Exercise 1 CAS – Geneva – 2015

$Q_x \simeq 18,7$ $\rightarrow$ $\bar{\beta}_x = \frac{R_{acc}}{Q_x} = \frac{172}{18,7} \simeq 9 \text{ m}$

$Q_y \simeq 18,8$ $\quad \bar{\beta}_y \simeq 9 \text{ m}$

$$N_{tot} = \frac{B_f}{f} \frac{(2\pi)^2 \varepsilon_0 m A \gamma^3 v^2}{e^2 Z^2} \tilde{\varepsilon}_x \left(1 + \sqrt{\frac{\bar{\beta}_y \tilde{\varepsilon}_y}{\bar{\beta}_x \tilde{\varepsilon}_x}}\right) |\Delta Q_{xf}| \quad (1)$$

Acceptance $A_x = 100$ mm-mrad
$A_y = 50$ mm-mrad

for Gaussian $3\tilde{\sigma}_x$ of the tails take 99,7% of particles $\rightarrow \tilde{\varepsilon}_x = \frac{A_x}{10}$, $\rightarrow \tilde{\varepsilon}_y = \frac{A_y}{10}$ (NO SAFETY MARGIN)

$\tilde{\varepsilon}_x \simeq 10$ mm-mrad

$\tilde{\varepsilon}_y \simeq 5$ mm-mrad

Energy $\rightarrow$ $E_k = 200$ MeV/u $\rightarrow$ $m\gamma c^2 = mc^2 + E_k = E_T$

$\rightarrow \beta = \sqrt{1 - \frac{1}{\gamma^2}}$ $\quad \gamma = E_T / mc^2$

$\gamma = 1,21 \quad \beta = 0,567$

Now for a Gaussian $f = 1/2$, we take $\Delta Q_{xf} = 0,25$

For a bunched beam we take $B_f = 1/3$:
this value of B_f is to reach maximum intensity.
Therefore Eq. 1 yields

$$x-\text{plane} \quad N_{tot} = 10^{12}$$
$$y-\text{plane} \quad N_{tot} = 7\times10^{11}$$

We now take into account the occupation $8/10$ $\Rightarrow$

$$x-\text{plane} \quad N_{tot} = 8\times10^{11}$$
$$y-\text{plane} \quad N_{tot} = 5{,}7\times10^{11}$$

Conclusion S.C. limit is reached for $N_{tot} \cong 5\times10^{11}$ ions without safety margin. By playing with B_f, f, $\tilde{\varepsilon}_x$, $\tilde{\varepsilon}_y$ safety margin can be manage a little. Or by being optimistic $|\Delta Q_{xe}| = 0{,}5$ (!!). All these analysis requires dedicated studies.

EXERCISE 2 CAS - Geneva - 2015

Consider the distribution

$$\alpha\, \delta\left(\frac{\varepsilon_x}{E_x} + \frac{\varepsilon_y}{E_y} - 1\right)$$

we compute $\rho(x,y) = \int \alpha\, \delta(\dots)\, dx'\, dy'$. We take $\alpha = \frac{1}{\pi^2 E_x E_y}$ which is a normalization factor.

$$\rho(x,y) = \frac{1}{\pi^2 E_x E_y} \int \delta\left(\frac{\varepsilon_x}{E_x} + \frac{\varepsilon_y}{E_y} - 1\right) dx'\, dy'$$

we transform in C-S coordinates $\begin{cases} x' = -\frac{\alpha_x}{\sqrt{\beta_x}}\hat{x} + \frac{1}{\sqrt{\beta_x}}\hat{x}' \\ x = \sqrt{\beta_x}\,\hat{x} \end{cases}$ (same for y-plane)

The Jacobian of the transformation is

$$\begin{vmatrix} \frac{1}{\sqrt{\beta_x}} & 0 \\ 0 & \frac{1}{\sqrt{\beta_y}} \end{vmatrix} = \frac{1}{\sqrt{\beta_x}\sqrt{\beta_y}}$$

Therefore

$$\rho(x,y) = \frac{1}{\pi^2 E_x E_y} \int \delta\left(\frac{\hat{x}^2 + \hat{x}'^2}{E_x} + \frac{\hat{y}^2 + \hat{y}'^2}{E_y} - 1\right) \frac{1}{\sqrt{\beta_x}\sqrt{\beta_y}}\, d\hat{x}'\, d\hat{y}'$$

Define now $\hat{x}^2 = E_x \check{x}^2$, $\hat{x}'^2 = E_x \check{x}'^2$ (same for y-plane) therefore

$$\rho = \frac{1}{\pi^2\sqrt{\beta_x\varepsilon_x\beta_y\varepsilon_y}}\int \delta\left(\check{x}^2+\check{x}'^2+\check{y}^2+\check{y}'^2-1\right) d\check{x}'\, d\check{y}'$$

with a last change of variable $\check{x}' = \sqrt{r}\cos\theta$

$\check{y}' = \sqrt{r}\sin\theta$

$$\Downarrow$$

$$\rho = \frac{1}{\pi^2\sqrt{\beta_x\beta_y\varepsilon_x\varepsilon_y}}\int \delta\left(r+\check{x}^2+\check{y}^2-1\right)\frac{1}{2}\, dr\, d\theta = \frac{1}{\pi\sqrt{\beta_x\varepsilon_x\beta_y\varepsilon_y}}$$

Conclusion: $\rho(x,y) = \dfrac{1}{\pi\sqrt{\beta_x\varepsilon_x\beta_y\varepsilon_y}}$ that is the density is constant. The same strategy allows to prove that in any plane $(x\text{-}y,\ x\text{-}y', \dots)$ the particle density is constant.

EXERCISE 3   CAS - Geneva - 2015

Consider the case without S.C.

We want to prove that

$$\frac{\partial}{\partial s_1} \delta\left(\frac{\varepsilon_x}{E_x} + \frac{\varepsilon_y}{E_y} - 1, s_1 \right) = 0 \leftarrow$$

This is the partial derivative only in $\delta(\ldots, s)$ which I call "temporary s_1" for CONVENIENCE

We then consider Vlasov equation and prove that

$$\frac{\partial}{\partial s}\delta(\,) + \frac{\partial}{\partial x}\delta(\ldots)\,x' + \frac{\partial}{\partial x'}\delta(\ldots)\,x'' + \frac{\partial}{\partial y}\delta(\,)\,y' + \frac{\partial}{\partial y'}\delta(\,)\,y'' = 0$$

this is the partial derivative only in $\varepsilon_x(s)$, $\varepsilon_y(s)$.

(1) We consider only x-plane (similar applies to y-plane).

$$\frac{\partial}{\partial x}\delta(\,) = \underbrace{\frac{\partial}{\partial \varepsilon_x}\delta(\,)}_{\equiv\,\Omega} \cdot \frac{\partial \varepsilon_x}{\partial x} = \Omega\left(2\gamma_x x + 2\alpha_x x'\right)$$

$$\frac{\partial}{\partial x'}\delta(\,) = \Omega \cdot \frac{\partial \varepsilon_x}{\partial x'} = \Omega\left(2\alpha_x x + 2\beta_x x'\right)$$

2) Now we sum

$$\frac{\partial}{\partial x}\delta(\,)\,x' + \frac{\partial}{\partial x'}\delta(\,)\,x'' = \Omega\Big[\left(2\gamma_x x + 2\alpha_x x'\right)x' +$$

$$+ \left(2\alpha_x x + 2\beta_x x'\right)x''\Big] = (*)$$

3) As no-nonlinearities are considered $x'' + k_x x = 0$

4) therefore

$$(*) = \Omega\left[(2\gamma_x x + 2\alpha_x x')x' + (2\alpha_x x + 2\beta_x x')(-k_x x)\right]$$

5) You now collect all terms which have the same coordinates

$$(*) = \Omega\left[2\gamma_x x x' + 2\alpha_x x'^2 - 2\alpha_x k_x x^2 - 2\beta_x k_x x x'\right]$$

6) Now we compute $\frac{\partial}{\partial s}\delta(\,)$ (in x-plane) (only in $\varepsilon_x(s)$)

this is

$$\frac{\partial}{\partial \varepsilon_x}\delta(\,)\frac{\partial \varepsilon_x}{\partial s} = \Omega\left[\gamma_x' x^2 + 2\alpha_x' x x' + \beta_x' x'^2\right]$$

7) All together

$$\frac{\partial}{\partial s_1}\delta(\ldots, s_1) + \Omega\Big[2\gamma_x x x' + 2\alpha_x x'^2 - 2\alpha_x k_x x^2 - 2\beta_x k_x x x' +$$
$$+ \gamma_x' x^2 + 2\alpha_x' x x' + \beta_x' x'^2\Big] = 0$$

8) Again: $\to$ collecting terms with the same order we get

$$\frac{\partial}{\partial s_1}\delta(\ldots, s_1) + \Omega\Big[(2\gamma_x + 2\alpha_x' - 2\beta_x k_x)x x' + (-2\alpha_x k_x + \gamma_x')x^2 +$$
$$+ (2\alpha_x + \beta_x')x'^2\Big] = 0$$

9) If

$$2\gamma_x + 2\alpha_x' - 2\beta_x k_x = 0$$

$$-2\alpha_x k_x + \gamma_x' = 0$$

$$2\alpha_x + \beta_x' = 0$$

then $\frac{\partial}{\partial s_1} \delta(\dots, s_1) = 0$ and the "type" of distribution does not change.

10) Using optic functions and their definitions you can prove that 9) are true.(*)

EXAMPLE: $2\alpha_x + \beta_x' = 2 \cdot \left(-\frac{1}{2}\beta_x'\right) + \beta_x' = 0$ (the same for the other equations)

11) All this can be repeated for y-plane

12) If space charge force is linear $\longrightarrow$ Beta function with space charge $\longrightarrow$ all can be repeated with the same steps. $\Rightarrow$ KV is not changing "type" (here we forget the instability of modes ...!.)

THE END

(*) use $\frac{1}{2}\beta_x\beta_x'' - \frac{1}{4}\beta_x'^2 + k_x\beta_x^2 = 1$.