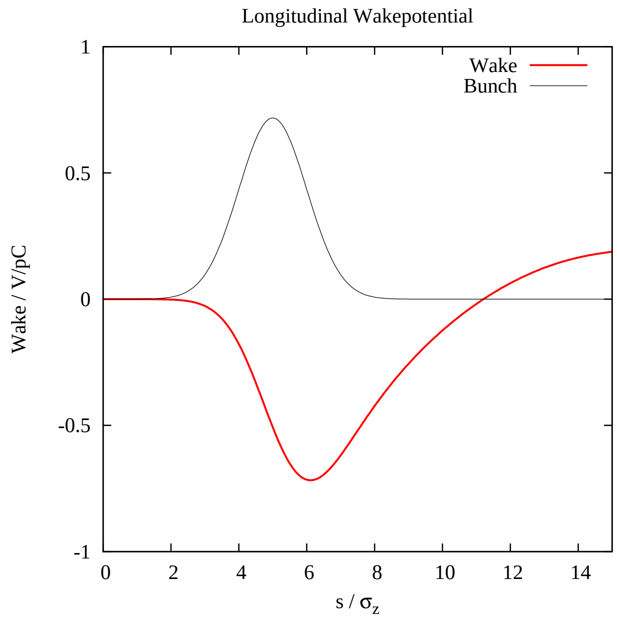
Transverse impedance

$$\boldsymbol{Z}_{\perp}(r_{2\perp},\omega) = \frac{-i}{c}\int_{-\infty}^{\infty} ds\, \boldsymbol{W}_{\perp}(r_{2\perp},s)\,\exp(-i\frac{\omega}{c}\,s)$$

phase factor $-i = \exp(-i\,\pi/2)$

Wakepotential of a Gaussian bunch

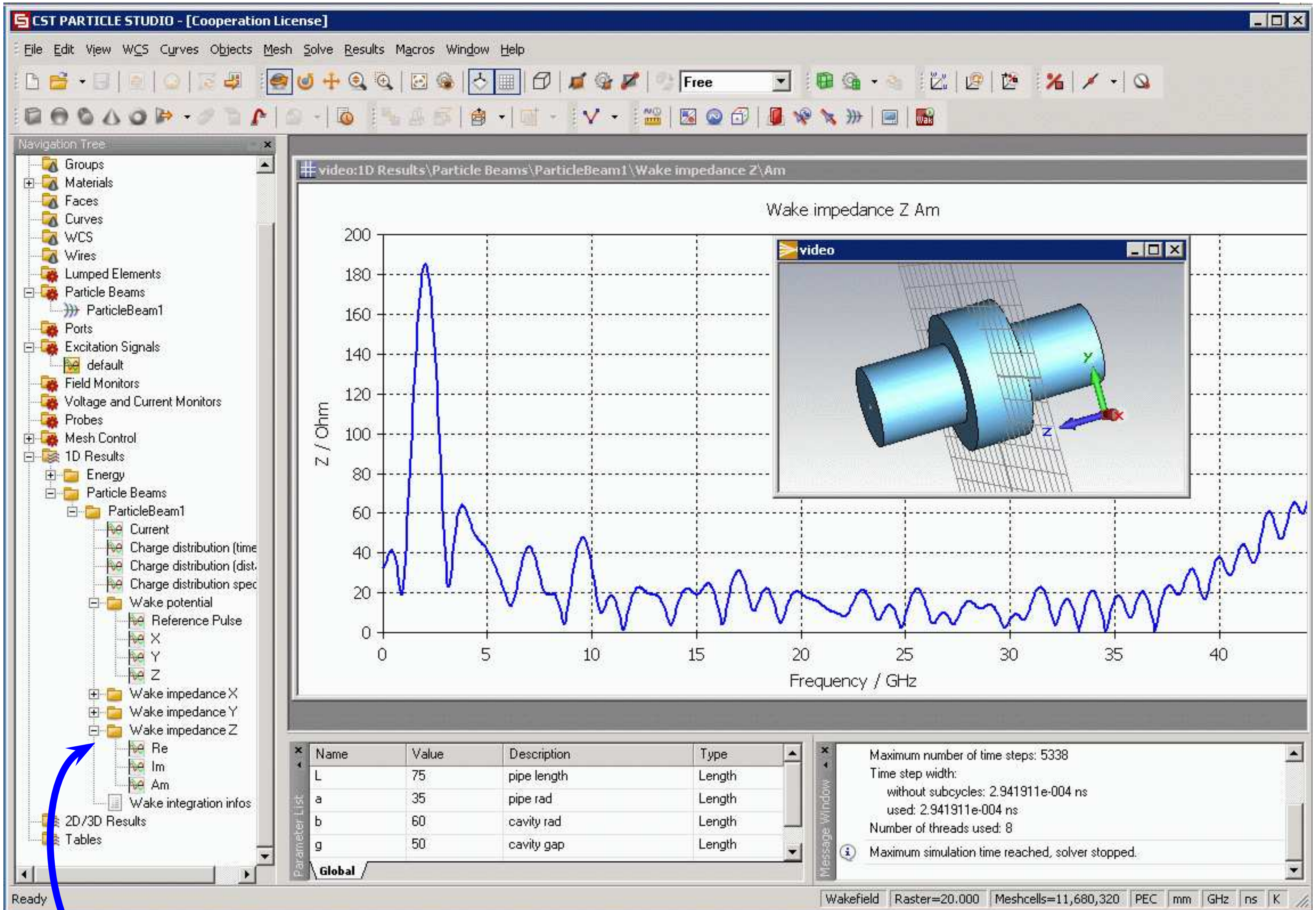
$$\mathcal{W}_{||}(r_{2\perp},s) = \int_{0}^{\infty} ds'\,\lambda(s-s')\,W_{||}(r_{2\perp},s')$$



Fourier Transform of the Wake

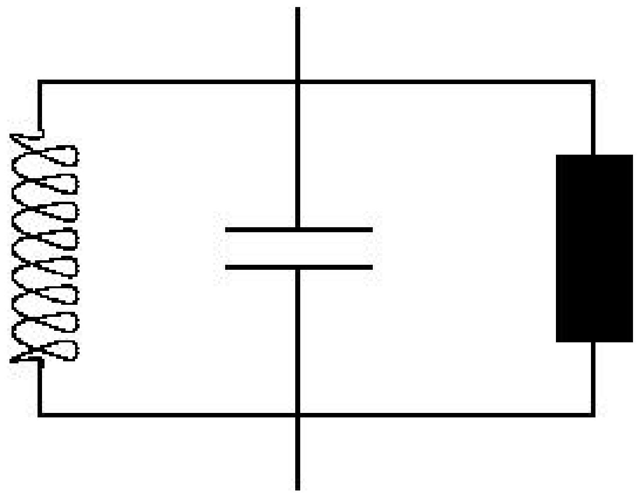
$$\widetilde{\lambda}(\omega)\;Z_{||}(r_{2\perp},\omega) = \frac{1}{c}\int_{-\infty}^{\infty} ds\,\mathcal{W}_{||}(r_{2\perp},s)\,\exp(-i\frac{\omega}{c}\,s)$$

You get the Impedance



if you click here !

Equivalent Circuit Model



$$2\,k_{\|} = \omega_r \frac{R}{Q} = \frac{1}{C}$$

$$\omega_r = \frac{1}{\sqrt{L\,C}}$$

$$Q = \omega_r\,R\,C$$

Impedance

$$Z_{\|}(\omega) = \frac{R}{1 + i\,Q\,\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$

Wake Potential $s > 0$ (point charge)

long range wakefield of one mode

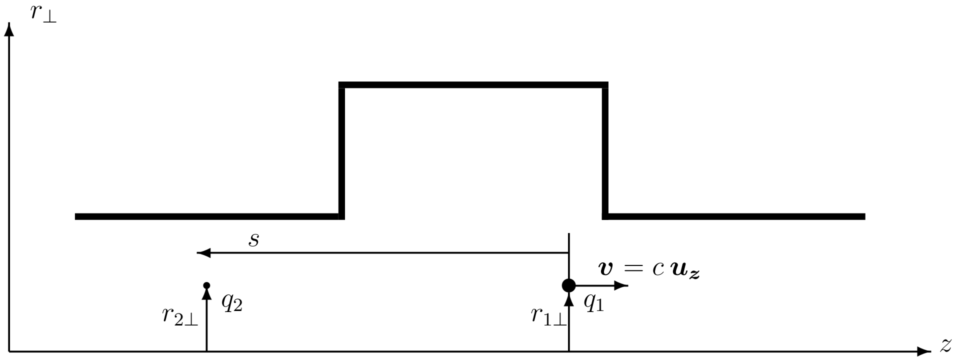
$$W_{\|}(s) \approx -2\,k_{\|}\,\cos(\omega_r\,\frac{s}{c})\,\exp(-\frac{\omega_r}{2\,Q}\,\frac{s}{c})$$

Loss parameter (Gaussian bunch)

$$k_{tot} \approx k_{\|}\,\exp\left(-{\omega_r}^2 (\frac{\sigma_z}{c})^2\right)$$

Panofsky-Wenzel-Theorem

$$\frac{\partial}{\partial s}\boldsymbol{W}_\perp(\boldsymbol{r}_{2\perp}, s) = -\boldsymbol{\nabla}_{2\perp} W_{||}(\boldsymbol{r}_{2\perp}, s)$$



$$\frac{\partial}{\partial s}\boldsymbol{W}_\perp(\boldsymbol{r}_{2\perp}, s) = \frac{1}{q_1}\int_{-\infty}^{\infty} dz \left[\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{E}_\perp(\boldsymbol{r}_{2\perp}, z, t) + c\,\boldsymbol{e}_z \times \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r}_{2\perp}, z, t)\right]_{t=(s+z)/c}$$

Maxwell Equation: $\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} \quad \Rightarrow \quad \boldsymbol{e}_z \times \frac{\partial}{\partial t}\boldsymbol{B} = \frac{\partial}{\partial z}\boldsymbol{E}_\perp - \boldsymbol{\nabla}_\perp E_z$

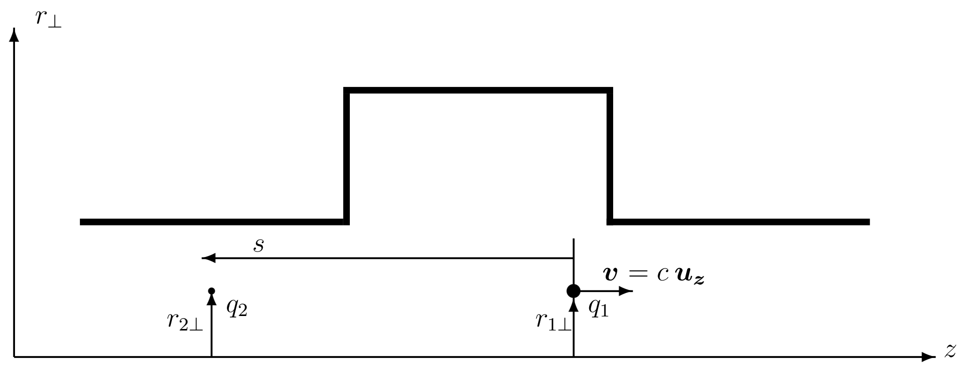
$$\frac{d}{dz}\boldsymbol{E}_\perp(\boldsymbol{r}_{2\perp}, z, \frac{z+s}{c}) = \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\boldsymbol{E}_\perp(\boldsymbol{r}_{2\perp}, z, \frac{z+s}{c})$$

$$\frac{\partial}{\partial s}\boldsymbol{W}_\perp(\boldsymbol{r}_{2\perp}, s) = \frac{1}{q_1}\int_{-\infty}^{\infty} dz \left((\frac{d}{dz}\boldsymbol{E}_\perp)(\boldsymbol{r}_2, \frac{z+s}{c}) - \boldsymbol{\nabla}_\perp E_z(\boldsymbol{r}_2, \frac{z+s}{c})\right)$$

$$\frac{\partial}{\partial s}\boldsymbol{W}_\perp(\boldsymbol{r}_{2\perp}, s) = -\boldsymbol{\nabla}_{2\perp} W_{||}(\boldsymbol{r}_{2\perp}, s) + \frac{1}{q_1}\int_{-\infty}^{\infty} dz(\frac{d}{dz}\boldsymbol{E}_\perp)(\boldsymbol{r}_2, \frac{z+s}{c})$$

Panofsky-Wenzel-Theorem (cont.)

$$\frac{\partial}{\partial s}\boldsymbol{W}_\perp(\boldsymbol{r}_{2\perp}, s) = -\boldsymbol{\nabla}_{2\perp} W_{||}(\boldsymbol{r}_{2\perp}, s)$$



Frequency domain version of the Panofsky-Wenzel-Theorem:

$$\frac{\omega}{c}\boldsymbol{Z}_\perp(\boldsymbol{r}_{2\perp}, \omega) = \boldsymbol{\nabla}_\perp Z_{||}(\boldsymbol{r}_{2\perp}, \omega)$$

Integration of the transverse gradient of the longitudinal wake potential provides the transverse wake potential:

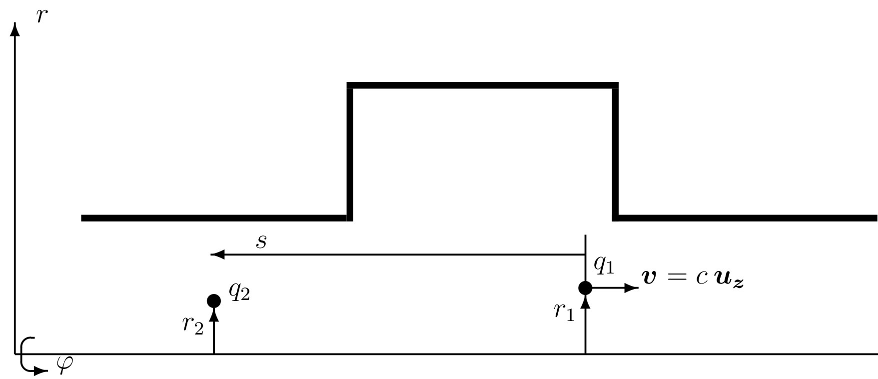
$$\boldsymbol{W}_\perp(\boldsymbol{r}_{2\perp}, s) = -\boldsymbol{\nabla}_\perp \int_{-\infty}^{s} ds'\, W_{||}(\boldsymbol{r}_{2\perp}, s').$$

W.K.H. Panofsky, W.A. Wenzel, *Some Considerations Concerning the Transverse Deflection of Charged Particles in Radio-Frequency Fields*, Rev. Sci. Instrum. 27 (1956), 947

Multipole Expansion of the Wake

Longitudinal Wakepotential

$$W_{||}(r_1, r_2, \varphi_1, \varphi_2, s) = \sum_{m=0}^{\infty} {r_1}^m \, {r_2}^m \; W_{||}^{(m)}(s) \;\; \cos\left(m\left(\varphi_2 - \varphi_1\right)\right)$$



Panofsky–Wenzel theorem

$$\frac{\partial}{\partial s}\boldsymbol{W}_{\perp}(x_2, y_2, x_1, y_1, s) = -\boldsymbol{\nabla}_{\perp_2} W_{||}(x_2, y_2, x_1, y_1, s)$$

Multipole expansion in Cartesian coordinates

$$\begin{aligned} W_{||}(x_1, y_1, x_2, y_2, s) \quad &\approx \quad W_{||}^{(0)}(s) \\ &+ \; (x_2 \; x_1 + y_2 \; y_1) \; W_{||}^{(1)}(s) \\ &+ \; \left(({x_2}^2 - {y_2}^2)\,({x_1}^2 - {y_1}^2) + 2\,x_2\,y_2 \; 2\,x_1\,y_1\right) \; W_{||}^{(2)}(s) \end{aligned}$$

$$\begin{aligned} \boldsymbol{W}_{\perp}(x_1, y_1, x_2, y_2, s) \quad &\approx \quad (x_1 \; \boldsymbol{u_x} + y_1 \; \boldsymbol{u_y}) \; W_{\perp}^{(1)}(s) \\ &+(x_2 \; \boldsymbol{u_x} - y_2 \; \boldsymbol{u_y}) \; 2 \; ({x_1}^2 - {y_1}^2) \; W_{\perp}^{(2)}(s) \\ &+(y_2 \; \boldsymbol{u_x} + x_2 \; \boldsymbol{u_y}) 2 \; (2\,x_1\,y_1) \; W_{\perp}^{(2)}(s). \end{aligned}$$

$$W_{\perp}^{(m)}(s) = -\int_{-\infty}^{s} ds' \; W_{||}^{(m)}(s')$$

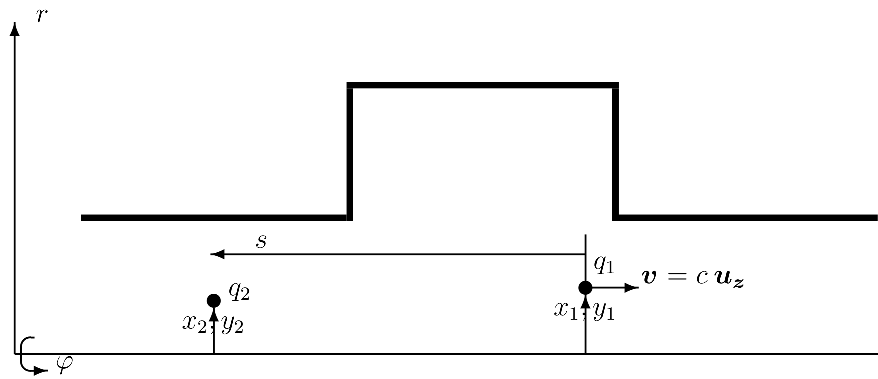
Monopole and Dipole Wake

Longitudinal Wakepotential

$$W_{||}(x_1, y_1, x_2, y_2, s) \approx$$
$$W_{||}^{(0)}(s) + \ (x_2\ x_1 + y_2\ y_1)\ W_{||}^{(1)}(s)$$

Transverse Wakepotential

$$\boldsymbol{W}_{\perp}(x_1, y_1, x_2, y_2, s) \approx (x_1\ \boldsymbol{u_x} + y_1\ \boldsymbol{u_y})\ W_{\perp}^{(1)}(s)$$



Monopole Wakepotential $\quad W_{||}^{(0)}(s)$

Longitudinal Dipole Wakepotential $\quad W_{||}^{(1)}(s)$

Transverse Dipole Wakepotential $\quad W_{\perp}^{(1)}(s) = -\int_{-\infty}^{s} ds'\, W_{||}^{(1)}(s')$

There is no transverse monopole wake potential.

There is no a priori relation between the wake potentials of different azimuthal order.

But for resistive wall impedance (wake) in a pipe with radius b there is a relation:

$$Z_{\perp}^{(1)}(\omega) = \frac{c}{\omega}\,\frac{2}{b^2}\, Z_{||}^{(0)}(\omega)$$

$W_{\|}(\boldsymbol{r}_\perp, s)$ as a harmonic function

$W_{\|}(\boldsymbol{r}_\perp, s)$ is a harmonic function of the transverse coordinates:

$$\boldsymbol{\nabla}_\perp^2 W_{\|}(\boldsymbol{r}_\perp, s) = 0$$

Line charge density $\lambda(z)$

Charge density

$\rho(\boldsymbol{r}, t) = q_1 \, \lambda_\perp(x, y) \, \lambda(z - ct)$

Current density

$\boldsymbol{j}(\boldsymbol{r}, t) = c \, \boldsymbol{u_z} \, \rho(\boldsymbol{r}, t)$

Maxwell Equation $+ \; \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \boldsymbol{\nabla}^2 \boldsymbol{E} \; \Rightarrow$

$$\boldsymbol{\nabla}^2 \boldsymbol{E} - \frac{1}{c^2} \frac{\partial^2}{\partial^2 t} \boldsymbol{E} = \frac{1}{\varepsilon_0} \left(\boldsymbol{\nabla} \rho + \frac{1}{c^2} \frac{\partial}{\partial t} \boldsymbol{j} \right)$$

ρ and $j_z = c \, \rho$ are functions of $z - ct \;\Rightarrow\; \frac{\partial}{\partial t} j_z = -c^2 \frac{\partial}{\partial z} \rho \;\Rightarrow$

$$\boldsymbol{\nabla}_\perp^2 E_z = \left(\frac{1}{c^2} \frac{\partial^2}{\partial^2 t} - \frac{\partial^2}{\partial^2 z} \right) E_z$$

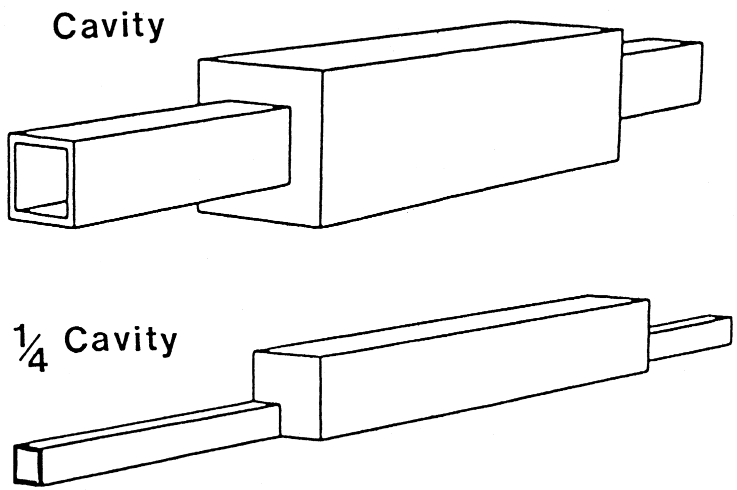
$$\boldsymbol{\nabla}_\perp^2 W_{\|}(x, y, s) = \frac{1}{q_1} \int_{-\infty}^{\infty} dz \, \boldsymbol{\nabla}_\perp^2 E_z(x, y, z, (s + z)/c) = 0$$

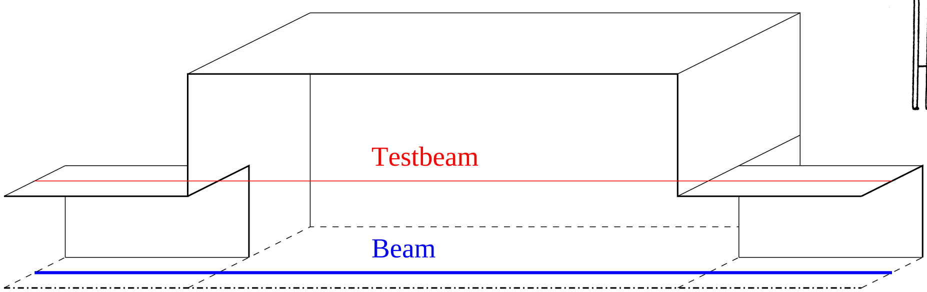
$$\boldsymbol{\nabla}_\perp^2 W_{\|} \sim \frac{1}{\gamma^2} \to 0, \quad \text{for} \quad \gamma \to \infty$$

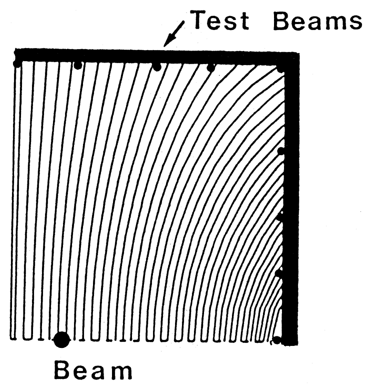
Indirect test beams

$\nabla_\perp^2 W_{||}(x, y, s) = 0 \quad \Rightarrow$

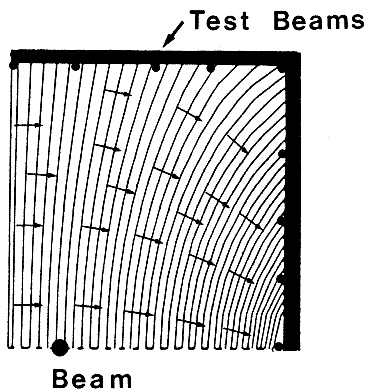
It is possible to find $W_{||}(x, y, s)$ for all possible beam positions (x, y) by a numerical solution of Poissons equation if one knows the wake potential on the boundary.







Beam and Test Beams on the boundary. Lines of constant longitudinal wake potential.

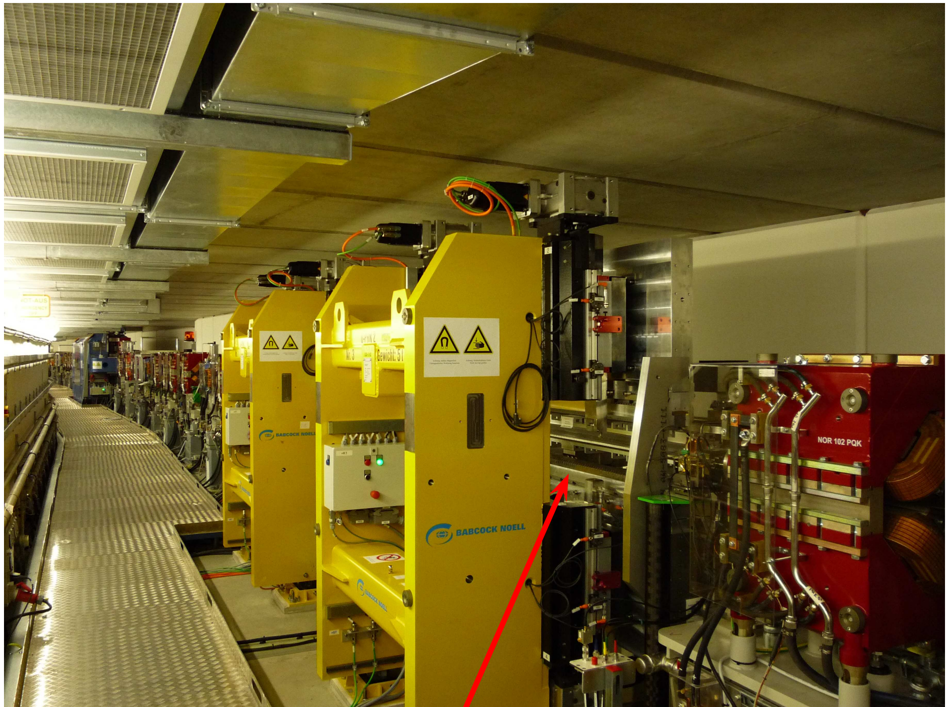


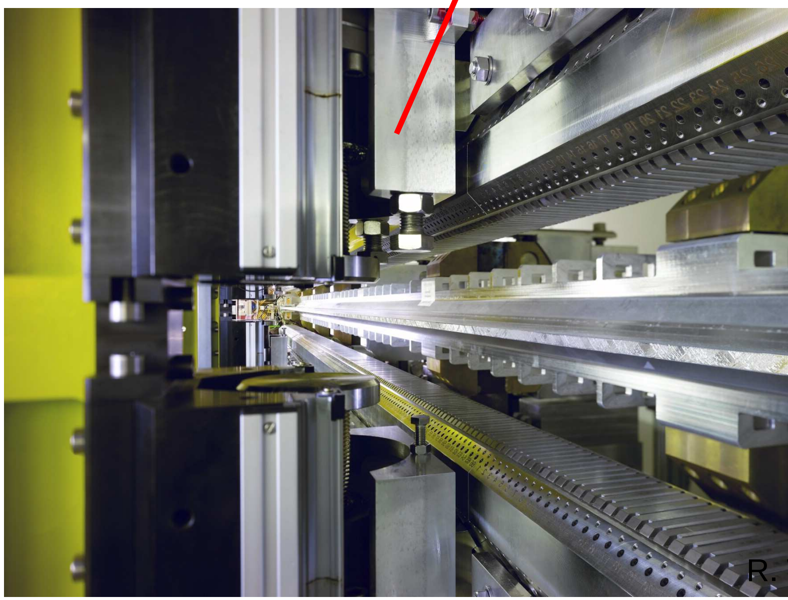
Gradient of the longitudinal wake potential. An integration provides the transverse wake potential according to the Panofsky–Wenzel theorem.

PETRA III

Application of Wake Field Calculations: The Light Source PETRA III

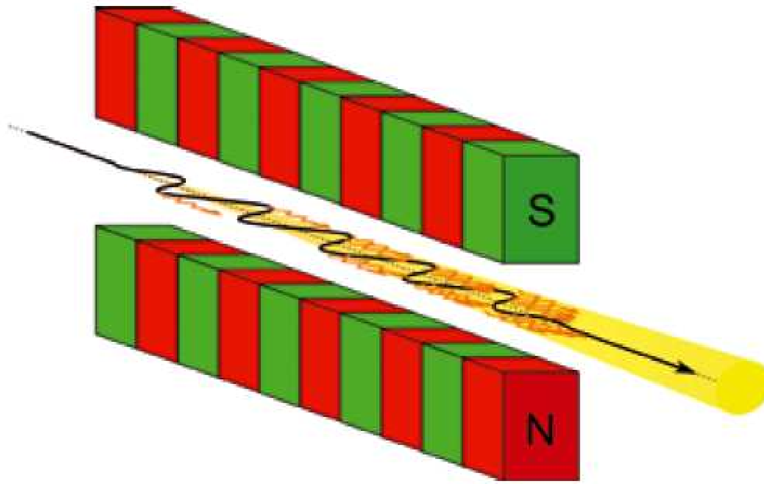
PETRA III undulator





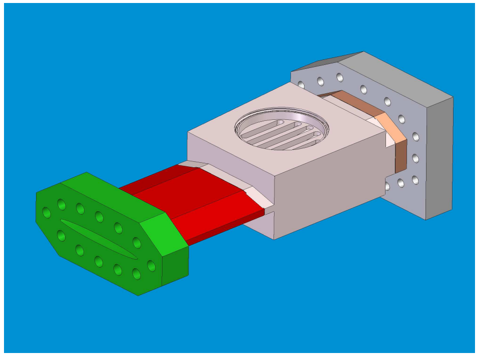
PETRA III undulators:

- Length: 10 m / 5 m / 2 m
- Mag. Field: $\sim$ 0.9 T
- Period: $\sim$ 30 mm

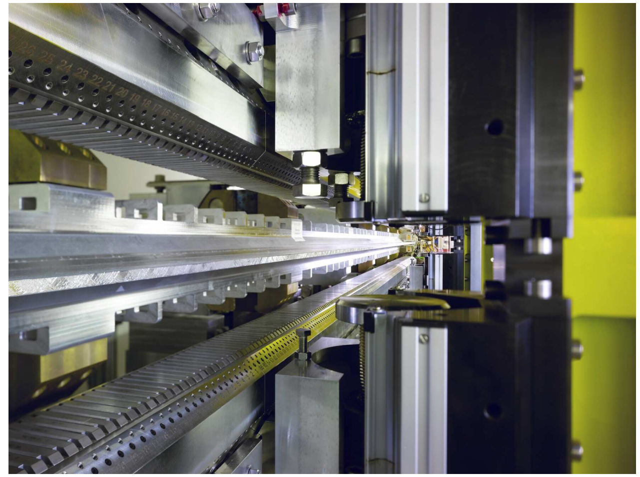


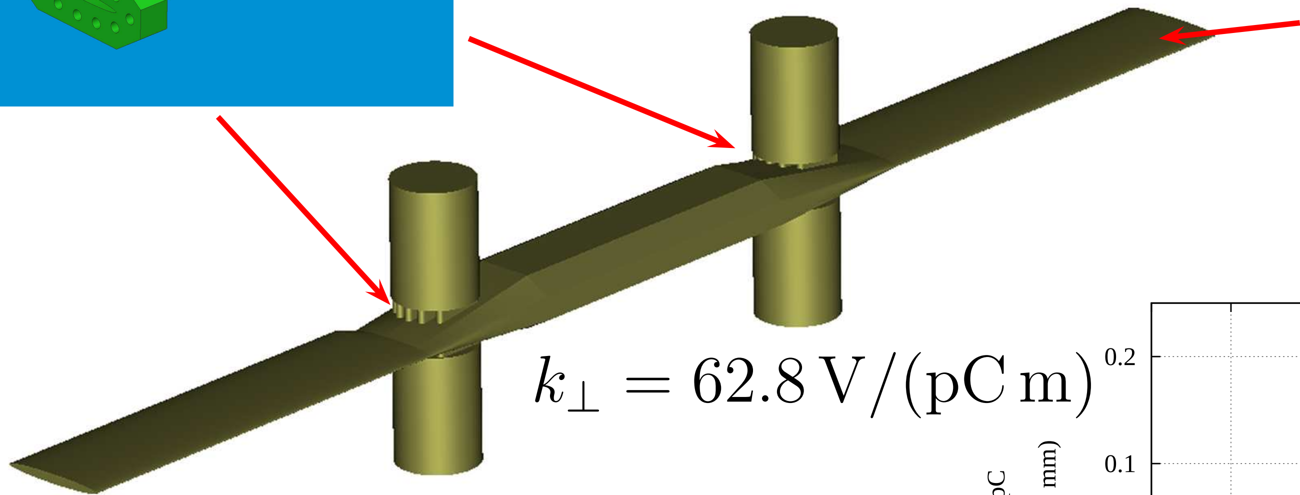
Synchrotron radiation from an undulator

Undulator Chamber Wakefields

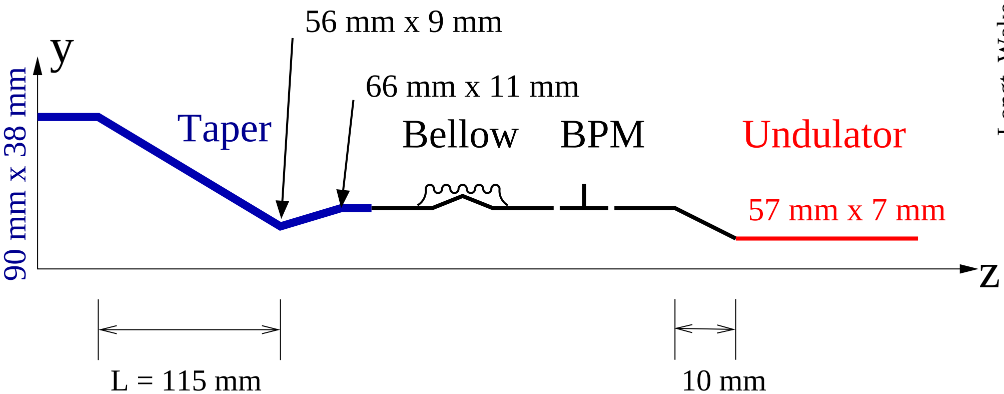


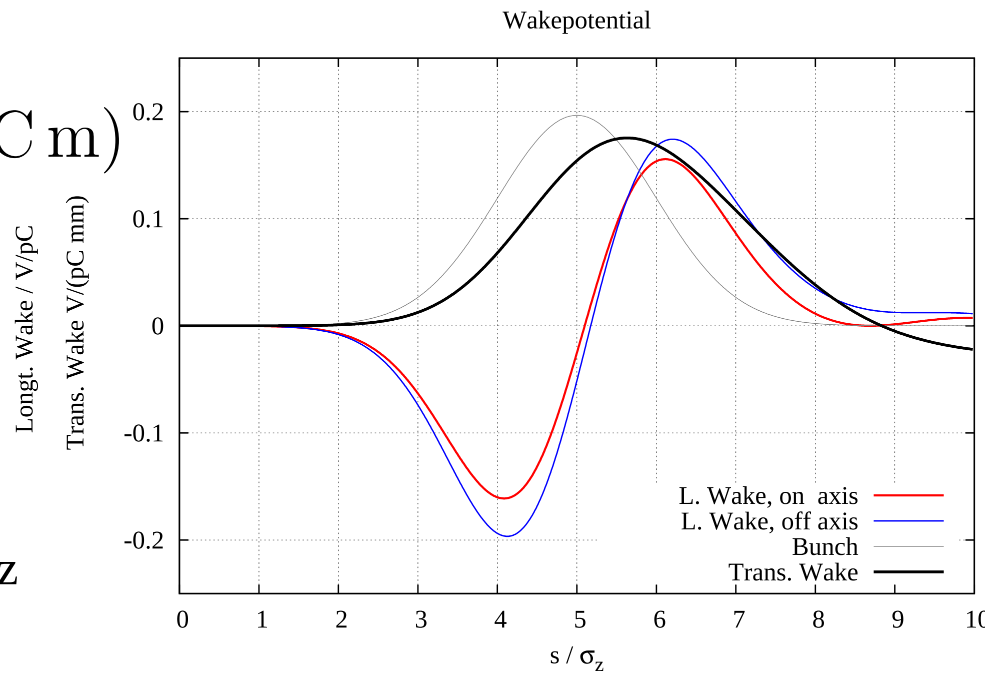
Model of the undulator chamber and the absorber





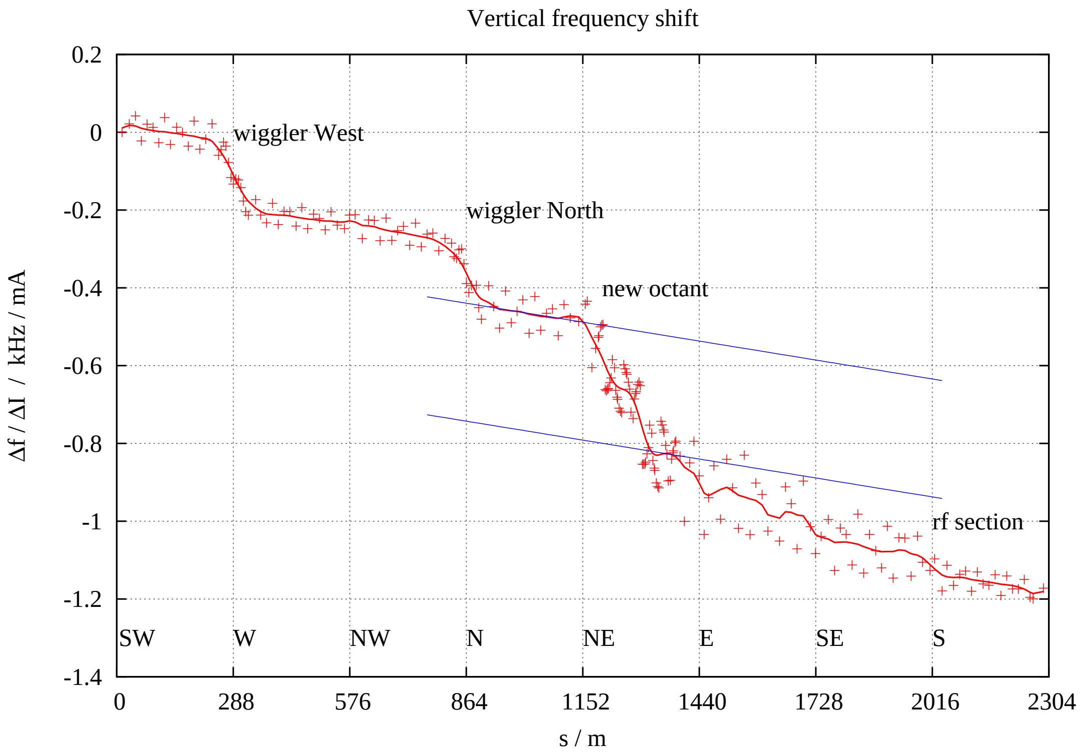
$$k_\perp = 62.8\,\mathrm{V/(pC\,m)}$$



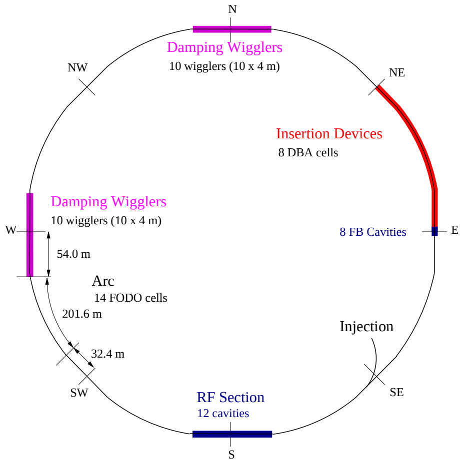


E. Gjonaj, et al. *Wake Computations for Undulator Vacuum Chambers of PETRA III*, PAC 07

Prediction versus Measurement



(ORM Data: J. Keil, DESY)



Betatron oszillations:

$$y = \sqrt{\epsilon_y \, \beta(s)} \, \cos(\phi(s))$$

$$\phi(C) = 2\pi \, Q_y$$

Measurement: phase $\phi(s)$

Orbit Responce Matrix (ORM)
with diffenrent single bunch intensities

- 240 x 0.083 mA = 20 mA
- 10 x 2 mA = 20 mA

$$\frac{\Delta f}{\Delta I} = \frac{1}{2 \, E/e} \sum_n \beta_n \, k_{\perp \, n}$$

New Octant: Intensity dependend phase shift

- -0.265 kHz/mA Prediction
- -0.325 kHz/mA Measurement ($\sim$ 20 % larger) ✔

Acknowledgment

Thank you for your attention !