BEAM INSTABILITIES IN LINEAR MACHINES 1

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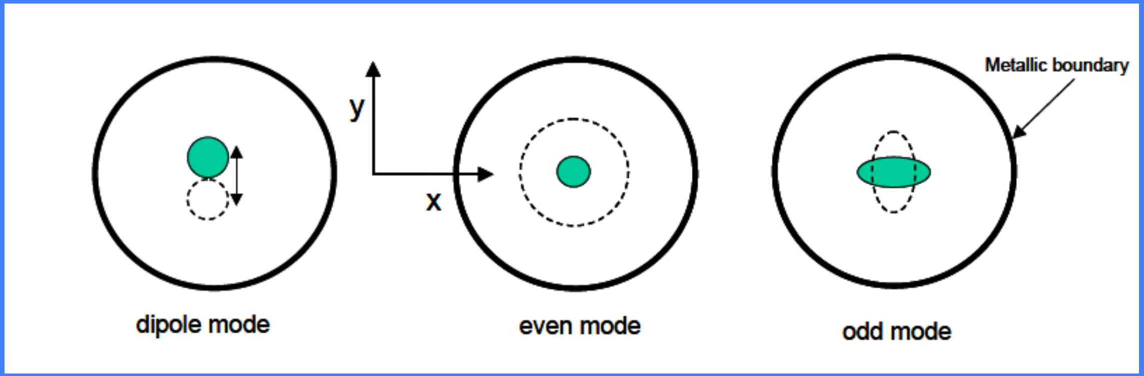


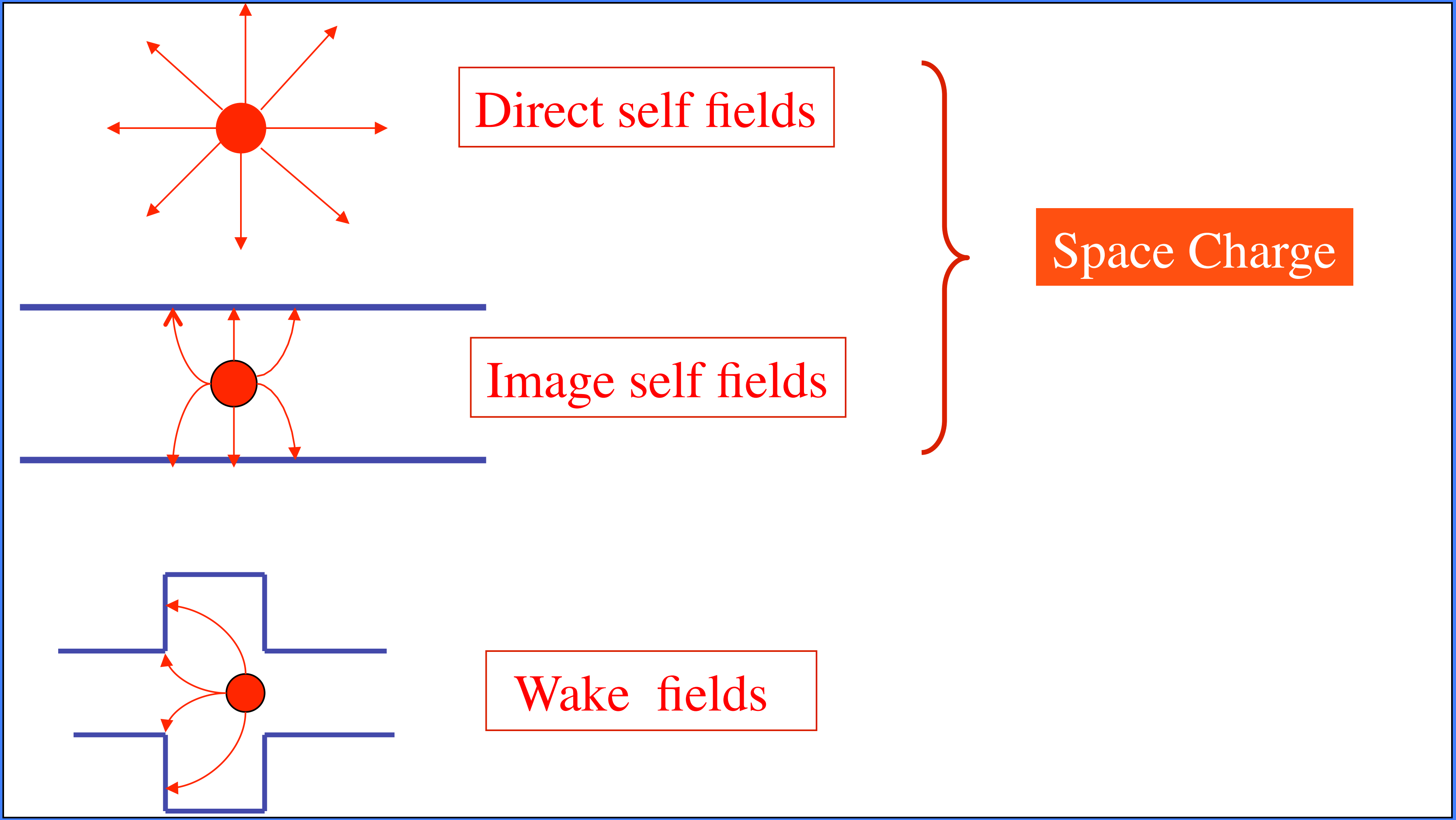
The CERN Accelerator School

CERN – 4 November 2015

CERN

SELF FIELDS AND WAKE FIELDS

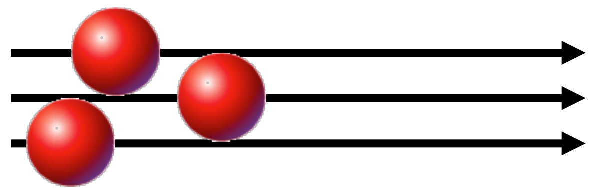
The realm of collective effects



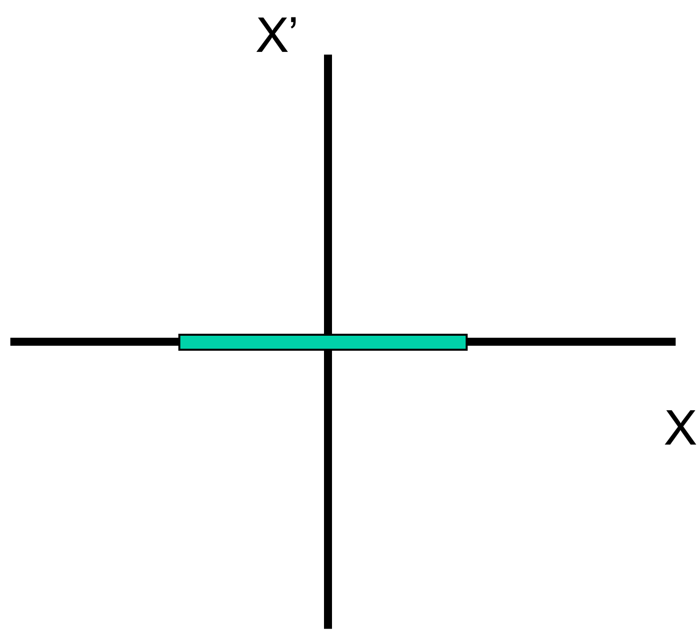
OUTLINE

- The rms emittance concept
- rms envelope equation
- Space charge forces
- Beam/Envelope emittance oscillations
- Matching conditions in a linac and emittance compensation

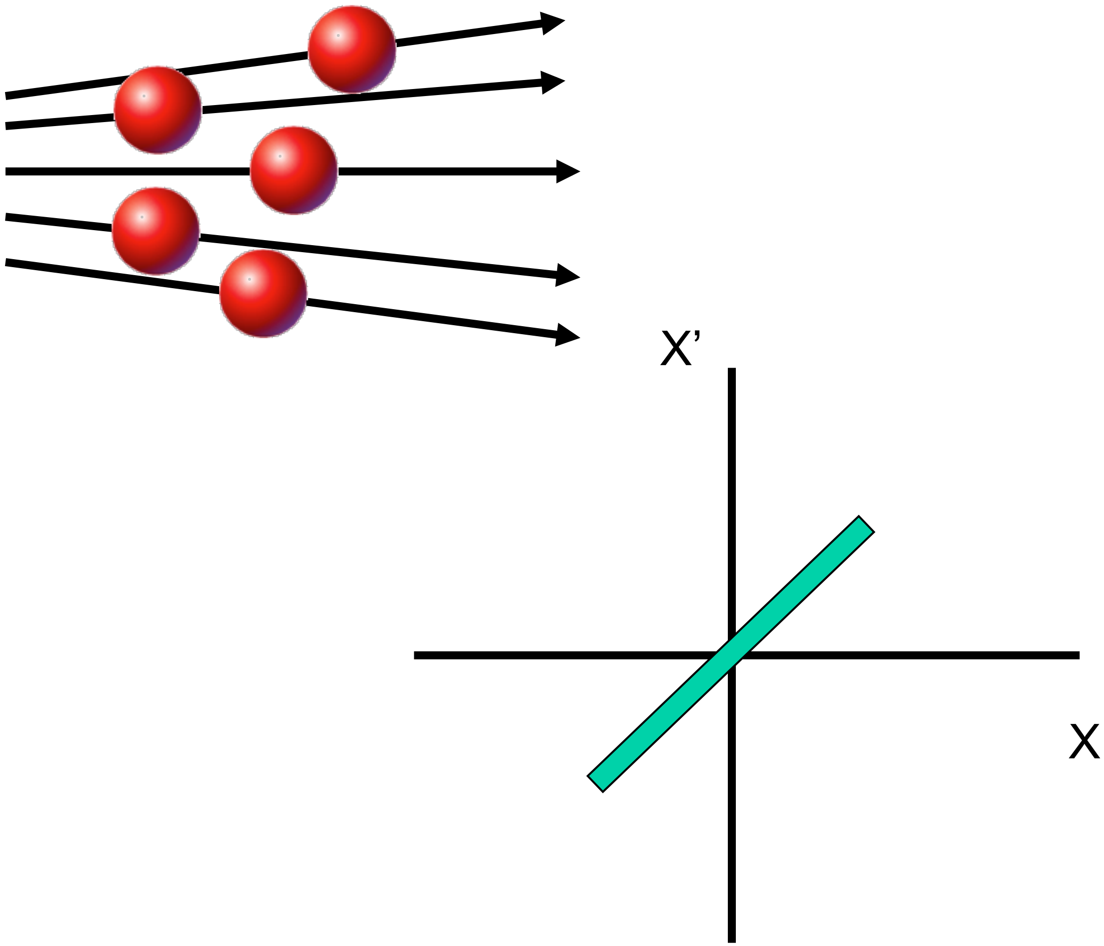
Trace space of an ideal laminar beam



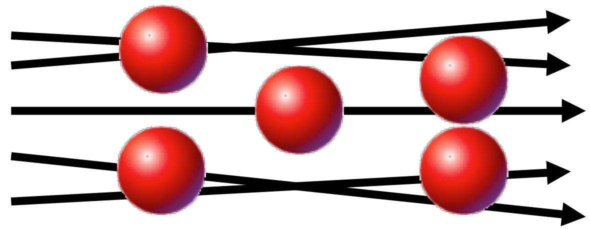
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases} \qquad p_x << p_z$$

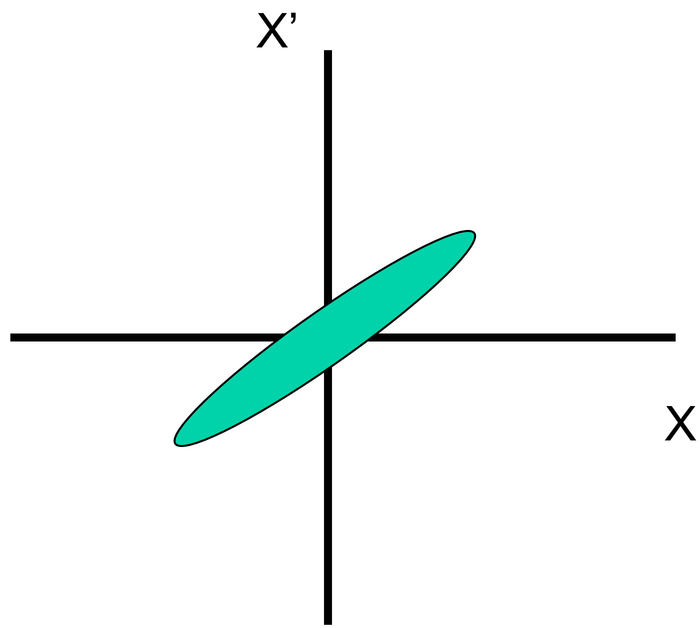


Trace space of a laminar beam



Trace space of non laminar beam



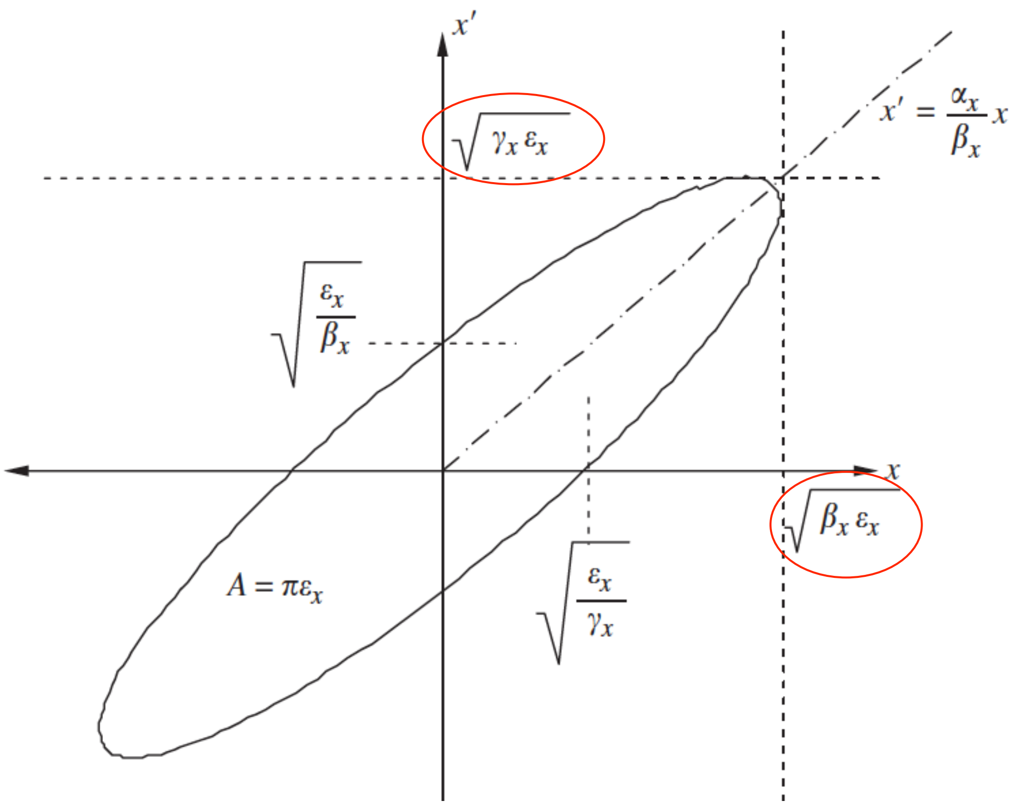


Geometric emittance: ε_g

Ellipse equation: $$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$$

Twiss parameters: $\beta\gamma - \alpha^2 = 1$ $\beta' = -2\alpha$

Ellipse area: $$A = \pi\varepsilon_g$$



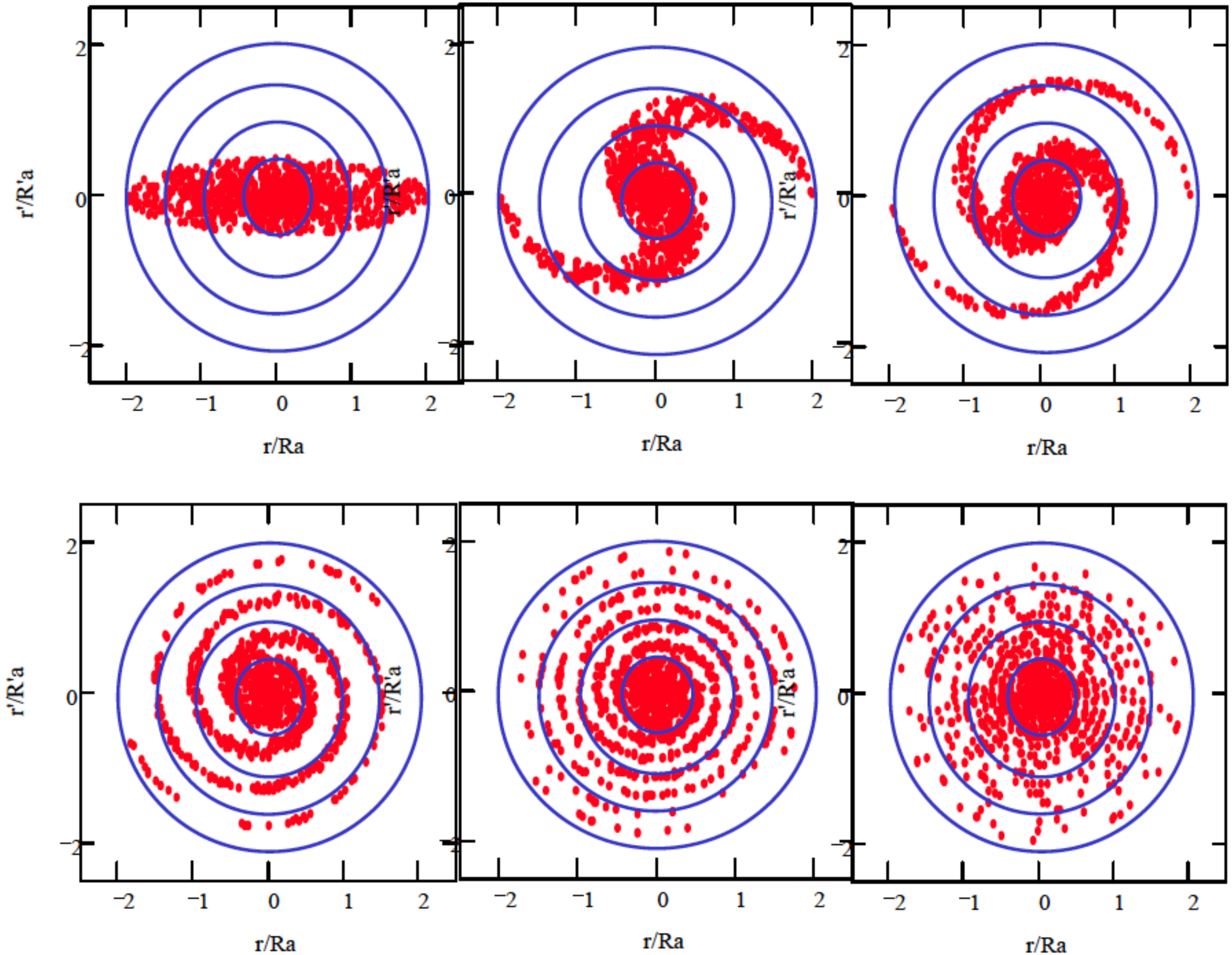
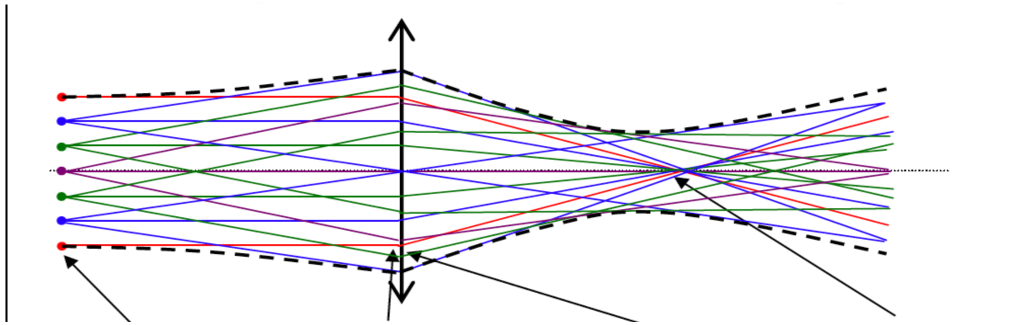
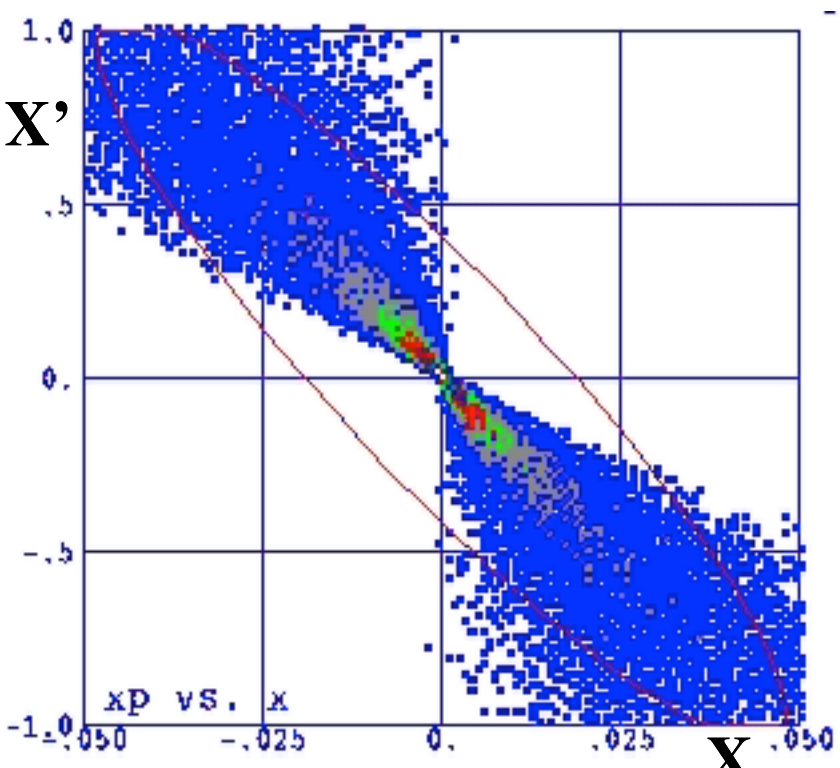


Fig. 17: Filamentation of mismatched beam in non-linear force

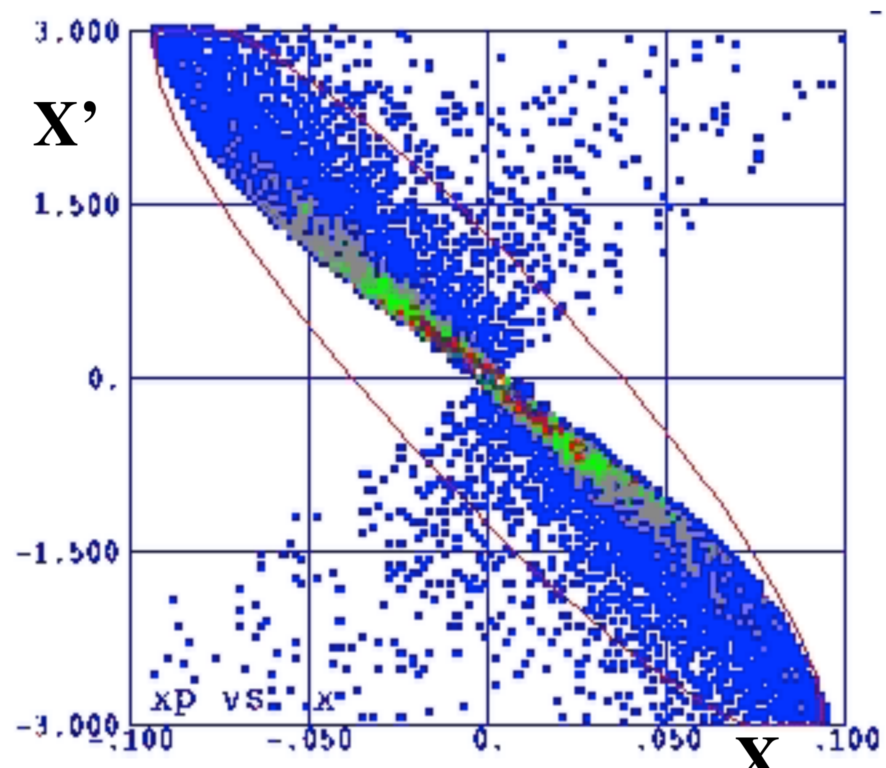
Trace space evolution



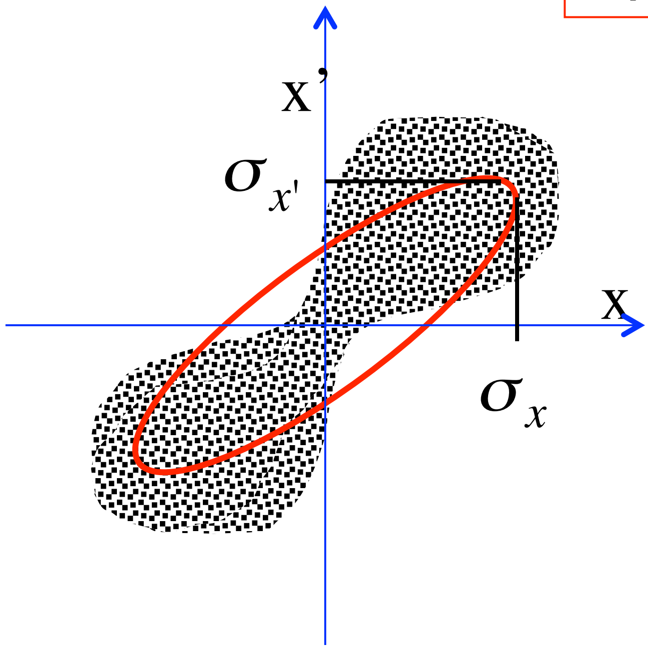
No space charge => cross over



With space charge => no cross over



rms emittance ε_{rms}



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1$$

$$f'(x,x') = 0$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\alpha = -\dfrac{\beta'}{2}$ $\beta = \dfrac{\langle x^2 \rangle}{\varepsilon_{rms}}$

it follows: $\alpha = -\dfrac{1}{2\varepsilon_{rms}} \dfrac{d}{dz} \langle x^2 \rangle = -\dfrac{\langle x x' \rangle}{\varepsilon_{rms}} = -\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}$

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\sigma_{xx'} = \langle xx' \rangle = -\alpha \varepsilon_{rms}$$

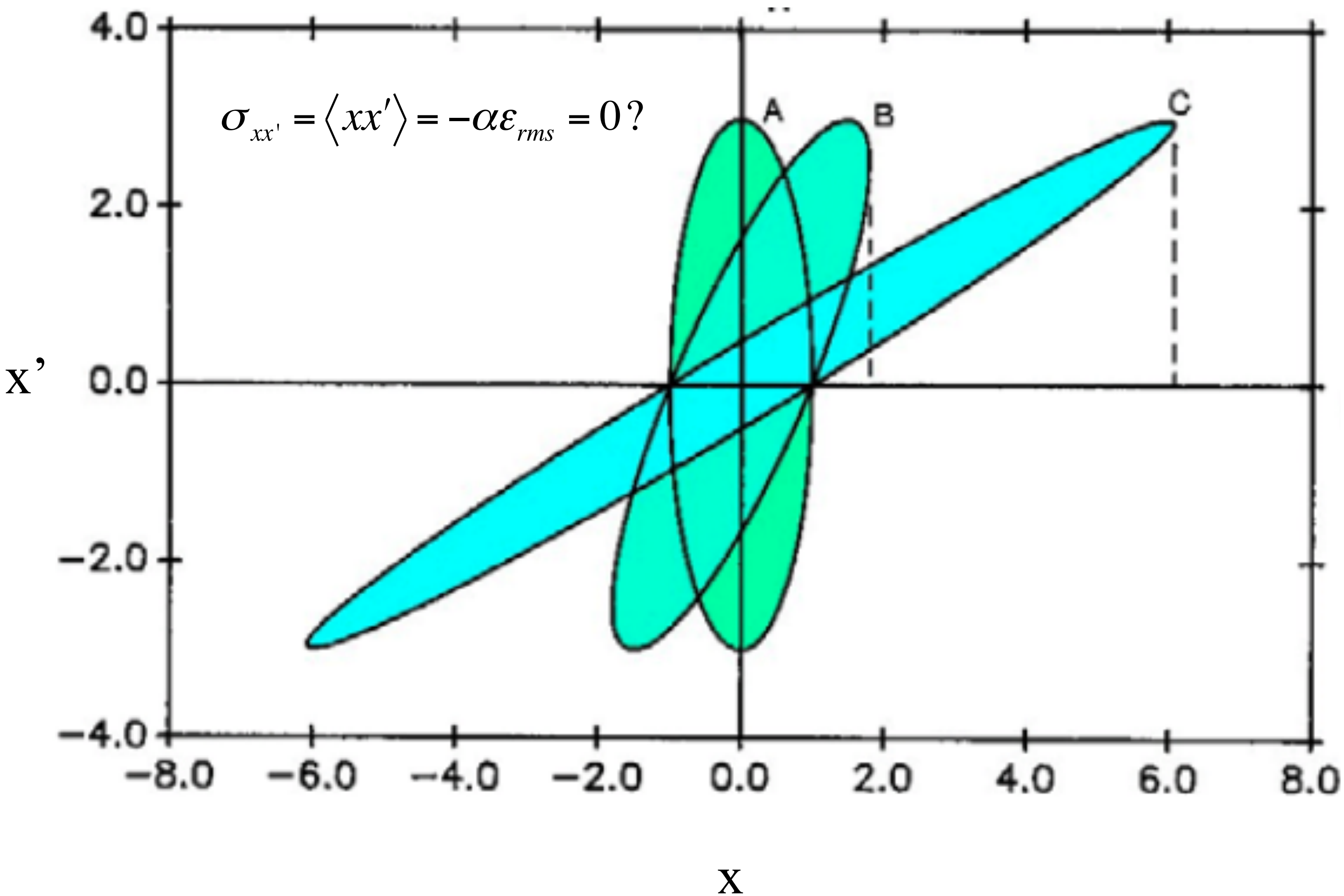
It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}} \right)^2 = 1$

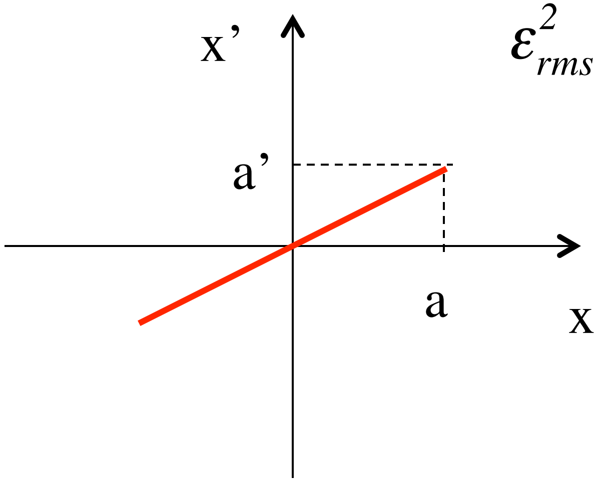
We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)}$$

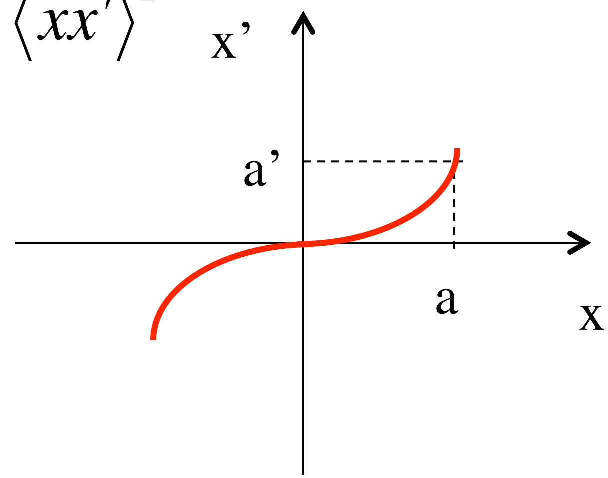
Which distribution has no correlations?



What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = Cx^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When $n = 1$ ==> $\varepsilon_{rms} = 0$

When $n \neq 1$ ==> $\varepsilon_{rms} \neq 0$

Constant under linear transformation only

$$\frac{\mathrm{d}}{\mathrm{d}z}\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\langle x'\rangle\langle x''\rangle - 2\langle xx''\rangle\langle xx'\rangle = 0$$

For linear transformations, $x'' = -k_x^2 x$, and the right-hand side of the equation is

$$2k_x^2\langle x^2\rangle\langle xx'\rangle - 2\langle x^2\rangle\langle xx'\rangle k_x^2 = 0,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}z}\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = 0$$

And without acceleration: $x' = \frac{p_x}{p_z}$

Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c\beta\gamma x'$ $\quad p_z \approx p$

$$\varepsilon_{n,rms} = \sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right)} \approx \langle\beta\gamma\rangle\varepsilon_{rms}$$

Liouville theorem: the density of particles *n*, or the volume V occupied by a given number of particles in phase space (x,p_x,y,p_y,z,p_z) remains invariant under conservative forces.

$$\frac{dn}{dt} = 0$$

It hold also in the projected phase spaces (x,p_x),(y,p_y)(,z,p_z) provided that there are no couplings

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Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\langle x^2 \rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\langle x^2 \rangle = \frac{1}{2\sigma_x}2\langle xx' \rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\langle x'^2 \rangle + \langle xx' \rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \langle xx'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify: $$\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

$$\sigma_x'' - \frac{\langle x x'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle x x'' \rangle = -k_x^2 \langle x^2 \rangle$

$$\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

We obtain the rms envelope equation with a linear focusing force in which the rms emittance enters as defocusing pressure like term.

$$\frac{\varepsilon_{rms}^2}{\sigma_x^3} \approx \frac{T}{V} \approx P$$

Beam temperature

Kinetic theory of gases defines temperature (in each direction and global) as

$$k\,T_{x,y,s} = m < v_{x,y,s}^2 > , \qquad T = \tfrac{1}{3}(T_x + T_y + T_s) \qquad (\tfrac{1}{2} m v^2 = \tfrac{3}{2} kT)$$

k: Boltzmann constant, m: mass of molecules, $v_{x,y,s}$: velocity components of molecules

Definition of beam temperature in analogy:

$$k\,T_{beam,x,y,s} = m_0 < v_{x,y,s}^2 > ,$$

where $v_{x,y,s}$ are the velocity spreads in the system moving with the beam.

The transverse velocity spread in the beam system is given by the r.m.s emittance:

$$< v_x^2 > = (\beta \gamma c)^2 < (x')^2 > = (\beta \gamma c)^2 \gamma_x \cdot \varepsilon_{x,r.m.s}$$

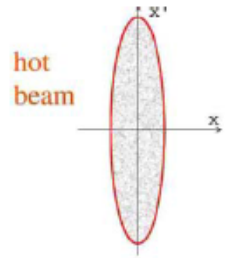
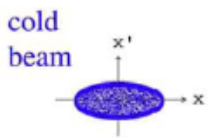
similar for y direction

βc: longitudinal beam velocity β, γ : relativistic parameter, $\gamma_x \approx 1/\beta_x$: Twiss (lattice) parameter

Hence

$$\Longrightarrow \quad k\,T_{beam,x,y} = m_0 c^2 (\beta\gamma)^2 \gamma_{x,y} \cdot \varepsilon_{x,y;rms}$$

$$\Longrightarrow \quad k\,T_{beam,x,y} = m_0 c^2 (\beta\gamma)^2 \gamma_{x,y} \cdot \varepsilon_{x,y;rms}$$

Property	Hot beam	Cold beam
ion mass (m_o)	heavy ion	light ion
ion energy ($\beta\gamma$)	high energy	low energy
beam emittance (ε)	large emittance	small emittance
lattice properties ($\gamma_{x,y} \approx 1/\beta_{x,y}$)	strong focus (low β)	high β
phase space portrait		

Electron Cooling: Temperature relaxation by mixing a hot ion beam with co-moving cold (light) electron beam.

Particle Accelerators
1973, Vol. 5, pp. 61–65

EMITTANCE, ENTROPY AND INFORMATION

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$$S = kN \log(\pi\varepsilon)$$

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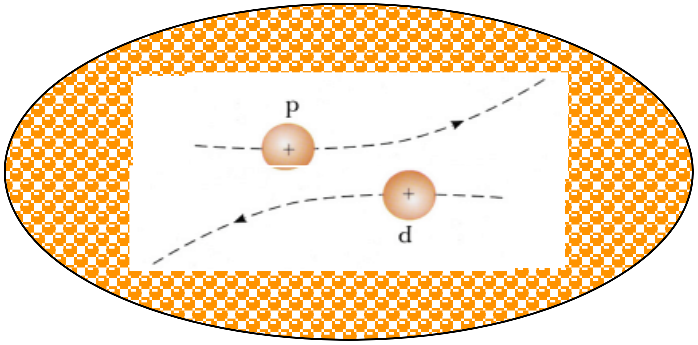
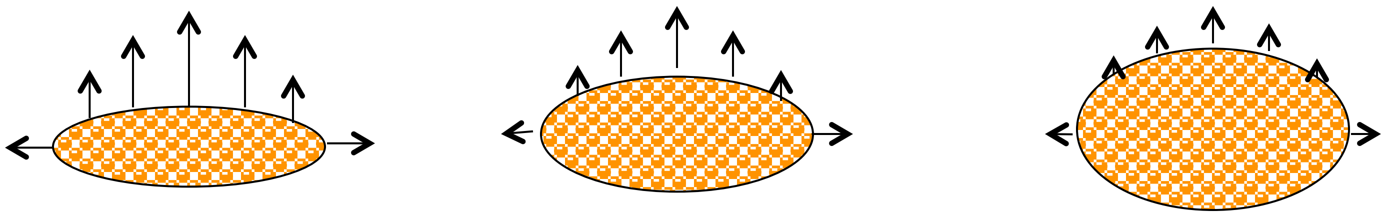
Space Charge: what does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

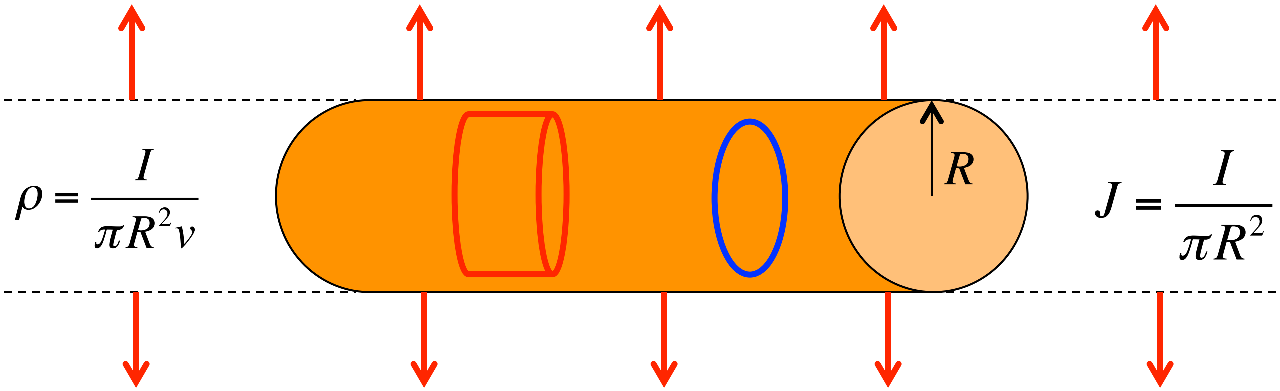
1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



Continuous Uniform Cylindrical Beam Model



$$\rho = \frac{I}{\pi R^2 v}$$

$$J = \frac{I}{\pi R^2}$$

Gauss' s law

$$\int \varepsilon_o E \cdot dS = \int \rho dV$$

$$E_r = \frac{I}{2\pi\varepsilon_o R^2 v} r \quad \text{for} \quad r \le R$$

$$E_r = \frac{I}{2\pi\varepsilon_o v} \frac{1}{r} \quad \text{for} \quad r > R$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

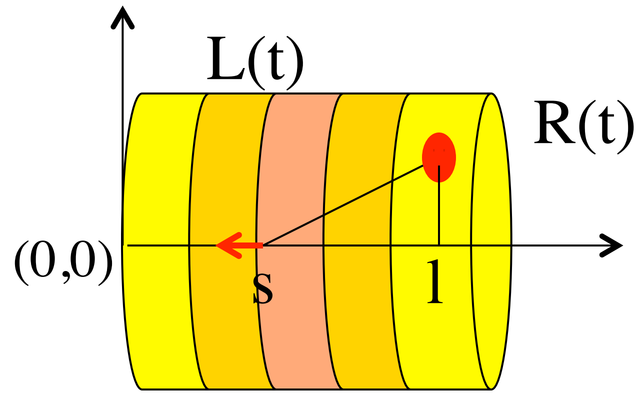
Ampere' s law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

$$B_\vartheta = \mu_o \frac{Ir}{2\pi R^2} \quad \text{for} \quad r \le R$$

$$B_\vartheta = \mu_o \frac{I}{2\pi r} \quad \text{for} \quad r > R$$

Bunched Uniform Cylindrical Beam Model



$$v_z = \beta c$$

Longitudinal Space Charge field in the bunch moving frame:

$$\tilde{\rho} = \frac{Q}{\pi R^2 \tilde{L}}$$

$$\tilde{E}_z(\tilde{s}, r=0) = \frac{\tilde{\rho}}{4\pi\varepsilon_o} \int_0^R \int_0^{2\pi} \int_0^{\tilde{L}} \frac{(\tilde{l} - \tilde{s})}{\left[(\tilde{l} - \tilde{s})^2 + r^2\right]^{3/2}} \, r dr d\varphi d\tilde{l}$$

$$\tilde{E}_z(\tilde{s}, r=0) = \frac{\tilde{\rho}}{2\varepsilon_0} \left[\sqrt{R^2 + (\tilde{L} - \tilde{s})^2} - \sqrt{R^2 + \tilde{s}^2} + (2\tilde{s} - \tilde{L})\right]$$

Radial Space Charge field in the bunch moving frame

by series representation of axisymmetric field:

$$\tilde{E}_r(r,\tilde{s}) \cong \left[\frac{\tilde{\rho}}{\varepsilon_0} - \frac{\partial}{\partial \tilde{s}}\tilde{E}_z(0,\tilde{s})\right]\frac{r}{2} + \left[\cdots\right]\frac{r^3}{16} +$$

$$\tilde{E}_r(r,\tilde{s}) = \frac{\tilde{\rho}}{2\varepsilon_0}\left[\frac{(\tilde{L}-\tilde{s})}{\sqrt{R^2+(\tilde{L}-\tilde{s})^2}} + \frac{\tilde{s}}{\sqrt{R^2+\tilde{s}^2}}\right]\frac{r}{2}$$

Lorentz Transformation to the Lab frame

$$E_z = \tilde{E}_z$$

$$E_r = \gamma \tilde{E}_r$$

$$\tilde{L} = \gamma L \quad \Rightarrow \quad \tilde{\rho} = \frac{\rho}{\gamma}$$

$$\tilde{s} = \gamma s$$

$$E_z(0,s) = \frac{\rho}{\gamma 2\varepsilon_0}\left[\sqrt{R^2 + \gamma^2 (L-s)^2} - \sqrt{R^2 + \gamma^2 s^2} + \gamma\left(2s - L\right)\right]$$

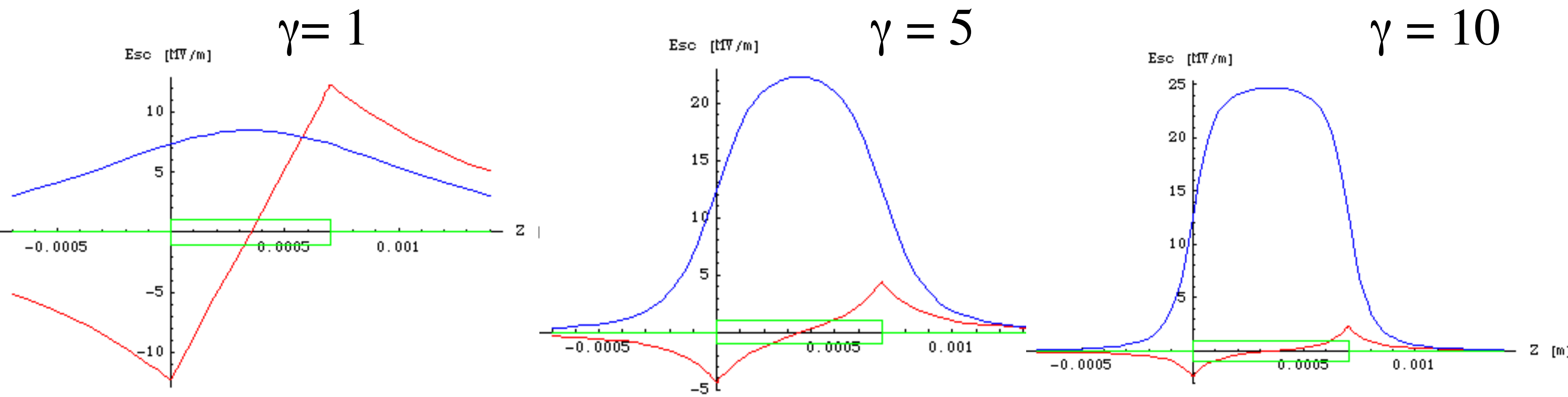
$$E_r(r,s) = \frac{\gamma\rho}{2\varepsilon_0}\left[\frac{(L-s)}{\sqrt{R^2 + \gamma^2 (L-s)^2}} + \frac{s}{\sqrt{R^2 + \gamma^2 s^2}}\right]\frac{r}{2}$$

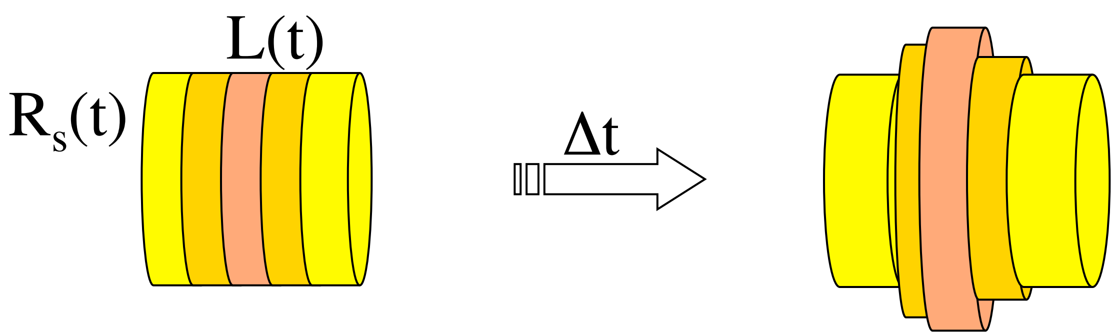
It is still a linear field with r but with a longitudinal correlation s

Bunched Uniform Cylindrical Beam Model

$$E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)$$

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$





Lorentz Force

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right)E_r = \frac{eE_r}{\gamma^2}$$

is a **linear** function of the transverse coordinate

$$\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g\left(s,\gamma\right)$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect.

$$F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g\left(s,\gamma\right)$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p\,x' = \beta\gamma m_o c x'$$

$$\frac{d}{dt}\left(px'\right) = \beta c \frac{d}{dz}\left(p\,x'\right) = F_x$$

$$x'' = \frac{F_x}{\beta c p}$$

$$F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0\sigma_x^2\beta c}\,g\left(s,\gamma\right)$$

$$x'' = \frac{k_{sc}\left(s,\gamma\right)}{\sigma_x^2}\,x$$

Now we can calculate the term $\langle xx'' \rangle$ that enters in the envelope equation

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}$$

$$x'' = \frac{k_{sc}}{\sigma_x^2} x$$

$$\langle xx'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\rho = \dfrac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}$

The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1 | Laminar Beam

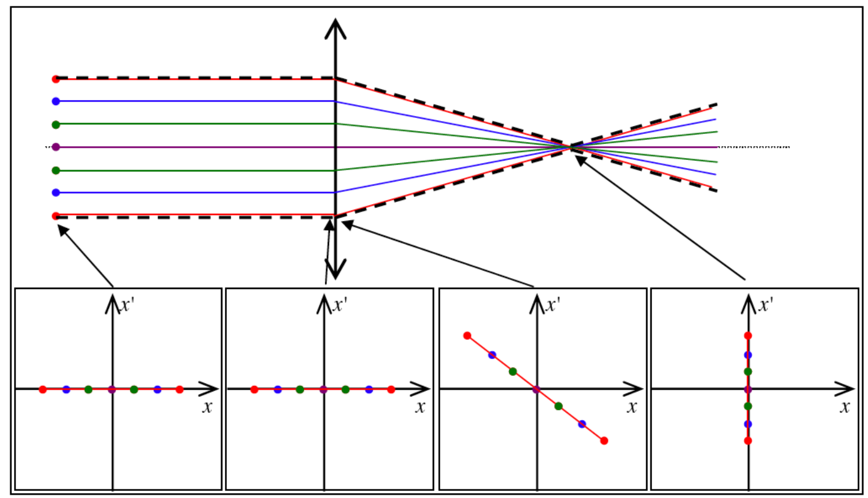


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

ρ<<1 | Thermal Beam

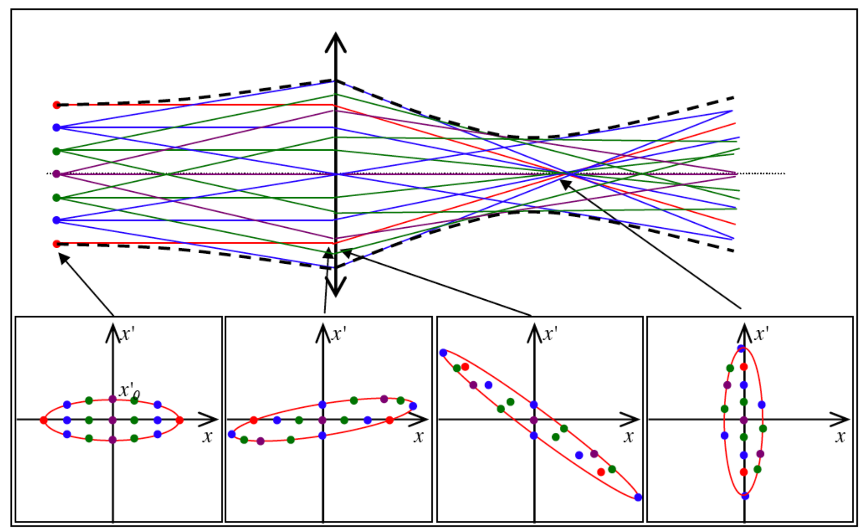


Fig. 11: Particle trajectories in non-zero emittance beam

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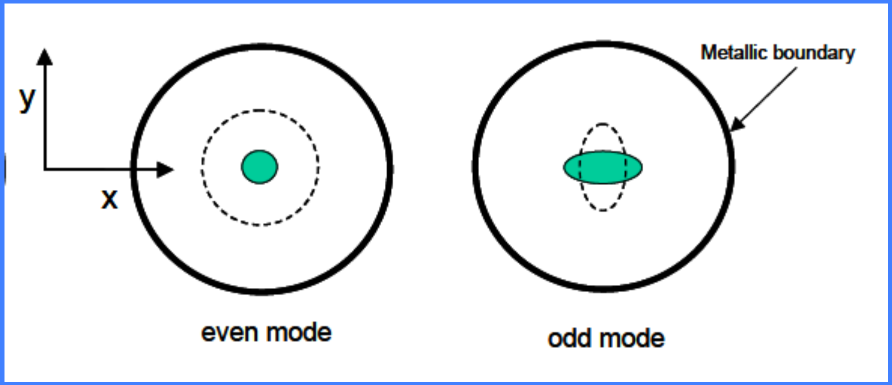
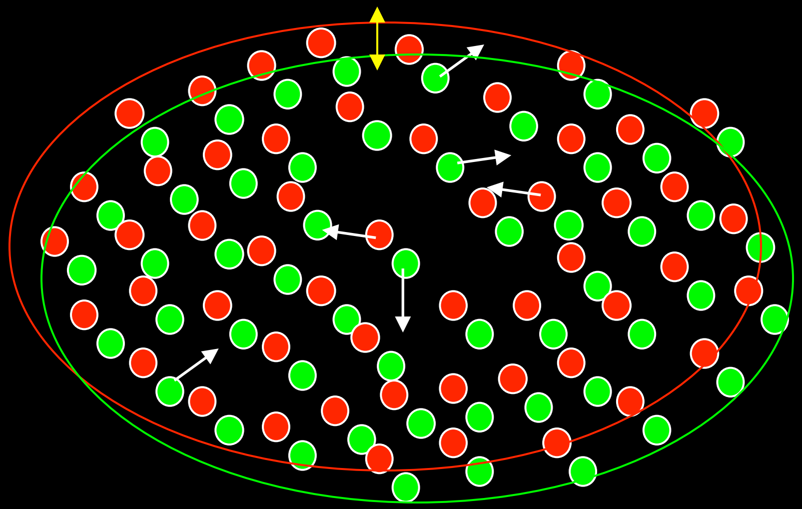


Surface charge density

$$\sigma = e\,n\,\delta x$$

δx



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\,n\,\delta x/\epsilon_0$$

Restoring force

$$m\,\frac{d^2\delta x}{dt^2} = e\,E_x = -m\,\omega_p^{\,2}\,\delta x$$

Plasma frequency

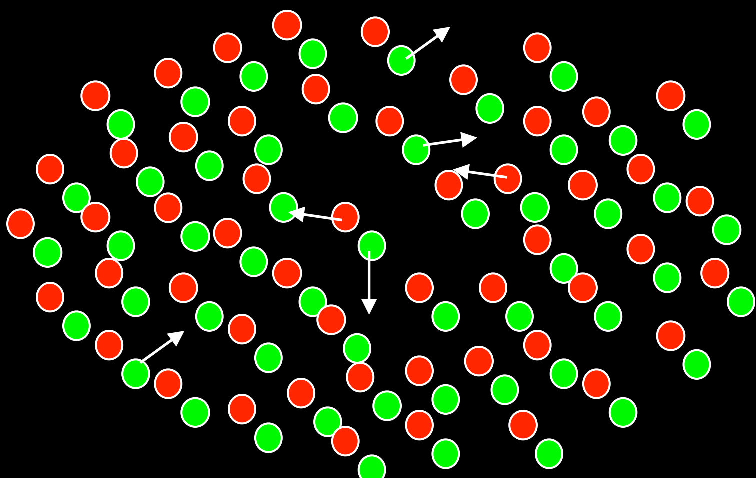
$$\omega_p^{\,2} = \frac{n\,e^2}{\epsilon_0\,m}$$

Plasma oscillations

$$\delta x = (\delta x)_0\,\cos\left(\omega_p\,t\right)$$

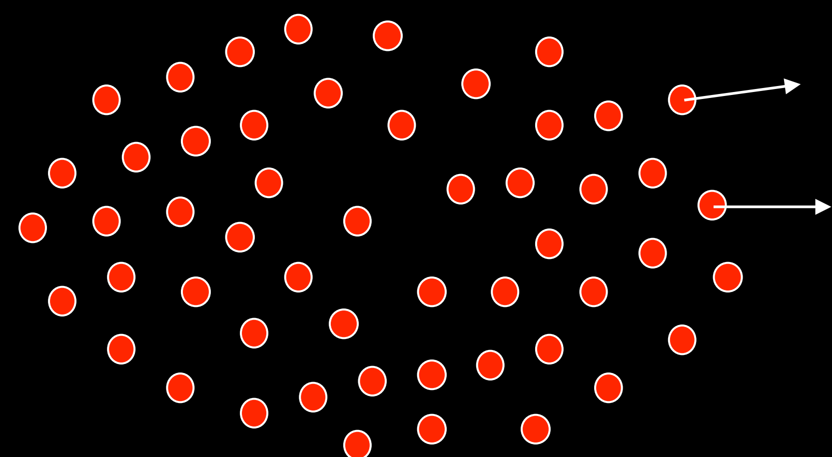
Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma

Magnetic focusing



Magnetic focusing

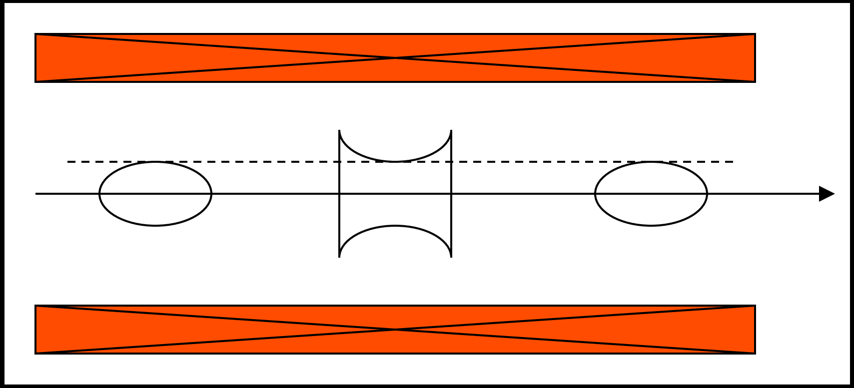
Single Component Relativistic Plasma

$$\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

$$k_s = \frac{qB}{2mc\beta\gamma}$$

Equilibrium solution:

$$\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}$$



Small perturbation:

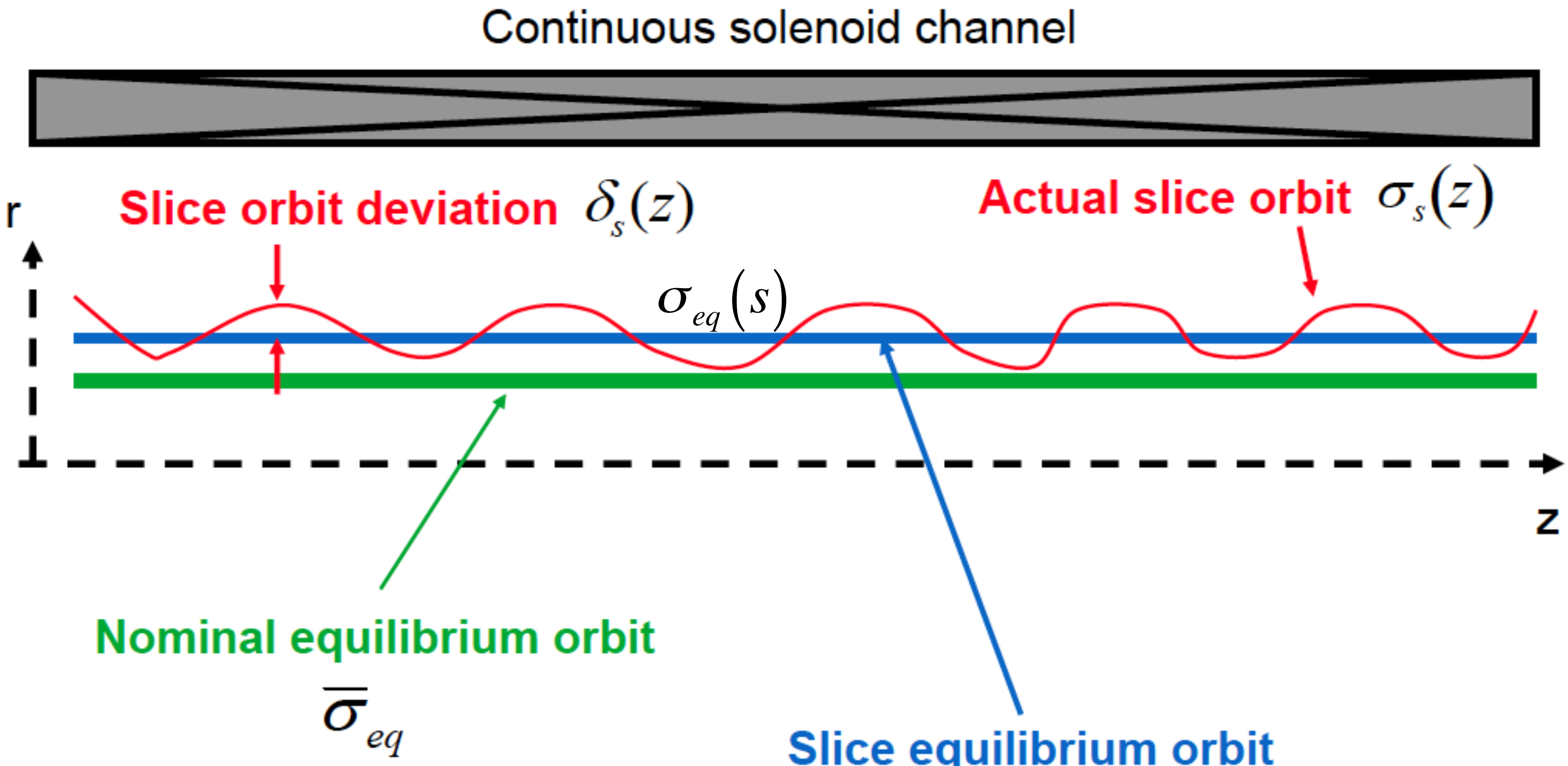
$$\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)$$

$$\delta\sigma''(s) + 2k_s^2 \delta\sigma(s) = 0$$

$$\delta\sigma(s) = \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$



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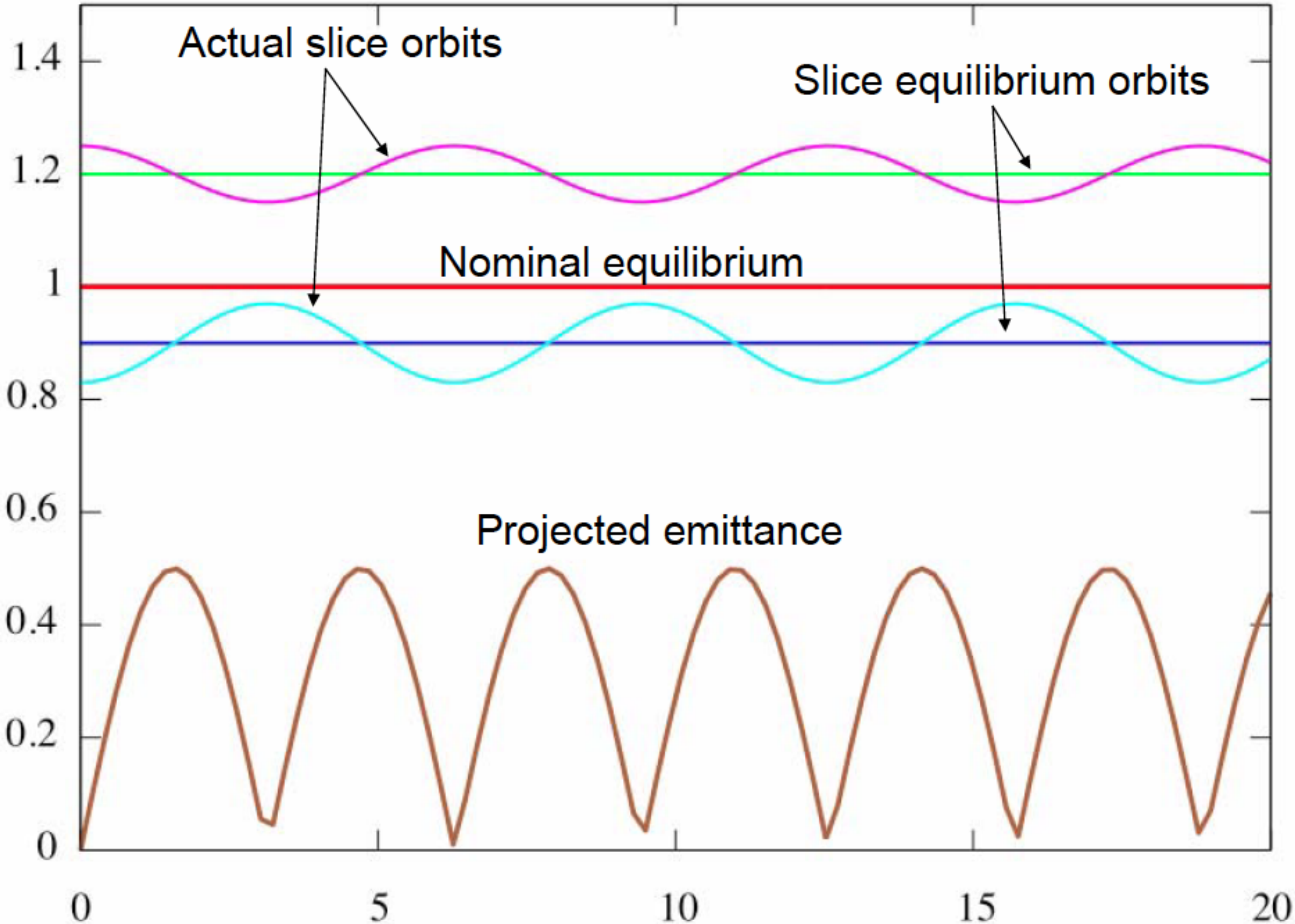
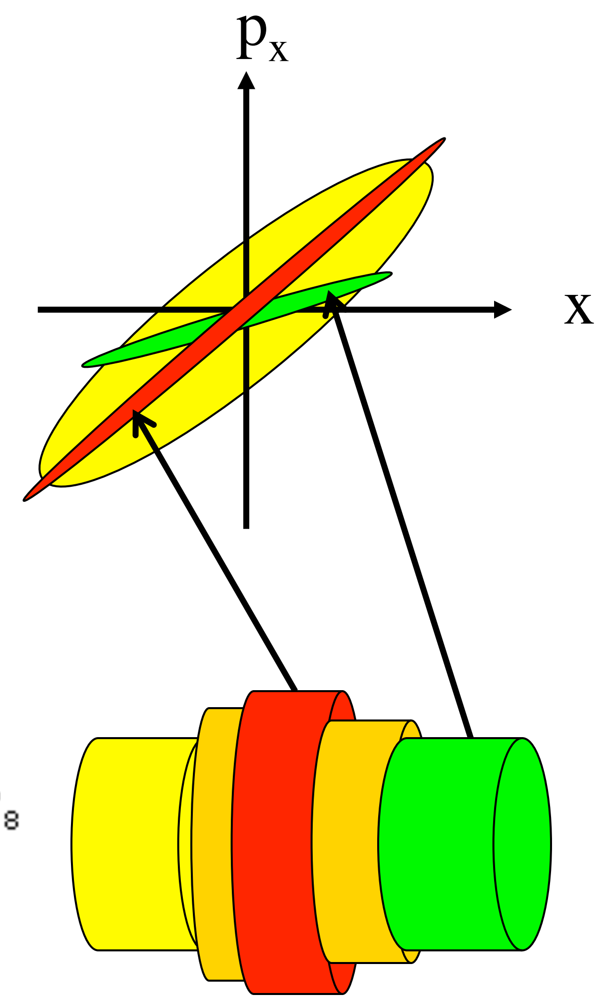


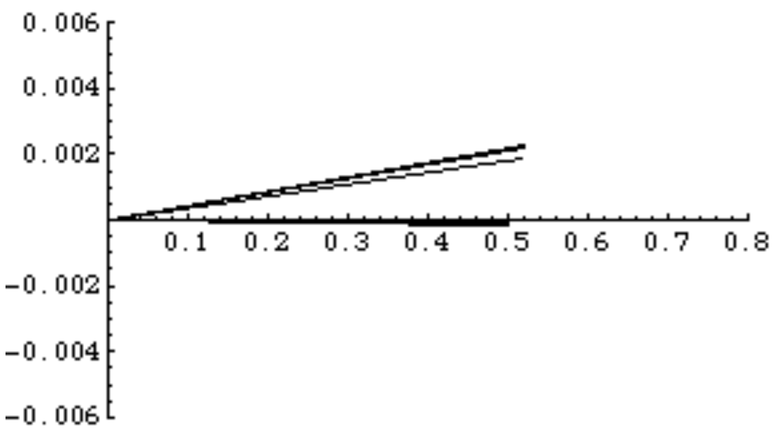
$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} \approx \left| \sin\left(\sqrt{2} k_s z\right) \right|$$

Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

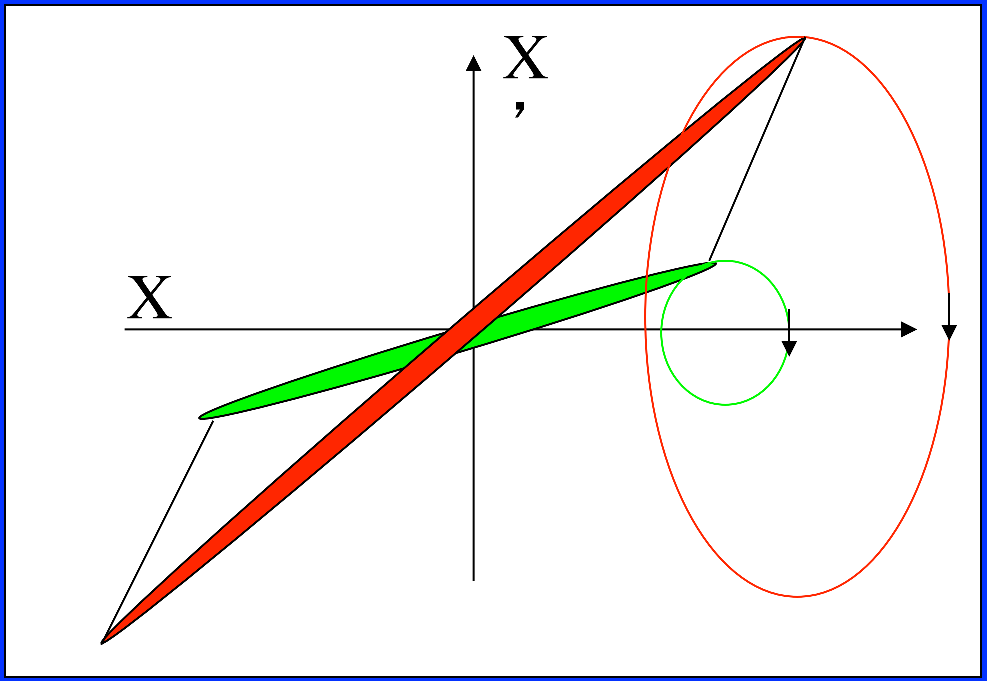
Projected Phase Space

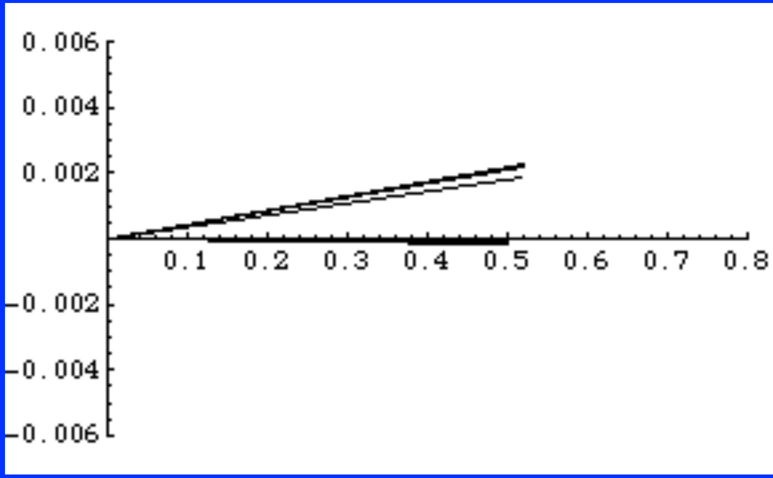


Slice Phase Spaces



Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes

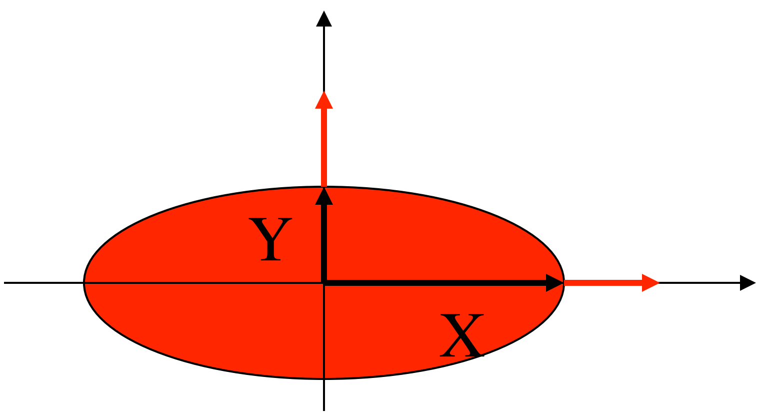




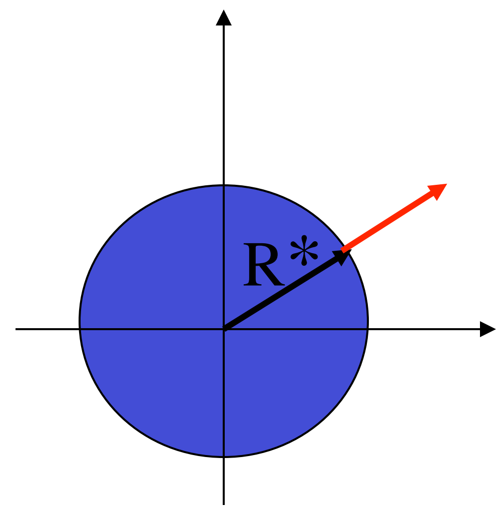
Elliptical cross section bunch

$$E_y = \frac{I}{\beta c \pi \varepsilon_o} \frac{y}{Y(X+Y)}$$

$$E_r = \frac{I}{2\beta c \pi \varepsilon_o R^2} r$$

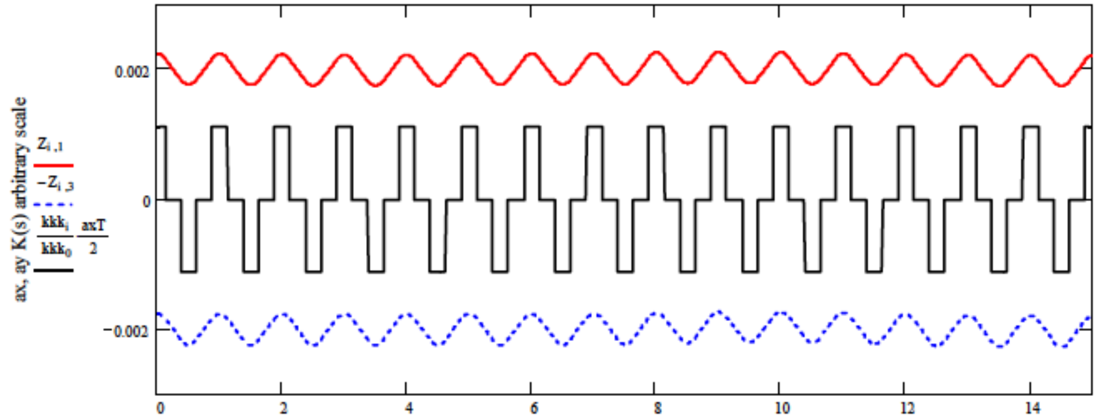


$$R^* = \frac{X+Y}{2}$$



$$E_x = \frac{I}{\beta c \pi \varepsilon_o} \frac{x}{X(X+Y)}$$

$$E_x(X,z) = E_y(Y,z) = E_r(R^*,z)$$


ax, ay K(s) arbitrary scale
0.002
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2
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6
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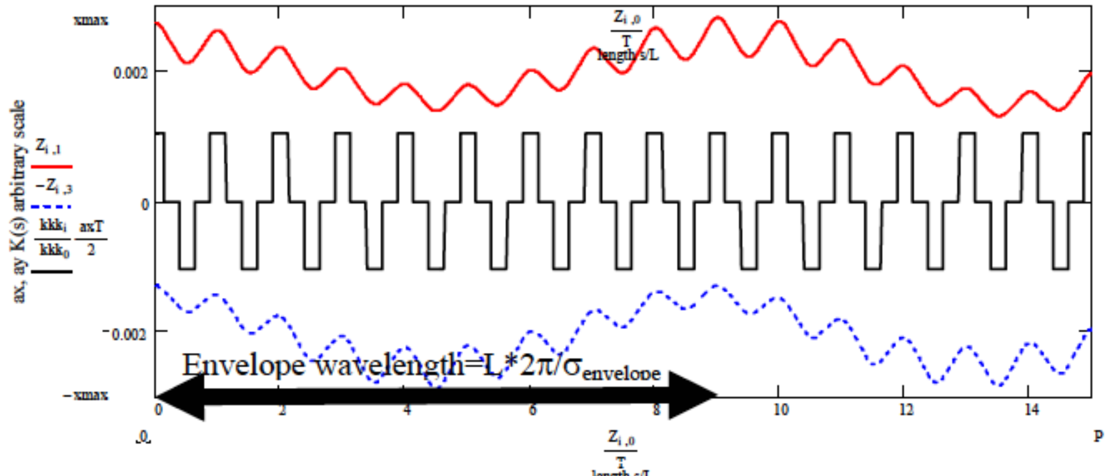
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0
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−xmax
ax, ay K(s) arbitrary scale
length s/L
Envelope wavelength=L*2π/σenvelope
0
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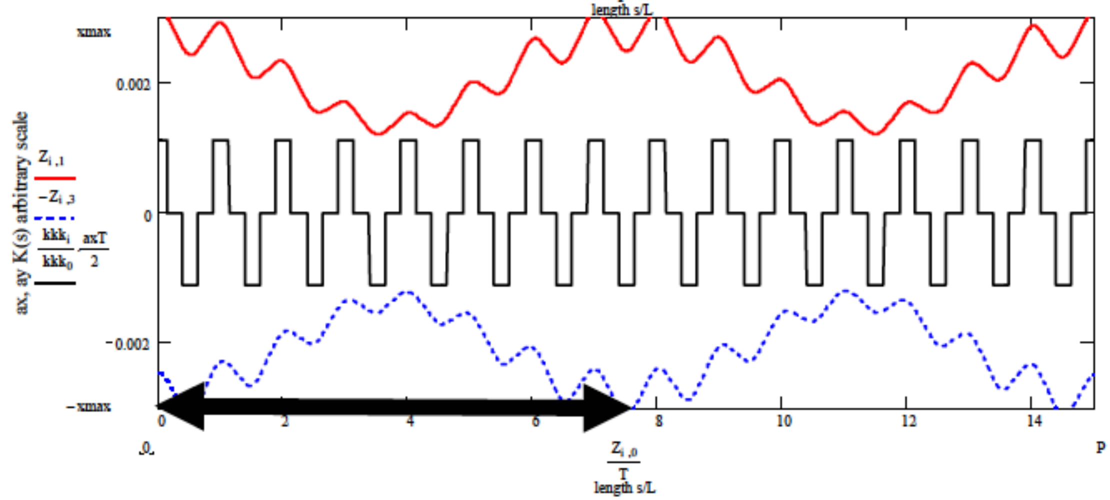
xmax
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ax, ay K(s) arbitrary scale
length s/L
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2
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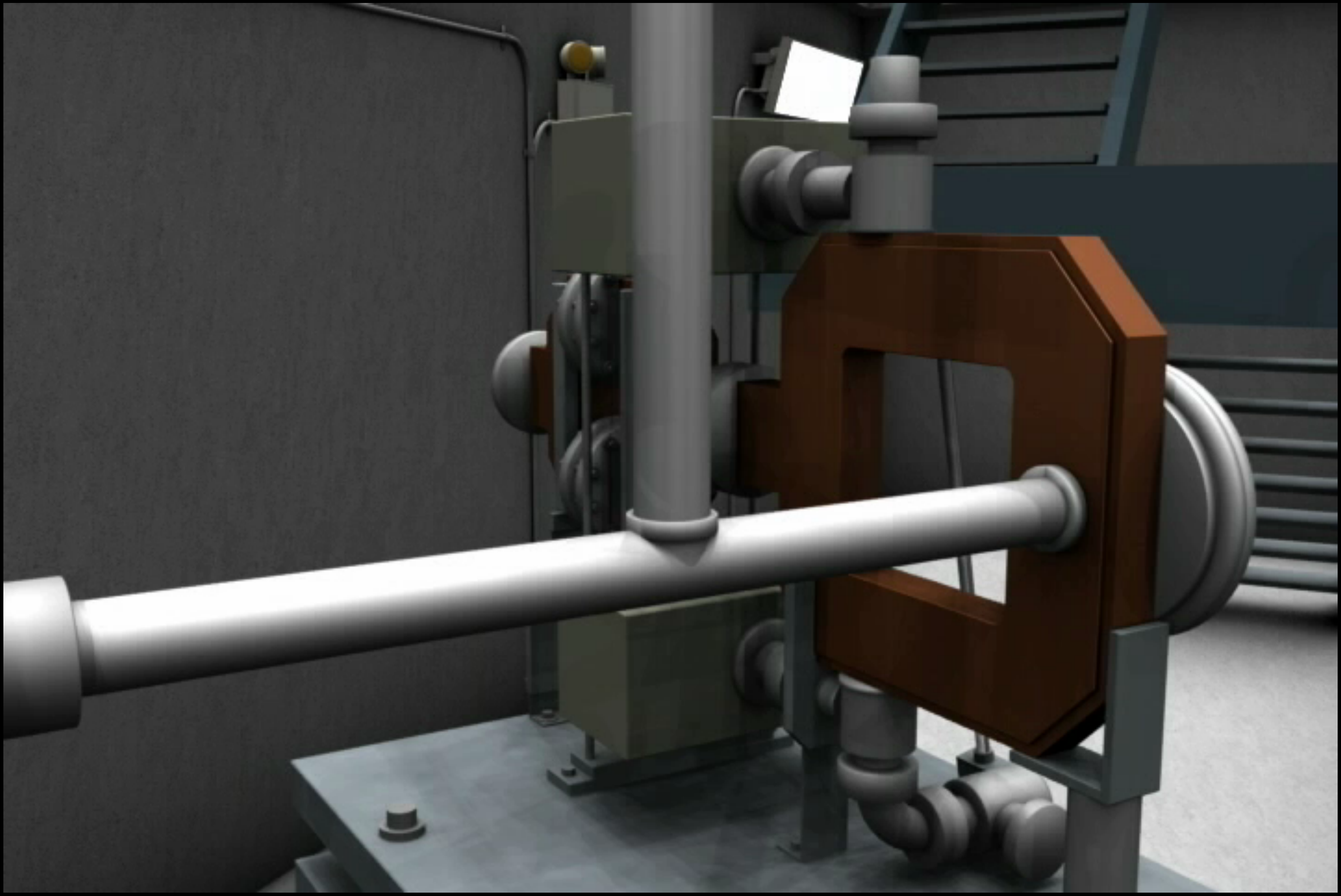
12
14

OUTLINE

- The rms emittance concept
- rms envelope equation
- Space charge forces
- Beam/Envelope emittance oscillations
- Matching conditions in a linac and emittance compensation

High Brightness Photo-Injector



Envelope Equation with Longitudinal Acceleration

$$p_o = \gamma_o m_o \beta_o c$$
$$p_x << p_o$$
$$p = p_o + p'z$$
$$p' = (\beta\gamma)' m_o c$$

$$\frac{dp_x}{dt} = \frac{d}{dt}(px') = \beta c \frac{d}{dz}(px') = 0$$

$$x'' + \frac{p'}{p}x' = 0$$

$$x'' = -\frac{(\beta\gamma)'}{\beta\gamma}x'$$

$$\langle xx'' \rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\langle xx' \rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\sigma_{xx'}$$

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}$$

$$\frac{d\sigma_x}{dz} = \sigma_x' = \frac{\sigma_{xx'}}{\sigma_x}$$

Space Charge De-focusing Force

$$\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

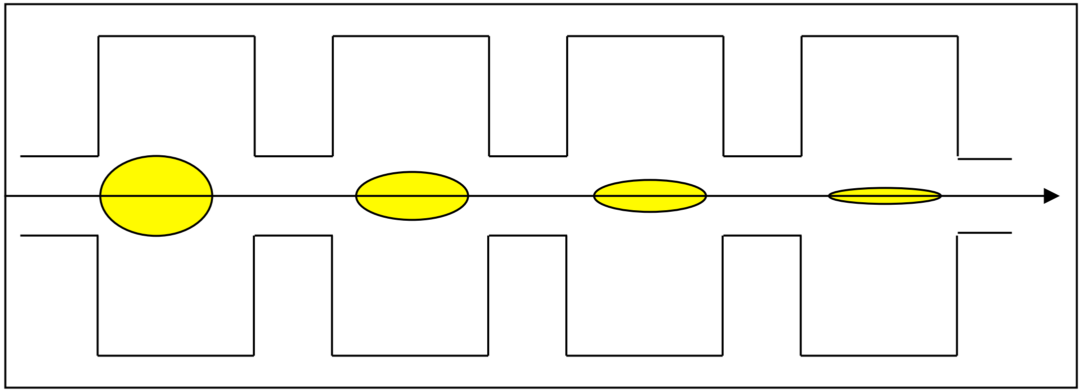
Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\varepsilon_n = \beta\gamma\varepsilon_{rms}$$

Beam subject to strong acceleration



$$\sigma_x'' + \frac{\gamma'}{\gamma}\sigma_x' + \frac{k_{RF}^2}{\gamma^2}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3} + \frac{k_{sc}^o}{\gamma^3\sigma_x}$$

We must include also the RF focusing force: $k_{RF}^2 = \frac{\gamma'^2}{2}$

$$k_{sc}^o = \frac{2I}{I_A} g(s,\gamma)$$

$$\sigma_x'' + \frac{\gamma'}{\gamma}\sigma_x' + \frac{k_{RF}^2}{\gamma^2}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3} + \frac{k_{sc}^o}{\gamma^3\sigma_x}$$

$\gamma = 1 + \alpha z$ ==> $\gamma'' = 0$

Looking for an "equilibrium" solution $\sigma_{inv} = \sigma_o \gamma^n$

==> all terms must have the same dependence on γ

$$\sigma_{inv}' = n\sigma_o\gamma^{n-1}\gamma'$$

$$\sigma_{inv}'' = n(n-1)\sigma_o\gamma^{n-2}\gamma'^2$$

$$n(n-1)\sigma_o\gamma^{n-2}\gamma'^2 + n\sigma_o\gamma^{n-2}\gamma'^2 + k_{RF}^2\sigma_o\gamma^{n-2} = \frac{k_{sc}^o}{\sigma_x}\gamma^{-3-n}$$

$$n - 2 = -3 - n \Rightarrow n = -\frac{1}{2}$$

$$\sigma''_x + \frac{\gamma'}{\gamma}\sigma'_x + \frac{k_{RF}^2}{\gamma^2}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2\sigma_x^3} + \frac{k_{sc}^o}{\gamma^3\sigma_x}$$

$\gamma = 1 + \alpha z$ ==> $\gamma'' = 0$

Looking for an "equilibrium" solution $\sigma_{inv} = \sigma_o \gamma^n$

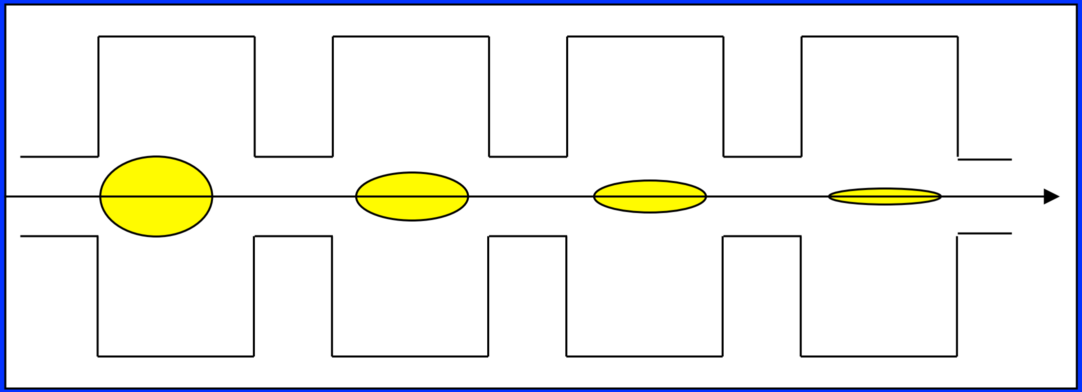
==> all terms must have the same dependence on γ

Laminar beam $\rho >> 1 \Rightarrow n = -\frac{1}{2}$ $\sigma_q = \frac{\sigma_o}{\sqrt{\gamma}}$

Thermal beam $\rho << 1 \Rightarrow n = 0$ $\sigma_\varepsilon = \sigma_o$

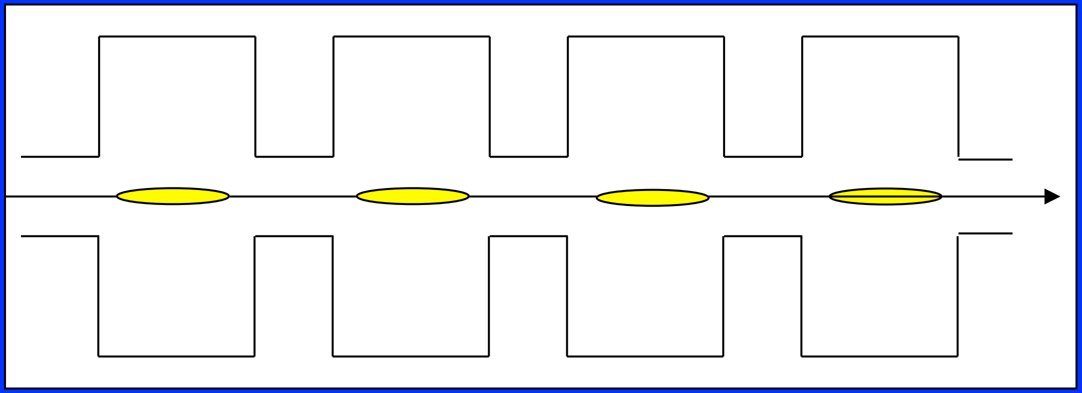
Space charge dominated beam (Laminar)

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}}$$

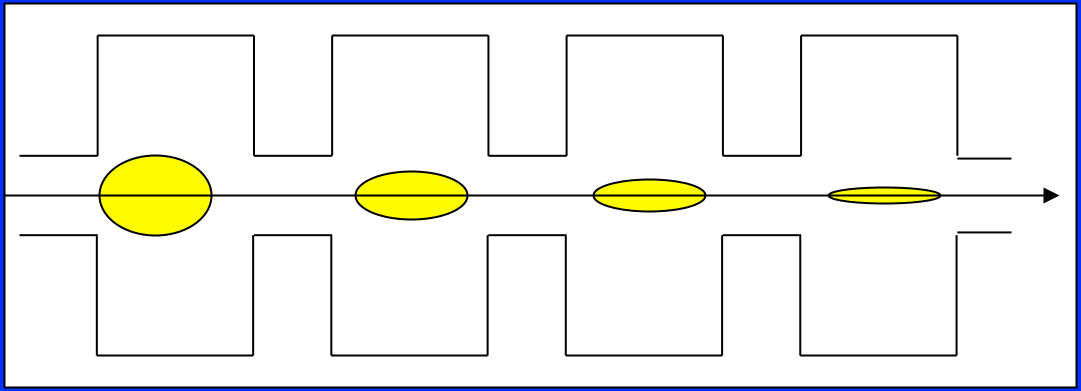


Emittance dominated beam (Thermal)

$$\sigma_\varepsilon = \sqrt{\frac{2\varepsilon_n}{\gamma'}}$$



$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}}$$



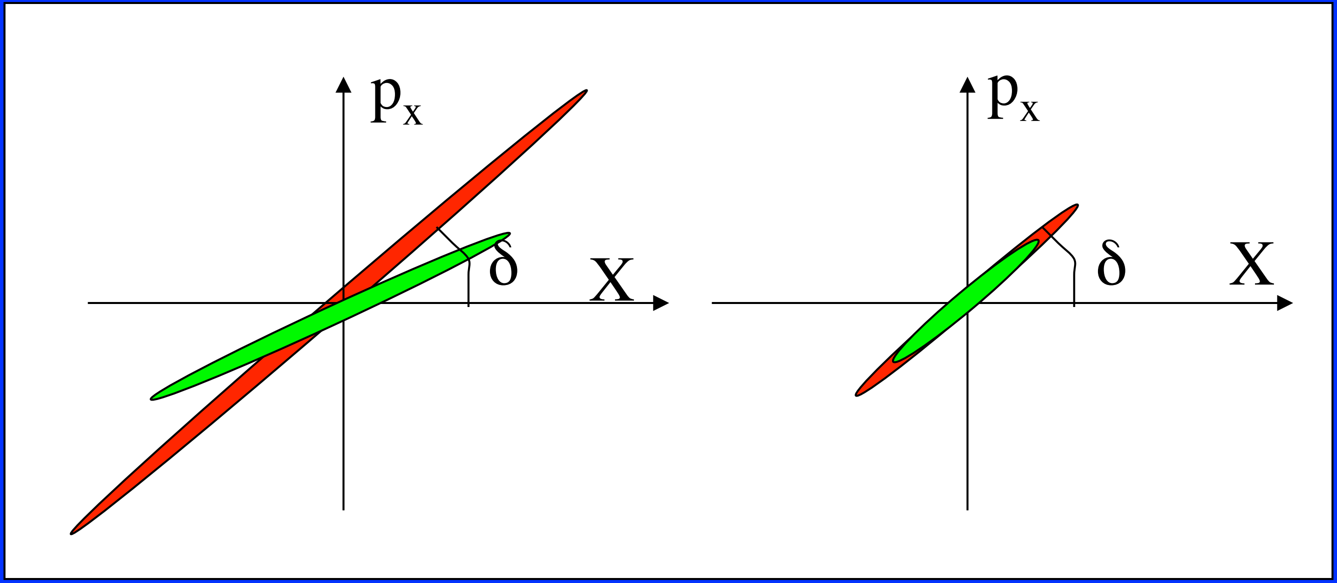
This solution represents a **beam equilibrium mode** that turns out to be the transport mode for achieving minimum emittance at the end of the **emittance correction process**

An important property of the laminar beam

$$\sigma_q = \frac{1}{\gamma'}\sqrt{\frac{2I}{I_A\gamma}}$$

$$\sigma'_q = -\sqrt{\frac{2I}{I_A\gamma^3}}$$

Constant phase space angle: $\delta = \frac{\gamma\sigma'_q}{\sigma_q} = -\frac{\gamma'}{2}$

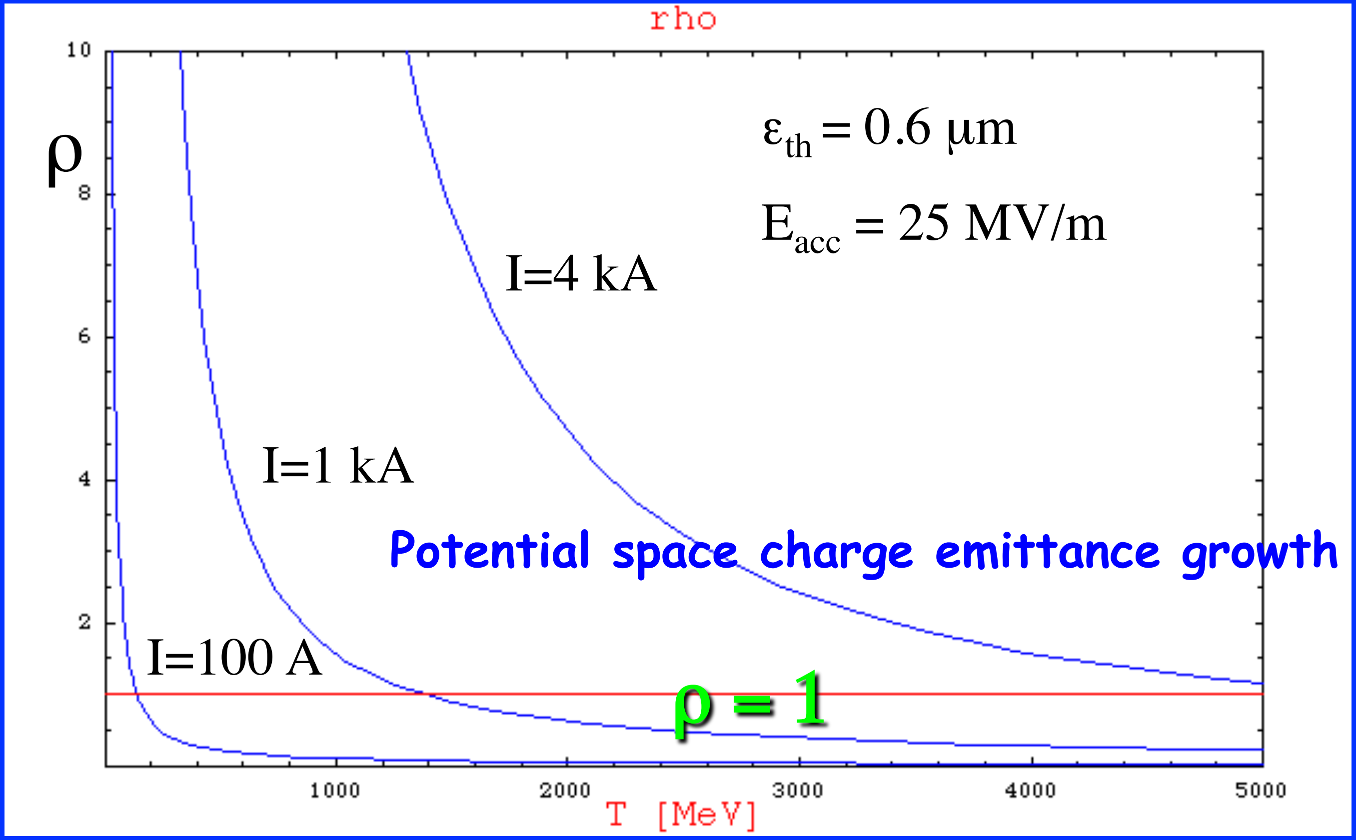


Laminarity parameter

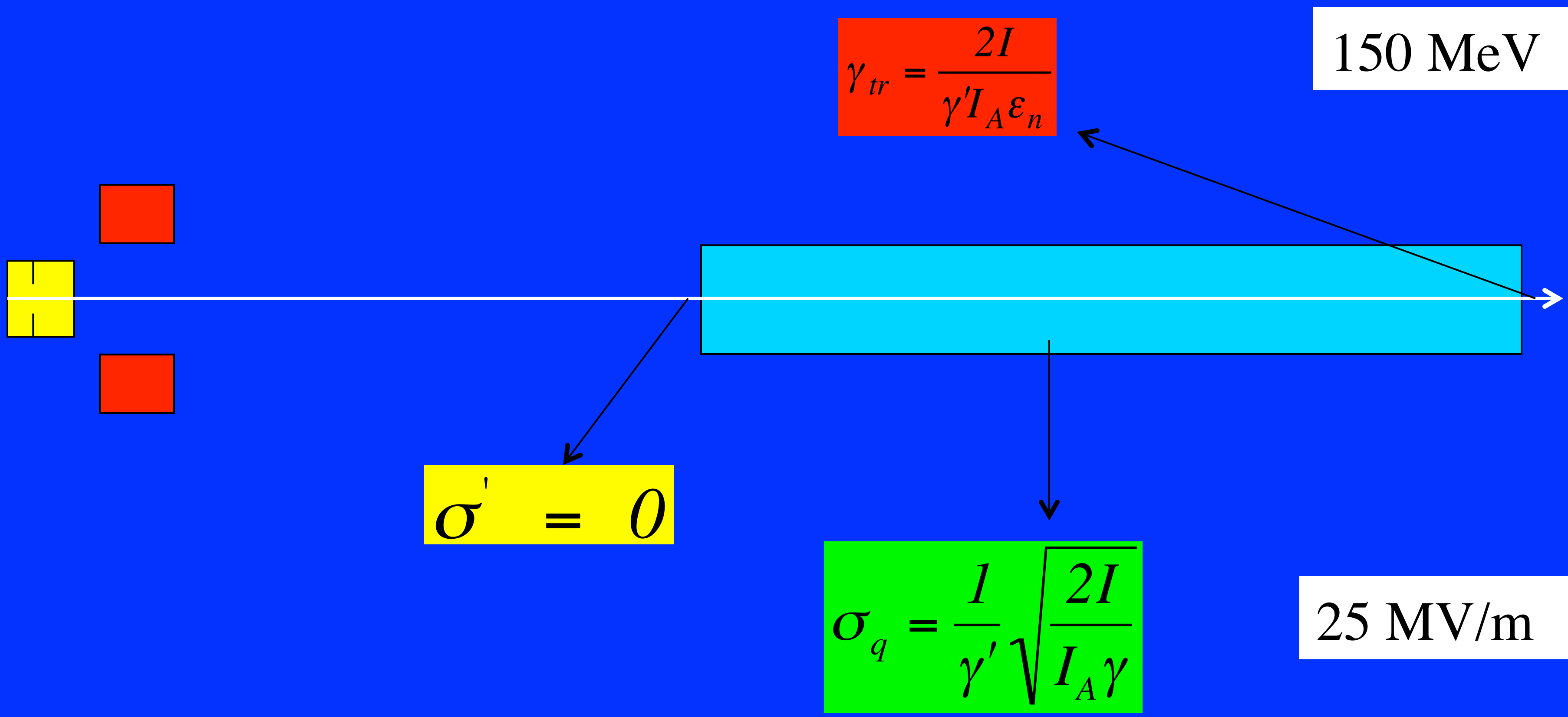
$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

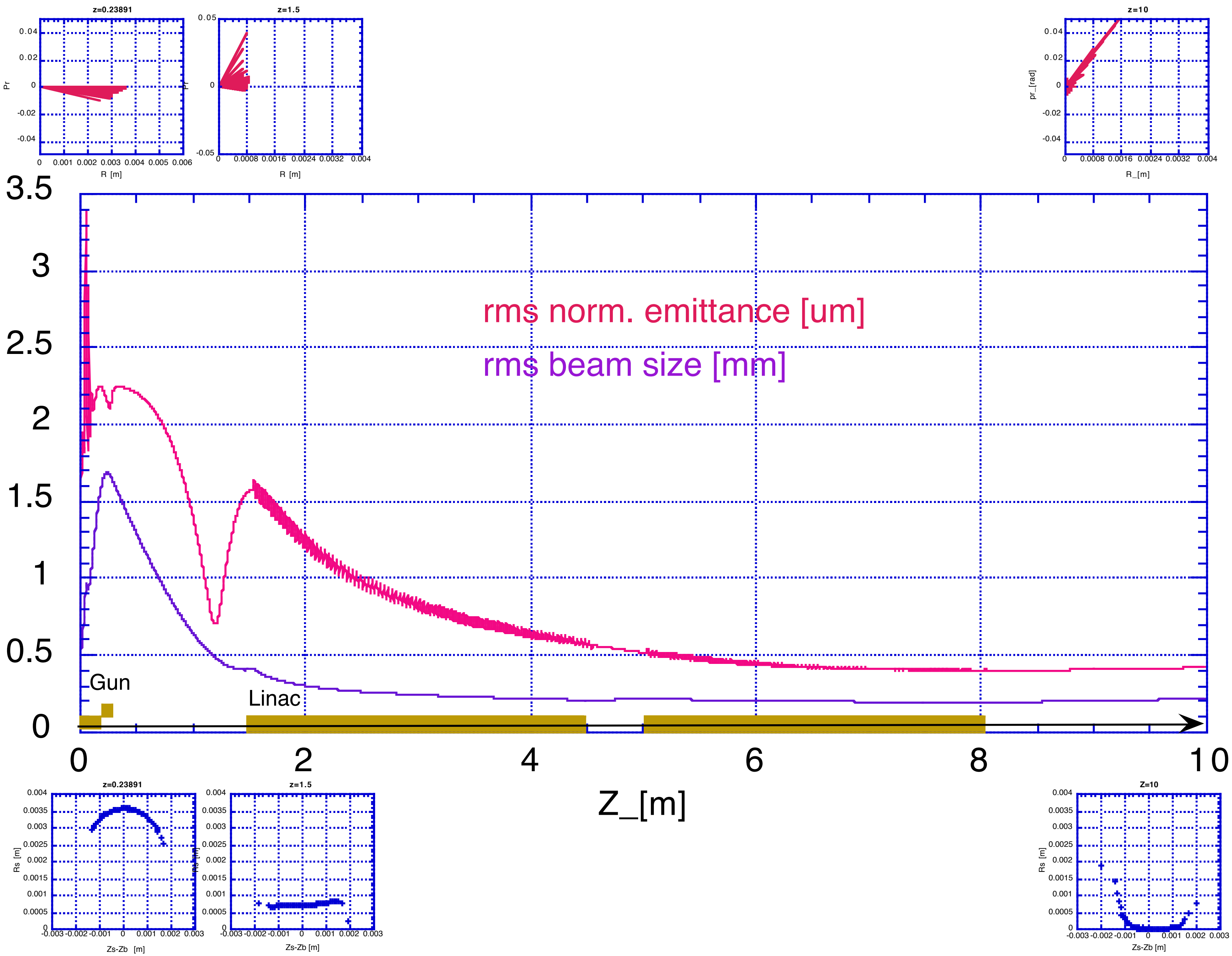
Transition Energy (ρ=1)

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$

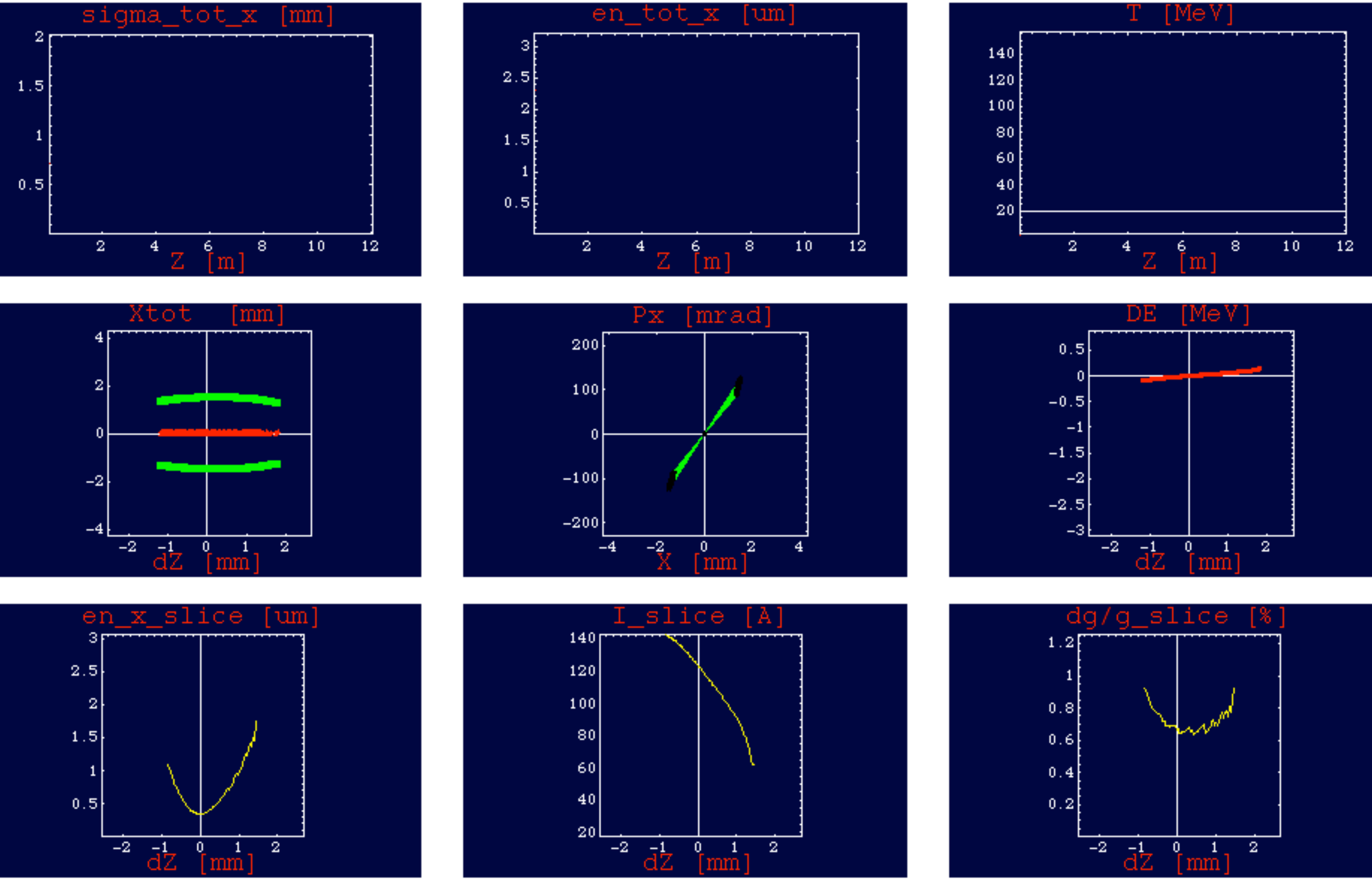


Matching Conditions with a TW Linac





z=0.23891
Pr
R [m]
z=1.5
R [m]
z=10
pr_[rad]
R_[m]
rms norm. emittance [um]
rms beam size [mm]
Gun
Linac
Z_[m]
z=0.23891
Rs [m]
Zs-Zb [m]
z=1.5
Zs-Zb [m]
Z=10
Rs [m]
Zs-Zb [m]


sigma_tot_x [mm]
Z [m]
en_tot_x [um]
Z [m]
T [MeV]
Z [m]
Xtot [mm]
dZ [mm]
Px [mrad]
X [mm]
DE [MeV]
dZ [mm]
en_x_slice [um]
dZ [mm]
I_slice [A]
dZ [mm]
dg/g_slice [%]
dZ [mm]

Emittance Compensation for a SC dominated beam: Controlled Damping of Plasma Oscillations

- ε_n oscillations are driven by Space Charge

- propagation close to the laminar solution allows control of ε_n oscillation "phase"

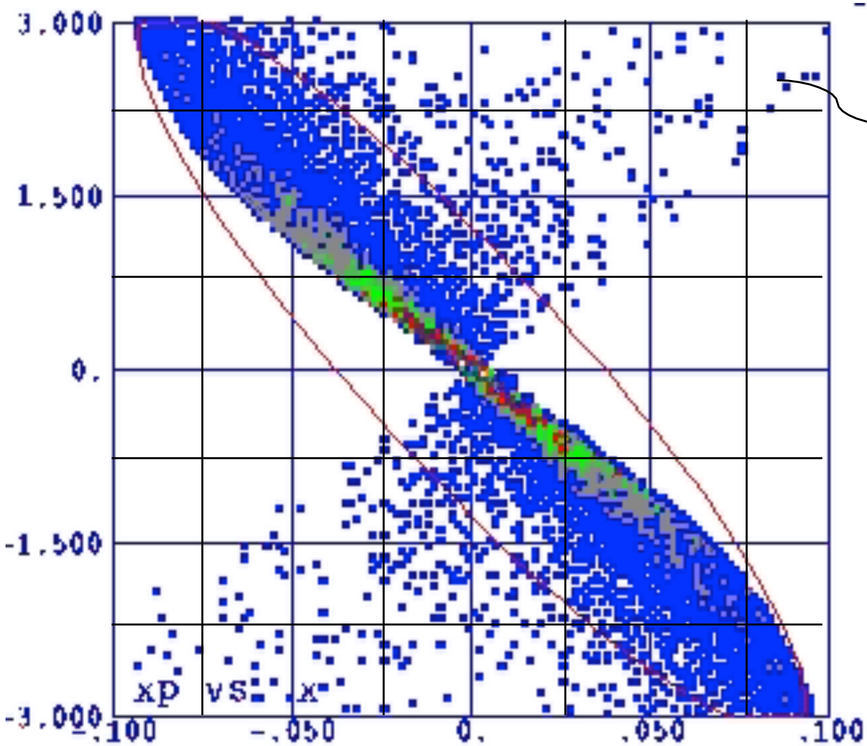
- ε_n sensitive to SC up to the transition energy

References:

[1] T. Shintake, Proc. of the 22nd Particle Accelerator Conference, June 25-29, 2007, Albuquerque, NM (IEEE, New York, 2007), p. 89.
[2] L. Serafini, J. B. Rosenzweig, PR E55 (1997) 7565
[3] M. Reiser, "Theory and Design of Charged Particle Beams" , Wiley, New York, 1994
[4] J. B. Rosenzweig, "Fundamentals of beam physics", Oxford University Press, New York, 2003
[5] T. Wangler, "Principles of RF linear accelerators", Wiley, New York, 1998
[6] S. Humphries, "Charged particle beams", Wiley, New York, 2002
[7] F. J. Sacherer, F. J., IEEE Trans. Nucl. Sci. NS-18, 1105 (1971).
[8] M. Ferrario et al., Int. Journal of Modern Physics A, Vol 22, No. 23, 4214 (2007)
[9] J. Buon, "Beam phase space and emittance", in CERN 94-01

THE END

Emittance and Entropy



$$A = \delta x \delta x'$$

The entropy of the distribution is by definition:

$$S = k \log W$$

$$W = \frac{N!}{n_1! n_2! \ldots n_M!}$$

is the number of ways in which the points can be assigned to the cells to produce the given distribution

For large N, n, Stirling's formula gives: $\log W = N \log N - \sum_{i=1}^{M} n_i \log n_i.$

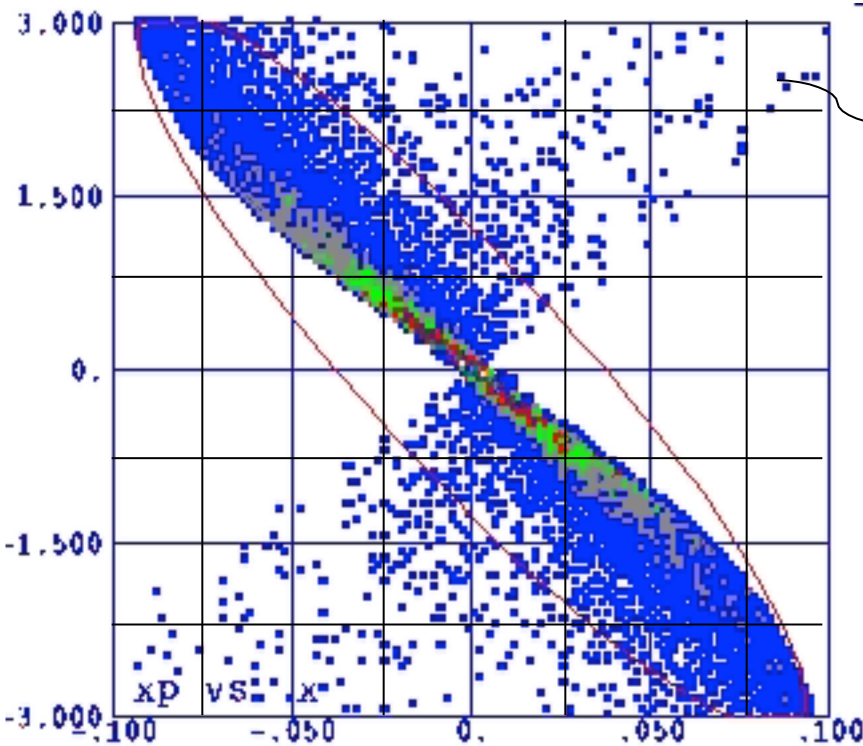
If A is sufficiently small, the summation may be replaced by an integral to give:

$$S/kN = S_0 = \log N - \frac{1}{N}\int \rho \log A\, \rho \,\mathrm{d}x\, \mathrm{d}x'$$

$$\rho = n/A$$

$$\int \rho \, dx \, dx' = N$$

Emittance and Entropy



$$A = \delta x \delta x'$$

The entropy of the distribution is by definition:

$$S = k \log W$$

Consider a distribution in which the density is uniform and bounded by an ellipse of area πε and:

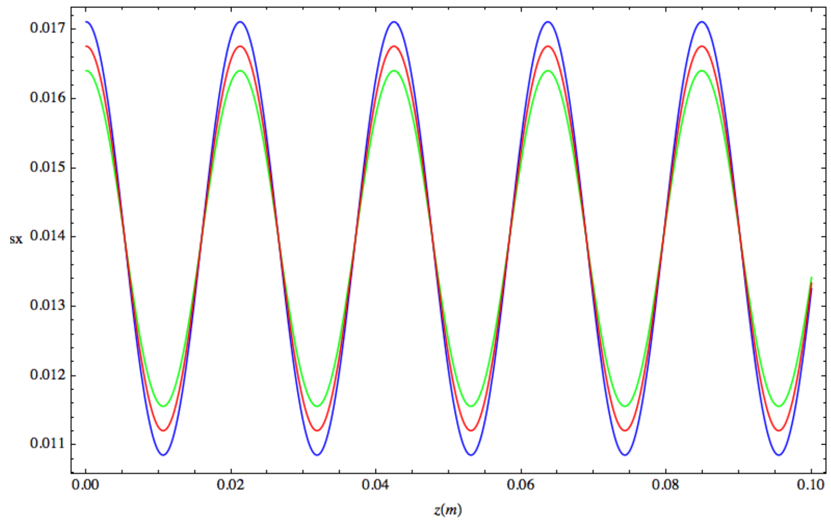
$$\rho = \frac{n}{A} = \frac{N}{\pi\varepsilon} \qquad \int \rho \, dx \, dx' = N$$

$$S/kN = S_0 = \log N - \frac{1}{N}\int \rho \log A \, \rho \, \mathrm{d}x \, \mathrm{d}x'$$

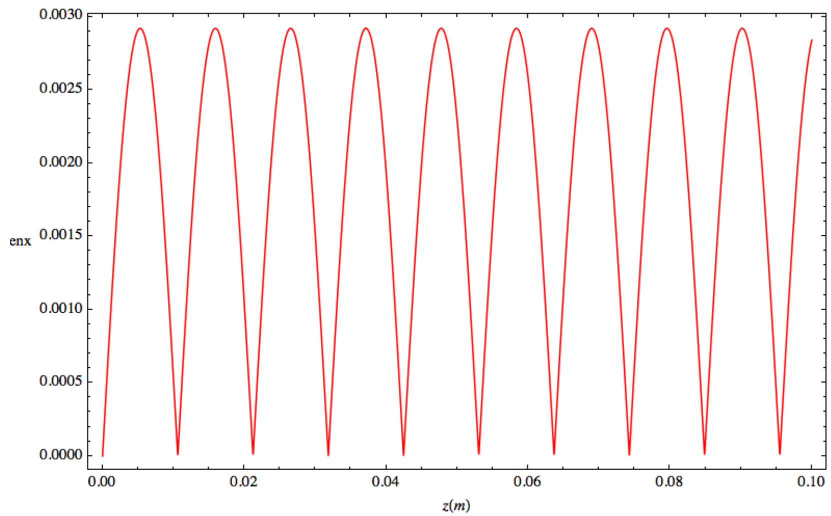
$$S_o = \log(N) - \log\left(\frac{AN}{\pi\varepsilon}\right) = \log(\pi\varepsilon) - \log(A)$$

Envelope oscillations drive Emittance oscillations

$\sigma(z)$



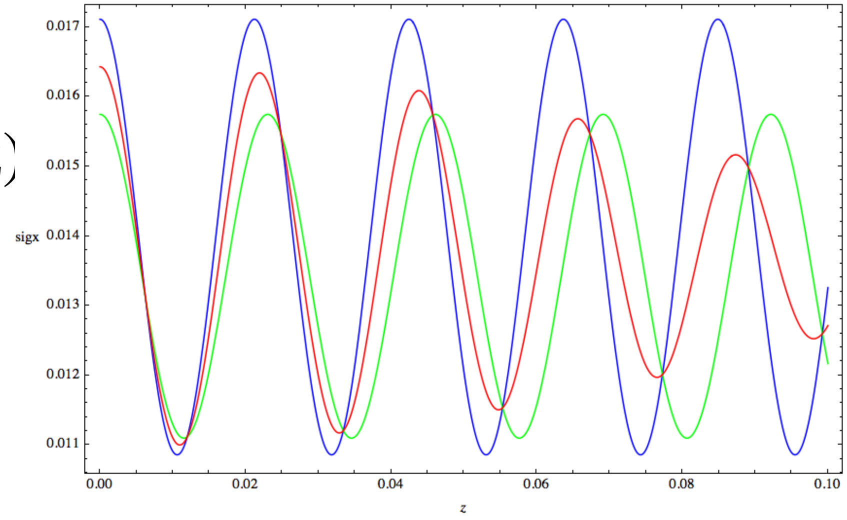
$\varepsilon(z)$



$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2\right)} \approx \left| \sin\left(\sqrt{2} k_s z\right) \right|$$

energy spread induces decoherence

σ(z)



ε(z)