

WAKE FIELDS AND INSTABILITIES IN LINEAR ACCELERATORS

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Abstract

When a charged particle travels across the vacuum chamber of an accelerator, it induces electromagnetic fields, which are left mainly behind the generating particle. These electromagnetic fields act back on the beam and influence its motion. Such an interaction of the beam with its surroundings results in beam energy losses, alters the shape of the bunches, and shifts the betatron and synchrotron frequencies. At high beam current the fields can even lead to instabilities thus limiting the performance of the accelerator in terms of beam quality and current intensity.

We discuss in this lecture the general features of the electromagnetic fields, introducing the concepts of wake fields and giving few simple examples of them in cylindrical geometry. We then show the effect of the wake fields on the dynamics of a beam in a LINAC, dealing in particular with the beam breakup instability and the way to cure it.

1. INTRODUCTION

Self induced electromagnetic (e.m.) forces in an accelerator, are generated by a charged particle beam which interacts with all the components of the vacuum chamber. These components may have a complex geometry: kickers, bellows, r.f. cavities, diagnostics components, special devices, etc. The study of the fields generally requires of solving the Maxwell's equations in a given structure taking the beam current as source of fields. This could result a quite complicated task, and therefore several dedicated computer codes, used to study and design accelerator devices, which solve the e.m. problem in the frequency or in the time domain, have been developed. These include, for example, CST Studio Suite [1], GDFIDL [2], ACE3P [3], ABCI [4], and others.

In this lecture discuss some general features of the self induced e.m. forces and introduce the concepts of wake fields and coupling impedances [5-12], showing some simple examples in cylindrical geometry. Although the space charge forces have been studied separately [13], they can be seen as a particular case of wake fields.

In the second part of the lecture we study the effects of the wake fields on the dynamics of a beam in a LINAC, such as energy loss and energy spread. We finally deal with the beam breakup (BBU) instability [14], and the way to cure it [15].

2. WAKE FIELDS AND POTENTIALS

2.1 Longitudinal and transverse wake fields

The self induced e.m. fields acting on a particle inside a beam depend on the whole charge distribution. However, if we know the fields in a given structure created by a single charge (i.e. we obtain the Green function of the structure), by using the superposition principle, we can easily reconstruct the fields produced by any charge distribution.

The e.m. fields created by a point charge act back on the charge itself and on any other charge of the beam. Referring to the coordinates' system of Fig. 1 let us call $q_0(s_0, \mathbf{r}_0)$ a charge, which we call source charge, traveling with constant longitudinal velocity $v=c$ (ultra-relativistic limit) along a

trajectory parallel to the axis of a given accelerator structure. Let us consider a test charge q , in a position $(s=s_0-z, \mathbf{r})$, which is moving with the same constant velocity on a parallel trajectory inside the structure.

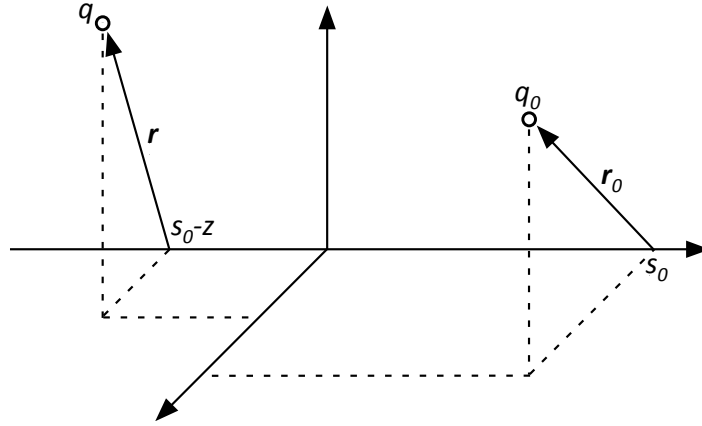


Fig. 1: Reference coordinates' system.

Let \mathbf{E} and \mathbf{B} be the electric and magnetic fields generated by q_0 inside the structure. Since the velocity of both charges is along z , the Lorentz force acting on q has the following components:

$$\mathbf{F} = q \left[E_z \hat{z} + (E_x - vB_y) \hat{x} + (E_y + vB_x) \hat{y} \right] \equiv \mathbf{F}_{\parallel} + \mathbf{F}_{\perp} \quad (1)$$

From the above equation we see that there can be two effects on the test charge: a longitudinal force which changes its energy, and a transverse force, which deflects its trajectory. If we consider a device of length L , the energy change in joule of q due to this force is:

$$U(\mathbf{r}, \mathbf{r}_0, z) = \int_0^L F_{\parallel} ds \quad (2)$$

while the transverse deflecting kick, expressed in [Nm], is:

$$\mathbf{M}(\mathbf{r}, \mathbf{r}_0, z) = \int_0^L \mathbf{F}_{\perp} ds \quad (3)$$

Note that the integration is performed over a given path of the trajectory. The quantities given by eqs. (2) and (3), normalised to the two charges q_0 and q , are called respectively longitudinal and transverse **wake fields**. In many cases, we deal with structures having particular symmetric shapes, generally cylindrical. It is possible to demonstrate that with a multipole expansion of the wake fields, the dominant term in the longitudinal wake field depends only on the distance z between the two charges, while the dominant one in the transverse wake field is still function of the distance z , but it is also linear with the transverse position of the source charge \mathbf{r}_0 . If we then divide the transverse wake field by r_0 we obtain the transverse dipole wake field, that is the transverse wake per unit of transverse displacement, depending only on z :

$$\text{Longitudinal wake field [V/C]}: \quad w_{\parallel}(z) = -\frac{U}{q_0 q} \quad (4)$$

$$\text{Transverse dipole wake field [V/Cm]}: \quad w_{\perp}(z) = \frac{1}{r_0} \frac{\mathbf{M}}{q_0 q} \quad (5)$$

The minus sign in the definition of the longitudinal wake field means that the test charge loses energy when the wake is positive. Positive transverse wake means that the transverse force is defocusing. The wake fields are properties of the vacuum chamber and the beam environment, but they are independent of the beam parameters (bunch size, bunch length, ...).

In order to study the effect of wake fields on the beam dynamics, it is convenient to distinguish between the wake fields that are synchronous with the same bunch that produced them, and influence the particles within the bunch, called short range wake fields, and those that influence the multi-bunch (or multi-turn) beam dynamics, which are generally resonant modes trapped inside a structure and are called long range wake fields.

As a first example of wake fields, let us consider the longitudinal wake field of “space charge”. Even if in the ultra-relativistic limit with $\gamma \rightarrow \infty$, there is no space charge effect, we can still define a wake field by considering a moderately relativistic beam with $\gamma \gg 1$ but not infinite. It turns out that the space charge forces can fit into the definition of wake field, and when that is done, we find that the wake depends on beam properties such as the transverse beam radius a and the beam energy γ . In Appendix 1 we show an example of such an interpretation. Let us consider here a relativistic beam with cylindrical symmetry and uniform transverse distribution of radius a . The longitudinal force acting on a charge q of the beam travelling inside a cylindrical pipe of radius b is given by [13]:

$$F_{//}(r, z) = \frac{-q}{4\pi\epsilon_0\gamma^2} \left(1 - \frac{r^2}{a^2} + 2 \ln \frac{b}{a} \right) \frac{\partial \lambda(z)}{\partial z} \quad (6)$$

with $\lambda(z)$ the longitudinal distribution ($z > 0$ at the bunch head). Note that, since the space charge forces move together with the beam, they are constant along the accelerator if the beam pipe cross section remains constant. We can therefore define the longitudinal wake field per unit length (V/Cm). To get the longitudinal wake field of a piece of pipe, we just multiply by the pipe length. Assuming $r \rightarrow 0$ (particle on axis), and a charge line density given by $\lambda(z) = q_0 \delta(z)$ we obtain:

$$\frac{dw_{//}(z)}{ds} = \frac{1}{4\pi\epsilon_0\gamma^2} \left(1 + 2 \ln \frac{b}{a} \right) \frac{\partial}{\partial z} \delta(z) \quad (7)$$

which has the peculiarity of being also dependent on the beam size a .

Another interesting case is the longitudinal wake potential of a resonant higher order mode (HOM) in an RF cavity, which is an example of long range wake field. When a charge crosses a resonant structure, as an RF cavity, it excites the fundamental mode and higher order modes. Each mode can be treated as an electric RLC circuit loaded by an impulsive current, as shown in Fig. 2.



Fig. 2: RF cavity and the equivalent RLC parallel circuit model driven by a current generator.

Just after the charge passage, the capacitor is charged with a voltage $V(0) = q_0/C$, and the longitudinal electric field is $E_z = V/l$, with l the length of the cavity. The time evolution of the electric field is then governed by the same differential equation of the voltage, which can be written as

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = \frac{1}{C}i \quad (8)$$

The passage of the impulsive current charges only the capacitor, which changes its potential by an amount $V_c(0)$. This potential will oscillate and decay producing a current flow in the resistor and inductance. After the charge passage, for $t > 0$ the potential satisfies the following equation and boundary conditions:

$$\begin{aligned} \ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V &= 0 \\ V(t=0^+) &= \frac{q_0}{C} = V_0 \\ \dot{V}(t=0^+) &= \frac{\dot{q}}{C} = -\frac{I(0^+)}{C} = -\frac{V_0}{RC} \end{aligned} \quad (9)$$

which has the following solution:

$$\begin{aligned} V(t) &= V_0 e^{-\Gamma t} \left[\cos(\bar{\omega}t) - \frac{\Gamma}{\bar{\omega}} \sin(\bar{\omega}t) \right] \\ \bar{\omega}^2 &= \omega_r^2 - \Gamma^2 \end{aligned} \quad (10)$$

where $\omega_r^2 = \frac{1}{LC}$ and $\Gamma = \frac{1}{2RC}$. For the HOM it is also convenient to define the quality factor $Q = \frac{\omega_r}{2\Gamma}$, from which we can write $C = \frac{Q}{R\omega_r}$.

Putting $z=ct$ (z is positive behind the source charge) we obtain the longitudinal wake field shown in Fig 3:

$$w_{||}(z) = \frac{V(z)}{q_0} = \frac{R\omega_r}{Q} e^{-\Gamma z/c} \left[\cos(\bar{\omega}z/c) - \frac{\Gamma}{\bar{\omega}} \sin(\bar{\omega}z/c) \right] \quad (11)$$

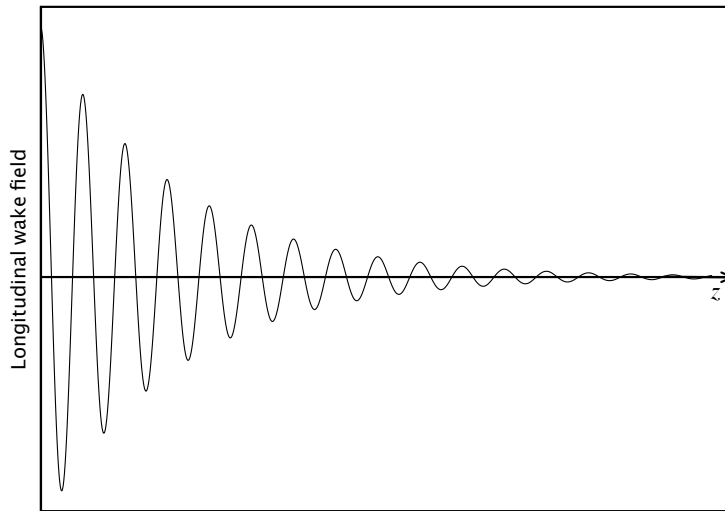


Fig. 3: Qualitative behavior of a resonant mode wake field.

In an analogous way, it is possible to obtain the transverse wake field of a HOM

$$w_{\perp}(z) = \frac{R_{\perp} \omega_r}{Q} e^{-\Gamma z/c} \sin(\bar{\omega} z / c) \quad (12)$$

with R_{\perp} expressed in (Ω/m).

We conclude this section by giving the longitudinal and transverse short range wake fields of a rectangular cell, as that shown in Fig. 4, under the hypothesis that the bunch length is much smaller than the pipe radius b . Its expression can be useful to study the effects of the short range wake fields of an accelerating structure in a LINAC.

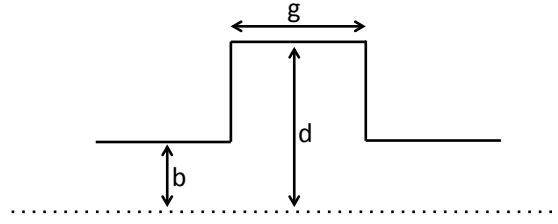


Fig. 4: Geometry of a single cell of a LINAC accelerating structure.

The model supposes each cell as a pill box cavity. When a bunch reaches the edge of the cavity, the electromagnetic field it creates is just the one that would occur when a plane wave passes through a hole; with this hypothesis it is possible to use the classical diffraction theory of optics to calculate the fields [7]. If the condition $g < (d-b)^2/(2\sigma)$ is satisfied, with g the cell gap, d the cell radius and σ the rms bunch length of a Gaussian bunch, then the longitudinal and transverse wake fields can be written respectively:

$$\begin{aligned} w_{||}(z) &= \frac{Z_0 c}{\sqrt{2\pi^2 b}} \sqrt{\frac{g}{z}} \\ w_{\perp}(z) &= \frac{2^{3/2} Z_0 c}{\pi^2 b^3} \sqrt{gz} \end{aligned} \quad (13)$$

For a collection of cavities, eqs. (13) cannot be used because the wake fields, along the cells, do not sum in phase and the result would be an overestimation of the effects. An asymptotic wake field, for a periodic collection of cavities of period p , obtained numerically at SLAC [16] and then fitted to a simple function, is used instead. Such wake fields are thus valid after a certain number of cavities given by:

$$N_{cr} = \frac{b^2}{2g \left(\sigma + \frac{2b}{\gamma} \right)} \quad (14)$$

Under these assumptions, the wake fields of eqs. (13) are modified into

$$\begin{aligned} w_{||}(z) &= \frac{Z_0 c p}{\pi b^2} e^{-\sqrt{z/s_1}} \\ w_{\perp}(z) &= \frac{4Z_0 c p s_2}{\pi b^4} \left[1 - \left(1 + \sqrt{\frac{z}{s_2}} \right) e^{-\sqrt{z/s_2}} \right] \end{aligned} \quad (15)$$

with

$$\begin{aligned} s_1 &= 0.41 \frac{b^{1.8} g^{1.6}}{p^{2.4}} \\ s_2 &= 0.17 \frac{b^{1.79} g^{0.38}}{p^{1.17}} \end{aligned} \quad (16)$$

2.2 Loss factor and beam loading theorem

A useful quantity for the effects of longitudinal wake field on the beam dynamics is the loss factor, defined as the normalised energy lost by the source charge q_0 :

$$k = -\frac{U(z=0)}{q_0^2} \quad (17)$$

For charges travelling with the light velocity, there is the problem that the longitudinal wake field is discontinuous at $z=0$, as shown in Fig. 5, giving an ambiguity for the evaluation of the loss factor. Indeed, when the source charge travels with the light velocity, it leaves the e.m. fields mainly on the back, reason why we call these fields “wake fields”. Any e.m. perturbation produced by the charge cannot overtake the charge itself. This means that the longitudinal wake field vanishes in the region $z < 0$. This property is a consequence of the “*causality principle*”. It is the causality that requires that the longitudinal wake field of a charge travelling with the velocity of light is discontinuous at the origin.

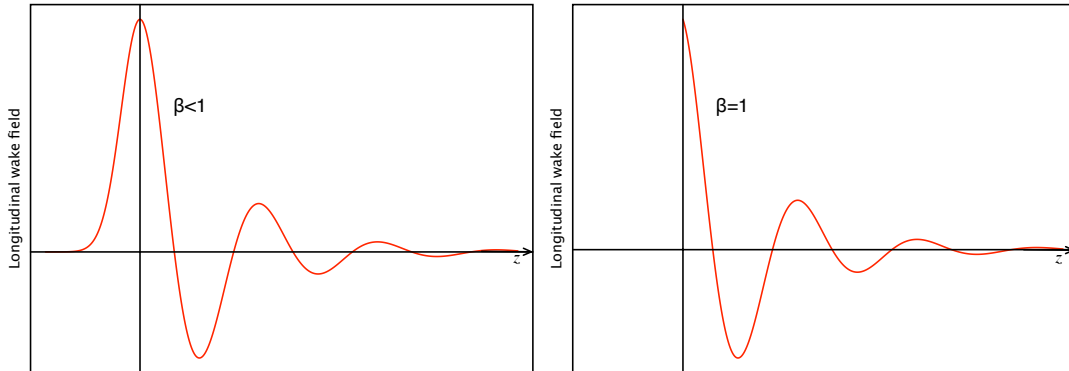


Fig. 5: Examples of longitudinal wake fields: left $\beta < 1$, right $\beta = 1$.

The exact relationship between k and $w_{//}(z \rightarrow 0)$ is, in this case, given by the beam loading theorem [17], which states that

$$k = \frac{w_{//}(z \rightarrow 0)}{2} \quad (18)$$

As example of verification of the beam loading theorem, let us consider the wake field of the resonant mode given by eq. (11). The energy lost by the charge q_0 loading the capacitor is $U = \frac{CV_0^2}{2} = \frac{q_0^2}{2C}$ giving $k = \frac{1}{2C}$, to compare with: $w_{//}(z \rightarrow 0) = \frac{1}{C}$.

2.3 Relationship between transverse and longitudinal forces

Another important feature worth mentioning here is the differential relationship existing between longitudinal and transverse forces and between the corresponding wake fields: the transverse gradient of the longitudinal force/wake is equal to the longitudinal gradient of the transverse force/wake, that is

$$\begin{aligned}\nabla_{\perp} F_{\parallel} &= \frac{\partial}{\partial z} F_{\perp} \\ \nabla_{\perp} w_{\parallel} &= \frac{\partial}{\partial z} w_{\perp}\end{aligned}\quad (19)$$

The above relations are known as “*Panofsky-Wenzel theorem*” [18].

2.4 Coupling impedance

The wake fields are generally used to study the beam dynamics in the time domain. If we take the equations of motion in the frequency domain, we need the Fourier transform of the wake fields. Since these quantities have ohms units they are called *coupling impedances*:

$$\text{Longitudinal impedance } [\Omega]: \quad Z_{\parallel}(\omega) = \frac{1}{v} \int_{-\infty}^{\infty} w_{\parallel}(z) e^{\frac{i\omega z}{v}} dz \quad (20)$$

$$\text{Transverse dipole impedance } [\Omega/\text{m}]: \quad Z_{\perp}(\omega) = -\frac{i}{v} \int_{-\infty}^{\infty} w_{\perp}(z) e^{\frac{i\omega z}{v}} dz \quad (21)$$

The longitudinal coupling impedance of the space charge wake given by eq. (7) in (Ω/m) is:

$$\frac{\partial Z_{\parallel}(\omega)}{\partial s} = \frac{1}{v} \int_{-\infty}^{\infty} \frac{\partial w_{\parallel}(z)}{\partial s} e^{\frac{i\omega z}{v}} dz = \frac{1+2\ln(b/a)}{v4\pi\epsilon_0\gamma^2} \int_{-\infty}^{\infty} \frac{d}{dz} \delta(z) e^{\frac{i\omega z}{v}} dz \quad (22)$$

since $\int_{-\infty}^{\infty} \delta'(z) f(z) dz = f'(0)$, we get:

$$\frac{\partial Z_{\parallel}(\omega)}{\partial s} = \frac{i\omega Z_0}{4\pi c\beta^2\gamma^2} \left(1 + 2\ln\frac{b}{a}\right) \quad (23)$$

The longitudinal coupling impedance of a resonant HOM, corresponding to the Fourier transform of eq. (11) is given by:

$$Z_{\parallel}(\omega) = \frac{R}{1+iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)} \quad (24)$$

where R is also called the shunt impedance of the longitudinal HOM. Note that the loss factor can be written as $k = \frac{\omega_r R}{2Q}$.

The transverse impedance obtained from eq. (12) is given by:

$$Z_{\perp}(\omega) = \frac{\bar{\omega}}{\omega} \frac{R_{\perp}}{1+iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)} \quad (25)$$

with R_{\perp} called the transverse shunt impedance.

2.5 Wake potential and energy loss of a bunched distribution

When we have a bunch with total charge q_0 and longitudinal distribution $\lambda(z)$, such that $q_0 = \int_{-\infty}^{\infty} \lambda(z') dz'$, we can obtain the amount of energy lost or gained by a single charge q in the beam by using the superposition principle.

To this end we calculate the effect on the charge by the whole bunch, as shown in Fig. 6, with the superposition principle, which gives the convolution integral:

$$U(z) = -q \int_{-\infty}^{\infty} w_{\parallel}(z'-z) \lambda(z') dz' \quad (26)$$

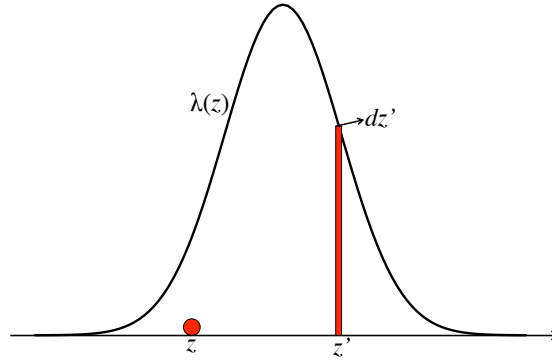


Fig. 6: Convolution integral for a charge distribution to obtain the energy loss of a particle due to the whole bunch.

Eq. (26) permits to define the longitudinal wake potential of a distribution:

$$W_{\parallel}(z) = -\frac{U(z)}{qq_0} = \frac{1}{q_0} \int_{-\infty}^{\infty} w_{\parallel}(z'-z) \lambda(z') dz' \quad (27)$$

The total energy lost by the bunch is computed summing up the energy loss of all particles:

$$U_{bunch} = \frac{1}{q} \int_{-\infty}^{\infty} U(z') \lambda(z') dz' = -q_0 \int_{-\infty}^{\infty} W_{\parallel}(z') \lambda(z') dz' \quad (28)$$

3. WAKE FIELDS EFFECTS IN LINEAR ACCELERATORS

3.1 Energy spread

The longitudinal wake forces change the energy of individual particles depending on their position in the beam, as given by eq. (26). As consequence the short range wake field can induce an energy spread in the beam.

For example the energy spread induced by the space charge force in a Gaussian bunch is given by:

$$\frac{dU(z)}{ds} = -q \int_{-\infty}^{\infty} \frac{dw_{\parallel}(z'-z)}{ds} \lambda(z') dz' = \frac{qq_0}{4\pi\epsilon_0\gamma^2 \sqrt{2\pi}\sigma_z^3} \left(1 + 2 \ln \frac{b}{a}\right) z e^{-(z^2/2\sigma_z^2)} \quad (29)$$

The bunch head gains energy ($z > 0$), while the tail loses energy. The total energy lost by the bunch U_{bunch} is zero.

In a similar way one can show that the energy loss induced by a resonant HOM inside a rectangular uniform bunch of length l_0 when $\Gamma \ll \bar{\omega}$ is given by:

$$U(z) = \frac{-qq_0 R \omega_r}{2Q} \frac{\sin\left[\frac{\omega_r}{c}\left(\frac{l_0}{2} - z\right)\right]}{\left(\frac{\omega_r l_0}{2c}\right)} \quad (30)$$

and the total energy loss obtained with eq. (28) is

$$U_{bunch} = -\frac{2q_0^2 R c^2}{\omega_r l_0^2 Q} \sin^2\left(\frac{\omega_r l_0}{2c}\right) \quad (31)$$

3.2 Single bunch beam break-up: two-particle model

A beam injected off-center in a LINAC, because for example of focusing quadrupoles misalignment, executes betatron oscillations. The bunch displacement produces a transverse wake field in all the devices crossed during the flight, which deflects the trailing charges (single bunch beam break-up), or other bunches following the first one in a multibunch regime (multibunch beam break-up). The first observation of the BBU was made at SLAC back in 1966 [19].

In order to understand the effect, we consider, as first example, a simple model with only two charges $q_1 = q_0/2$ (leading = half bunch) and $q_2 = q$ (trailing = single charge) travelling with $\beta=1$.

The leading charge executes free betatron oscillations of the kind:

$$y_1(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right) \quad (32)$$

The trailing charge, at a distance z behind, over a length L_w experiences an average deflecting force proportional to the displacement y_1 , and dependent on the distance z , which, from the definition of the transverse dipole wake field is:

$$\langle F_y(z, y_1) \rangle = \frac{qq_0}{2L_w} w_{\perp}(z) y_1(s) \quad (33)$$

Notice that L_w is the length of the device for which the transverse wake has been computed. For example, in the case of a cavity cell L_w is the length of the cell. This force drives the motion of the trailing charge:

$$y_2'' + \left(\frac{\omega_y}{c}\right)^2 y_2 = \frac{qq_0 w_{\perp}(z)}{2E_0 L_w} \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right) \quad (34)$$

This is the typical equation of a resonator driven at the resonant frequency.

The solution is given by the superposition of the “free” oscillations and “forced” ones, which, being driven at the resonant frequency, grow linearly with s , as shown in Fig. 7:

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y}{c} s\right) + y_2^{forced} \quad (35)$$

$$y_2^{forced} = \frac{cq q_0 w_{\perp}(z)s}{4\omega_y E_o L_w} \hat{y}_1 \sin\left(\frac{\omega_y}{c} s\right) \quad (36)$$

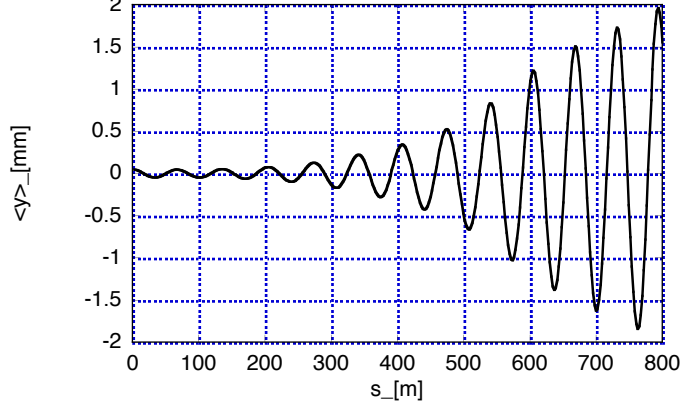


Fig. 7: HOMDYN [20] simulation of a typical BBU instability, 50 μm initial offset, no energy spread.

At the end of the LINAC of length L_L , the oscillation amplitude is grown by ($\hat{y}_1 = \hat{y}_2$):

$$\left(\frac{\Delta \hat{y}_2}{\hat{y}_2}\right)_{\max} = \frac{c N e w_{\perp}(z) L_L}{4 \omega_y (E_o / e) L_w} \quad (37)$$

If the transverse wake is given per cell, the relative displacement of the tail with respect to the head of the bunch depends on the number of cells. It depends, of course, also on the focusing strength through the betatron frequency ω_y .

3.3 BNS damping

The BBU instability is quite harmful and hard to take under control even at high energy with a strong focusing, and after a careful injection and steering. A simple method to cure it has been proposed observing that the strong oscillation amplitude of the bunch tail is mainly due to the “resonant” driving head. If the tail and the head move with a different frequency, this effect can be significantly removed [15].

Let us assume that the tail oscillates with a frequency $\omega_y + \Delta\omega_y$, so that eq. (34) becomes:

$$y_2^n + \left(\frac{\omega_y + \Delta\omega_y}{c}\right)^2 y_2 = \frac{N e^2 w_{\perp}(z)}{2 E_o L_w} \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right) \quad (38)$$

the solution of which is:

$$y_2(s) = \hat{y}_2 \cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) - \frac{c^2 N e^2 w_{\perp}(z)}{4 \omega_y \Delta\omega_y E_o L_w} \hat{y}_1 \left[\cos\left(\frac{\omega_y + \Delta\omega_y}{c} s\right) - \cos\left(\frac{\omega_y}{c} s\right) \right] \quad (39)$$

In this case we observe that the amplitude of the oscillation is limited and does not grow up linearly with s any more. Furthermore, by a suitable choice of $\Delta\omega_y$, it is possible to fully depress the oscillations of the tail. Indeed, by setting:

$$\Delta\omega_y = \frac{c^2 N e^2 w_{\perp}(z)}{4\omega_y E_o L_w} \quad (40)$$

if $\hat{y}_2 = \hat{y}_1$, from eq. (39) we get:

$$y_2(s) = \hat{y}_1 \cos\left(\frac{\omega_y}{c} s\right) \quad (41)$$

that is the tail oscillates with the same amplitude of the head and with the same betatron frequency. This method of curing the single bunch BBU instability is called BNS damping by the names of the authors Balakin, Novokhatsky, and Smirnov who proposed it [15].

In order to have the BNS damping, eq. (40) imposes an extra focusing at the tail, which must have a higher betatron frequency than the head. This extra focusing can be obtained by: 1) using a RFQ, where head and tail see a different focusing strength, 2) create a correlated energy spread across the bunch which, because of the chromaticity, induces a spread in the betatron frequency. An energy spread correlated with the position is attainable with the external accelerating voltage or with the wake fields.

In Fig. 8 we show the betatron oscillation corresponding to Fig. 7 but with a 2% of energy spread.

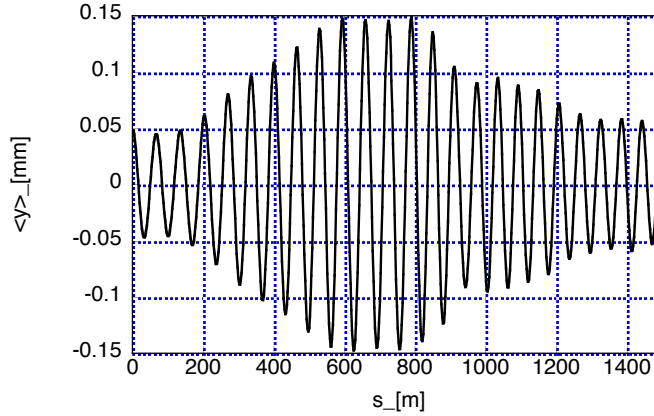


Fig. 8: HOMDYN simulation of a typical BNS damping, 50 μm initial offset, 2% energy spread.

3.4 Single bunch beam break-up: general distribution

To extend the analysis we did in section 3.2 to a particle distribution, we write the transverse equation of motion of a single charge q with the inclusion of the transverse wake field effects as [14]:

$$\frac{\partial}{\partial s} \left[\gamma(s) \frac{\partial y(z, s)}{\partial s} \right] + k_y^2(s) \gamma(s) y(z, s) = \frac{q}{m_0 c^2 L_w} \int_z^{\infty} y(s, z') w_{\perp}(z' - z) \lambda(z') dz' \quad (42)$$

where $\gamma(s)$ is the relativistic parameter, which varies along the LINAC, and $1/k_y(s)$ the betatron function. We remember that the integral of the longitudinal distribution function $\lambda(z)$ is the total charge of the bunch q_0 .

The solution of the equation in the general case is unknown. We can however apply a perturbation method to obtain the solution at any order in the wake field intensity. Indeed we write:

$$y(z, s) = \sum_n y^{(n)}(z, s) \quad (43)$$

with n representing the n^{th} order solution. The first order solution is found without the wake field effect from the equation

$$\frac{\partial}{\partial s} \left[\gamma(s) \frac{\partial y^{(0)}(z, s)}{\partial s} \right] + k_y^2(s) \gamma(s) y^{(0)}(z, s) = 0 \quad (44)$$

It is important to notice that the above equation does not depend on z any more. This means that the bunch distribution remains constant along the structure.

If the s -dependence of $\gamma(s)$ and $k_y^2(s)\gamma(s)$ is moderate, we can use the WKB approximation [5], and the solution of the above equation with the starting conditions $y(0) = \hat{y}$, $y'(0) = 0$ is

$$y^{(0)}(s) = \sqrt{\frac{\gamma_0 k_{y0}}{\gamma(s) k_y(s)}} \hat{y} \cos[\psi(s)] \quad (45)$$

where

$$\psi(s) = \int_0^s k_y(s') ds' \quad (46)$$

Eq. (45) represents the unperturbed transverse motion of the bunch in a LINAC.

The differential equation of the second order solution is obtained by substituting the first order solution (45) in the right side of eq. (42) thus giving

$$\frac{\partial}{\partial s} \left[\gamma(s) \frac{\partial y^{(1)}(z, s)}{\partial s} \right] + k_y^2(s) \gamma(s) y^{(1)}(z, s) = \frac{q}{m_0 c^2 L_w} y^{(0)}(s) \int_z^\infty w_\perp(z' - z) \lambda(z') dz' \quad (47)$$

We are interested in the forced solution of the above equation that can be written in the form

$$y^{(1)}(z, s) = \hat{y} \frac{q}{m_0 c^2 L_w} \sqrt{\frac{\gamma_0 k_{y0}}{\gamma(s) k_y(s)}} G(s) \int_z^\infty w_\perp(z' - z) \lambda(z') dz' \quad (48)$$

where

$$\begin{aligned} G(s) &= \int_0^s \frac{1}{\gamma(s') k_y(s')} \sin[\psi(s) - \psi(s')] \cos[\psi(s')] ds' = \\ &= \frac{1}{2} \int_0^s \frac{\sin[\psi(s) - 2\psi(s')]}{\gamma(s') k_y(s')} ds' + \frac{1}{2} \sin[\psi(s)] \int_0^s \frac{1}{\gamma(s') k_y(s')} ds' \end{aligned} \quad (49)$$

The first integral undergoes several oscillations with s and, if $\gamma(s)$ and $k_y(s)$ do not vary much, it is negligible, so that we can finally write

$$y^{(1)}(z, s) = \hat{y} \frac{q}{2m_0 c^2 L_w} \sqrt{\frac{\gamma_0 k_{y0}}{\gamma(s) k_y(s)}} \sin[\psi(s)] \int_0^s \frac{ds'}{\gamma(s') k_y(s')} \int_z^\infty w_\perp(z' - z) \lambda(z') dz' \quad (50)$$

Note that the last integral in the above equation is proportional to the transverse wake potential produced by the whole bunch, defined in a similar way of eq. (27). This solution can then be substituted again in the right side of eq. (42) to obtain a third order solution and so on. If we consider constant $\gamma(s)$ and $k_y(s)$, eq. (50) gives the same result of the two-particle model of eq. (36) when we substitute $\lambda(z)$ with $q_0/2$ representing the leading half bunch affecting a trailing charge q .

If the BBU effect is strong, it is necessary to include higher order terms in the perturbation expansion. Under the assumption of:

- rectangular bunch distribution $\lambda(z) = q_0 / l_0$, $-l_0 / 2 < z < l_0 / 2$, l_0 bunch length;
- monoenergetic beam;
- constant acceleration gradient $dE_0 / ds = \text{const}$;
- constant beta function;
- linear wake function inside the bunch $w_{\perp}(z) = w_{\perp 0} z / l_0$;

the sum of eq. (43) can be written in terms of powers of the adimensional parameter η also called BBU strength

$$\eta = \frac{qq_0}{k_y (dE_0 / ds)} \frac{w_{\perp 0}}{L_w} \ln \left(\frac{\gamma_f}{\gamma_i} \right) \quad (51)$$

with γ_i and γ_f respectively the initial and final relativistic parameter.

By using the method of the steeping descents [8], it is possible to obtain the asymptotic expression of $y(z,s)$ thus finding, at the end of the LINAC,

$$y(L_L) = y_m \sqrt{\frac{\gamma_i}{6\pi\gamma_f}} \eta^{-1/6} \exp \left[\frac{3\sqrt{3}}{4} \eta^{1/3} \right] \cos \left[k_y L_L - \frac{3}{4} \eta^{1/3} + \frac{\pi}{12} \right] \quad (52)$$

that, differently from the two-particle model and from the first order solution, gives a tail displacement growing exponential with η .

3.5 Multi-bunch beam break-up

We have seen in the previous sections that when a bunch passes off-axis (due, for example, to betatron oscillations) in an axis-symmetric accelerating structure, it excites transverse wake fields which may cause the tail of the bunch to oscillate with increasing amplitude as the bunch goes along the LINAC. In the same way, the whole bunch may excite deflecting trapped modes in the RF cavities of the LINAC that may cause trailing bunches to be deflected, whether they are on axis or not. These angular deflections are transformed into transverse displacements through the transfer matrices of the focusing system and the displaced bunches will themselves create similar wake fields in the downstream accelerating structures of a LINAC. The subsequent bunches will be further deflected leading to a beam blow-up. Due to the long range wake fields, there is a coupling in the motion of the bunches that are more and more deflected as they proceed along the LINAC in a process that is called multi-bunch BBU. Even if the bunches are not lost, the transverse beam emittance can be greatly increased, leading to a significant luminosity reduction.

We summarize here the analytical study of multi-bunch BBU performed with the formalism used in [14]. All the bunches are considered to be rigid macro-particles, like delta-functions, separated by period T , and we assume all bunches injected with the same initial offset x_0 . We consider the transverse equation of motion of a bunch as a whole, ignoring internal structures; the beam is

therefore made of a train of bunches with same charge (Q_b) evenly spaced by period T , which is an integer number of the RF period of the accelerating mode.

We also consider all the cells of the LINAC accelerating structure identical and with the same dipole trapped mode in each cell of length L_w . Rigorously the analytical approach requires that many betatron oscillations are performed in the LINAC and the BBU remains moderate within a betatron oscillation. Moreover, the theory is valid if the beam energy does not change too much in a betatron wavelength. This last hypothesis is also called adiabatic acceleration.

The transverse wake field force experienced by the k^{th} bunch, spaced kT from the first bunch, depends on the transverse wake field generated by the preceding bunches (and thus by their transverse displacement). The dipole long range wake field is produced by a high order deflecting mode, identical in all the cavities of the structure, and it is described in terms of its resonant frequency ω_r , the quality factor Q and the dipole shunt resistance R_{\perp} (expressed in ohm/meter).

The equations of motion are then written in terms of the Z-transform [21] since the displacement $x(kT, s)$ of the k^{th} bunch at the position s is a discrete function of time. The solution can be retrieved with a perturbation method, which considers its expansion into a series of the driving wake field force.

The 0^{th} order solution is given for a vanishing driving force, i.e. a pure betatron oscillation (unperturbed motion). It represents the motion of the first bunch, which is not affected by any wake field because of the causality principle (the wake field cannot travel ahead of the bunch itself). The n^{th} order solution is driven by the wake field excited by the solution of the order $n-1$. Thus the 1^{st} order solution is computed from the motion of the first bunch and it affects all the bunches, except the first one; it means that the n^{th} order solution affects only bunches of index larger than n . Therefore the summation of the series can be stopped at the M^{th} order of a train of M bunches. The n^{th} order solution in the Z-domain can be written as [14]

$$x_n(z, s) = \sqrt{\frac{\gamma_0 k_{y0}}{\gamma(s) k_y(s)}} x_0 e^{i\psi(s)} \frac{a^n(s)}{i^n n!} G_n(z) \quad (52)$$

where $a(s)$ is the so called dimensionless BBU strength given, in case of constant $k_y(s)$, by

$$a(s) = \frac{Q_b}{2k_{y0} G} \omega_r \frac{R_{\perp}}{L_w Q} \ln \left[\frac{\gamma(s)}{\gamma_0} \right] \quad (53)$$

with G is the accelerating gradient (in V/m), and

$$G_n(z) = \frac{z}{z-1} \tilde{w}_{\perp}^n(z) \quad (54)$$

with

$$\tilde{w}_{\perp}(z) = \frac{1}{2i} \left(\frac{z}{z-z_1} - \frac{z}{z-z_2} \right) \quad (55)$$

and

$$z_{1,2} = e^{\frac{T\omega_r}{2Q}} e^{\pm i\omega_r T} \quad (56)$$

The inverse Z-transform of $x_n(z, s)$, that is $x_n(kT, s)$, can then be summed to get the transverse displacement of the k^{th} bunch as

$$x(kT, s) = \sum_{n=0}^{\infty} x_n(kT, s) \quad (57)$$

We remember that the sum can be stopped at the M^{th} term for a beam containing M bunches.

For $a(s) \ll 1$ the series expansion can be stopped at the first order term, while, if the BBU strength parameter a is moderate, it is sufficient to keep only few terms of the summation.

In the z -domain the n^{th} order solution, given by equation (52), has been determined analytically, and the same is possible with its infinite sum, but its inverse z -transform (57) is, in general, not possible to write in a closed analytical form. It is however possible to compute the exact solution for the n^{th} bunch as a sum of n terms if the BBU instability is moderate in a betatron period. Moreover, it is possible to use an asymptotic technique, valid when the blow-up is strong, to have an expression of the transverse displacement that puts in evidence the main parameters playing an important role in the instability.

The asymptotic transverse displacement of the k^{th} bunch, expressed in terms of the oscillation amplitude only, is [14]

$$x(kT, s) = x_{\infty}(s) + \frac{\sqrt{\frac{\gamma_0 k_{y0}}{\gamma(s) k_y(s)}} x_0 e^{-k \frac{T\omega_r}{2Q}}}{\sqrt{2\pi} \left(2e^{\frac{T\omega_r}{2Q}}\right)^{1/2} \left[\cosh\left(\frac{T\omega_r}{2Q}\right) - \cos(\omega_r T)\right]^{1/2}} \frac{[2a(s)k]^{1/4}}{2k} e^{[2a(s)k]^{1/2}} \quad (58)$$

where $x_{\infty}(s)$ is the steady state solution that is reached when long (rigorously infinite) train of bunches are accelerated.

In Fig. 9 we show a comparison between the analytical solution obtained by numerically solving eq. (57) and a simple tracking code that considers the bunches in the train as rigid macroparticles, but which can also take into account the contribution of several resonant modes, and different initial offsets and displacements of the bunches. The parameters used for the calculations are given in table 1. They refer to a C-band LINAC with the BBU effect produced by a HOM. In the vertical axis the normalized transverse position, evaluated at the exit of the LINAC, is defined as:

$$\frac{x(kT, s)}{x_0} \sqrt{\frac{\gamma(s) k_y(s)}{\gamma_0 k_{y0}}} \quad (58)$$

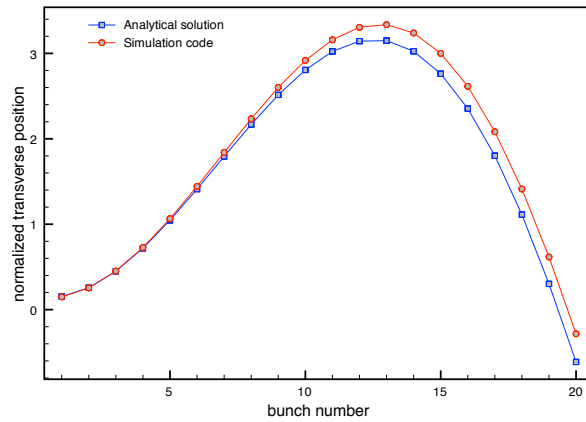


Fig. 9: Normalized transverse position as a function of the bunch number: comparison between the analytical solution and a tracking code.

Linac length	30 m
Initial energy	80 MeV
Energy gradient	30 MeV/m
Betatron function $1/k_y$	1 m
Bunch spacing T	15 ns
Bunch charge	1 nC
HOM resonant frequency f_r	8.4 GHz
HOM transverse impedance R_{\perp}	50 M Ω /m
HOM quality factor	11000
Cell length	17.5 cm

Table1: Beam parameters used for comparing the analytical solution of multi-bunch BBU with the results of a tracking code.

From equation (52) we see that one possible way to reduce the BBU instability is to act on the dimensionless BBU strength given by equation (53). For example we can reduce the bunch charge Q_b or the betatron function, i.e. increase the focusing strength. A better approach is to remove the source of the instability by damping the transverse dipole mode, for example with an improved electromagnetic design of the accelerating cells.

The other main approach to the BBU instability suppression is to detune the cell frequencies in order to introduce a spread in the resonance frequency of the dangerous mode so that it will no longer be excited coherently by the beam. Indeed by properly detuning each cell, a damping of the BBU instability is produced by a decoherence of the various cell wake fields. It has been demonstrated [22] that a Gaussian distribution of the cell frequencies, which provides a rapid drop in the wake field for a given total frequency spread, would be optimal. The analytical approach to determine the effectiveness of this detuning technique for the BBU multi-bunch instability can be found in ref [14], where it is also shown that the damping increases with the amplitude of the frequency spread.

REFERENCES

- [1] CST AG, Darmstadt, Germany, www.cst.com.
- [2] W. Bruns, "GdfidL: A Finite Difference Program with Reduced Memory and CPU Usage", 1997 Particle Accelerator Conference, Vancouver, B.C., Canada, (1997), pp. 2651-2653.
- [3] C. Ng, et al., "State of the Art in EM Field Computation", Proceedings of EPAC 2006, Edinburgh, Scotland, pp. 2763-2767.
- [4] Y. H. Chin, KEK Report No. 2005-06, KEK, Tsukuba, Japan, 2005.
- [5] K. Yokoya, DESY 86-084, 1986.
- [6] K. Bane, SLAC-PUB-4169 (1986).
- [7] K. Bane and M. Sands, SLAC-PUB-4441 (1987).
- [8] A. Chao, "Physics of Collective Beam Instabilities in High Energy Accelerators", John Wiley and Sons Inc., (1993).
- [9] L. Palumbo, V. G. Vaccaro, M. Zobov, "Wake Fields and Impedance", CAS Accelerator School, CERN 95-06 (1995).
- [10] B. W. Zotter and S. A. Kheifets, "Impedances and Wakes in High-Energy Particle Accelerators", (World Scientific, Singapore, 1998).
- [11] G. V. Stupakov, "Wake and Impedance", SLAC-PUB-8683 (2000).
- [12] K. Bane, "Emittance Control For Very Short Bunches", Proc of EPAC 2004, Lucerne, Switzerland.
- [13] M. Ferrario, M. Migliorati, L. Palumbo, "Space Charge Effects", these proceedings.
- [14] A. Mosnier, "Instabilities in linacs", CAS Accelerator School, CERN 95-06 (1995), Vol. I, pp. 459-514.
- [15] V. E. Balakin, A.V. Novokhatsky, V. P. Smirnov, Proc. 12th Int. Conf. on High Energy Accelerators, Batavia, 1983.
- [16] K. Bane, SLAC-PUB 9663, (2003).
- [17] P.B. Wilson, SLAC PUB 2884 (1982).
- [18] W.K.H. Panofsky, W.A. Wenzel, "Transverse Deflection of Charged particles in Radio-Frequency Fields", Rev. Sci. Instrument, Vol 27 (1956).
- [19] H. Altenmueller, et al., "Beam Break-Up experiments at SLAC", 1966 Linear Accelerator Conference, Los Alamos, October 1966, also SLAC-PUB-224 October 1966.
- [20] M. Ferrario, V. Fusco, B. Spataro, M. Migliorati, L. Palumbo, "Wake Fields Effects In The Sparc Photoinjector", Proc of EPAC 2004, Lucerne, Switzerland.
- [21] See, e.g., E. I. Jury, "Theory and application of the Z-transform method", John Wiley & Sons, New York (1964).
- [22] K. L. F. Bane, R. L. Gluckstern, SLAC-PUB-5783 (1992).
- [23] N. A. Vinokurov, "Space Charge", in High Quality Beams, ed. By S. I. Kurokawa et al., 2001 American Institute of Physics.

APPENDIX 1 – POWER RADIATED BY A BUNCH PASSING THROUGH A TAPER

In the case of uniform charge distribution, and $\gamma \rightarrow \infty$, the electric field lines of a beam passing inside a perfectly conducting circular pipe are perpendicular to the direction of motion and travel together with the charge [9], as shown in Fig. A1. In other words, the field map does not change during the charge flight, as long as the trajectory is parallel to the pipe axis. Under this condition the transverse fields intensity can be computed like in the static case, applying the Gauss's and Ampere's laws:

$$\int_S \epsilon_0 \mathbf{E} \cdot \mathbf{n} dS = \int_V \rho dV, \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} dS \quad (\text{A1})$$

Let us consider a cylindrical bunch of radius a and current I , with uniform charge density $\rho = \frac{I}{\pi a^2 v}$ and current density $J = \frac{I}{\pi a^2}$, propagating with relativistic speed $v = \beta c$ along the axis of a cylindrical perfectly conducting pipe of radius b , as shown in Fig. A1.

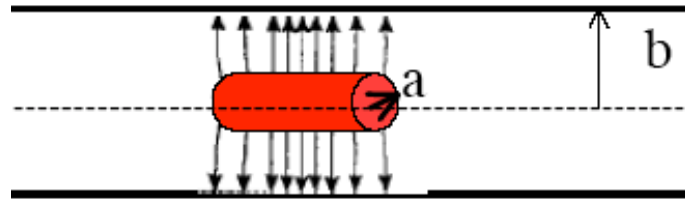


Fig. A1: Cylindrical bunch of radius a propagating inside a cylindrical perfectly conducting pipe of radius b .

By applying the relations (A1), one can obtain for the radial component of the electric field:

$$E_r = \frac{I}{2\pi\epsilon_0 a^2 v} r \quad \text{for } r \leq a$$

$$E_r = \frac{I}{2\pi\epsilon_0 v} \frac{1}{r} \quad \text{for } r > a$$

and the relation $B_\theta = \frac{\beta}{c} E_r$ holds.

The electrostatic potential satisfying the boundary condition $\varphi(b) = 0$ is given by:

$$\varphi(r, z) = \int_r^b E_r(r', z) dr' = \begin{cases} \frac{I}{4\pi\epsilon_0 v} \left(1 + 2 \ln \frac{b}{a} - \frac{r^2}{a^2} \right) & \text{for } r \leq a \\ \frac{I}{2\pi\epsilon_0 v} \ln \frac{b}{r} & \text{for } a \leq r \leq b \end{cases}$$

How can a perturbation of the boundary conditions affect the beam dynamics? Let us consider the following example: a smooth transition of length L (taper) from a beam pipe of radius b to a larger beam pipe of radius d is experienced by the beam [9]. To satisfy the boundary condition of a perfectly conducting pipe also in the tapered region the field lines are bent as shown in Fig. A2. Therefore there must be a longitudinal electric field $E_z(r, z)$ in the transition region.

A test particle moving outside the beam charge distribution will experience along the transition of length L a voltage difference given by [21]:

$$V = - \int_z^{z+L} E_z(r, z') dz' = - [\varphi(r, z+L) - \varphi(r, z)] = - \frac{I}{2\pi\epsilon_0 v} \ln \frac{d}{b}$$

that is decelerating if $d > b$. The power lost by the beam in order to sustain the induced voltage is given by:

$$P_{lost} = VI = \frac{I^2}{2\pi\epsilon_0 v} \ln \frac{d}{b} \quad (\text{A2})$$

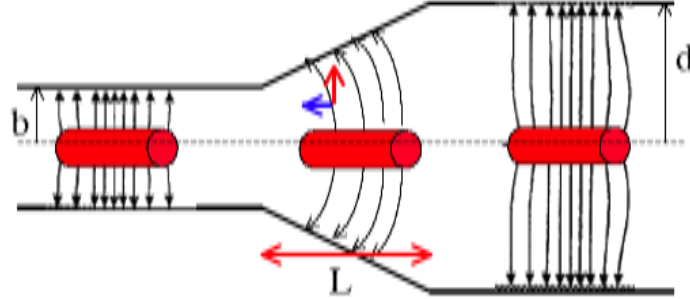


Fig. A2: Smooth transition of length L (Taper) from a beam pipe of radius b to a larger beam pipe of radius d .

It means that for $d > b$ the power is deposited into the energy of the fields: moving from left to right of the transition the beam induces the fields in the additional space around the bunch (i.e. in the region $b < r < d$, $0 < z < l_0$) at the expenses of the only available energy source that is the kinetic energy of the beam itself.

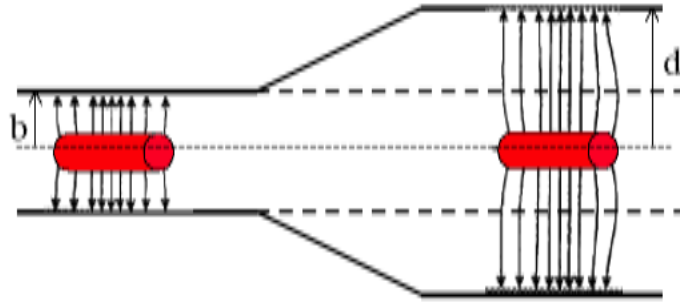


Fig. A3: During the beam propagation in the taper additional e.m. power flow is required to fill up the new available space.

To verify such interpretation let us now compute the electromagnetic power radiated by the beam to fill up the additional space available around the bunch as shown in Fig A3. Integrating the Poynting vector through the surface $\Delta S = \pi(d^2 - b^2)$ representing the additional power passing through the right part of the beam pipe, one obtains:

$$P_{em} = \int_{\Delta S} \left(\frac{1}{\mu} \mathbf{E} \times \mathbf{B} \right) \cdot \mathbf{n} dS = \int_b^d \frac{E_r B_\theta}{\mu} 2\pi r dr = \frac{I^2}{2\pi\epsilon_0 v} \ln \frac{d}{b}$$

that is exactly the same expression of eq. (A2). Notice that if $d < b$ the beam gains energy. If $d \rightarrow \infty$ the power goes to infinity. Such an unphysical result is nevertheless consistent with the original assumption of an infinite energy beam ($\gamma \rightarrow \infty$).