A Bottom-Up Approach to Lepton Flavor and CP Symmetries

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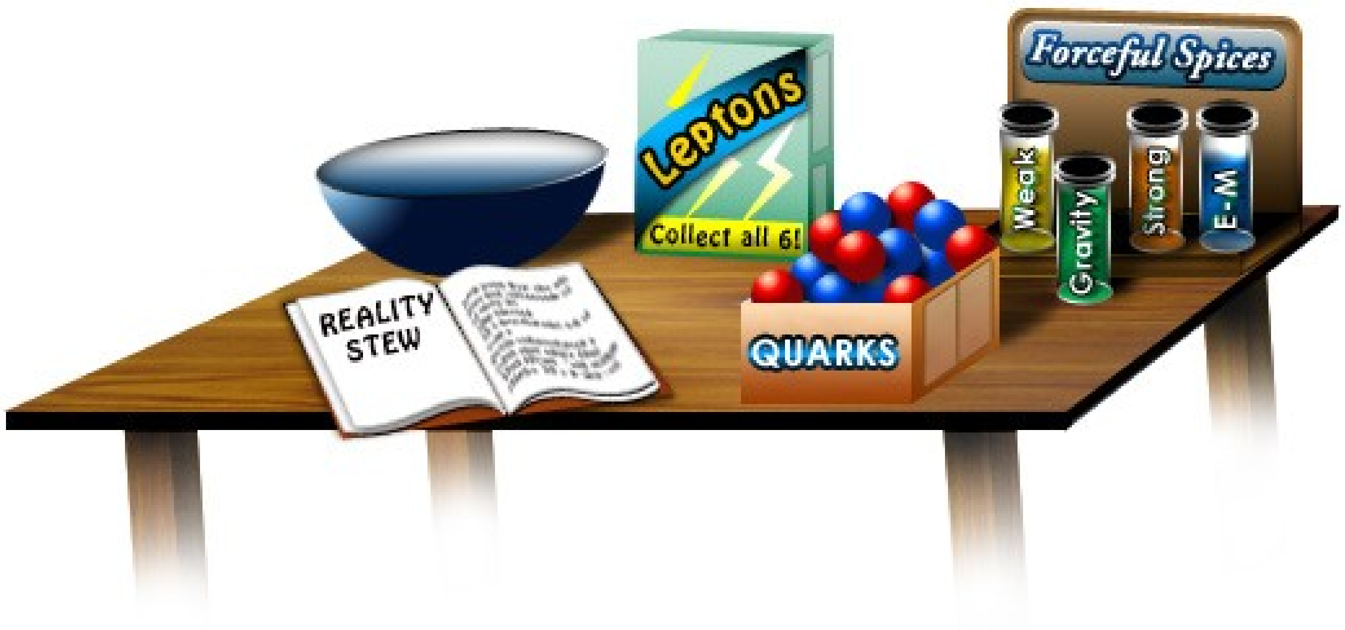
Based on: L.L. Everett, T. Garon, and AS, JHEP **1504**, 069 (2015) [arXiv:1501.04336]

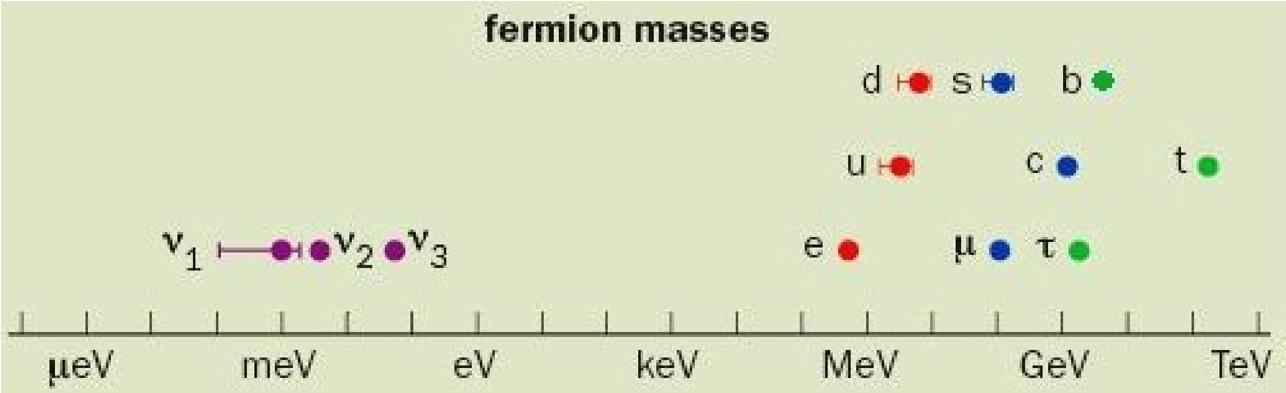
The Standard Model

Triumph of modern science, but incomplete-
Fails to predict the measured fermion masses and mixings.



http://www.particleadventure.org/standard_model.html

What We Taste



Quark Mixing

$$U_{CKM} = R_1(\theta_{23}^{CKM}) R_2(\theta_{13}^{CKM}, \delta_{CKM}) R_3(\theta_{12}^{CKM})$$

$$\theta_{13}^{CKM} = 0.2^\circ \pm 0.1^\circ$$

$$\theta_{23}^{CKM} = 2.4^\circ \pm 0.1^\circ$$

$$\theta_{12}^{CKM} = 13.0^\circ \pm 0.1^\circ$$

$$\delta_{CKM} = 60^\circ \pm 14^\circ$$

Lepton Mixing

$$U_{PMNS} = R_1(\theta_{23}) R_2(\theta_{13}, \delta_{CP}) R_3(\theta_{12}) P$$

$$\theta_{13} = 8.50^\circ \left(^{+0.20^\circ}_{-0.21^\circ}\right)$$

$$\theta_{23} = 42.3^\circ \left(^{+3.0^\circ}_{-1.6^\circ}\right)$$

$$\theta_{12} = 33.48^\circ \left(^{+0.78^\circ}_{-0.75^\circ}\right)$$

$$\delta_{CP} = 306^\circ \left(^{+39^\circ}_{-70^\circ}\right)$$

M.C. Gonzalez-Garcia et al: 1409.5439

http://lbne.fnal.gov/how-work.shtml

Focus on leptons.

Residual Charged Lepton Symmetry

Since charged leptons are Dirac articles, consider $M_e = m_e m_e^\dagger$.
When **diagonal**, this combination is left invariant by a phase matrix

$$Q_e = \mathrm{Diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3})$$

Because $Q_e^\dagger M_e Q_e = M_e$

$$\text{Let } T = Q_e \text{ and } \beta_i = 2\pi k_i/m \text{ with } k_i = 0, 1, \ldots, m-1$$

$$m \text{ an integer}$$

Notice that *T* generates a Z_m abelian symmetry.

Assume M_e diagonal. Then, $U_e = 1$ and *all* mixing comes from neutrino sector.

$$U_{MNSP} = U_e^\dagger U_\nu$$

To this end, what are the possible residual symmetries in the neutrino sector?

Residual Neutrino Flavor Symmetry

Key: Assume neutrinos are Majorana particles

$$U_\nu^T M_\nu U_\nu = M_\nu^{\text{Diag}} = \text{Diag}(m_1, m_2, m_3) = \text{Diag}(|m_1|e^{-i\alpha_1}, |m_2|e^{-i\alpha_2}, |m_3|e^{-i\alpha_3})$$

Notice $U_\nu \to U_\nu Q_\nu$ with $Q_\nu = \text{Diag}(\pm1, \pm1, \pm1)$ also diagonalizes the neutrino mass matrix. Restrict to $\text{Det}(Q_\nu) = 1$ and define $G_0^{\text{Diag}} = 1$

$$G_1^{\text{Diag}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G_2^{\text{Diag}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, G_3^{\text{Diag}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe non-trivial relations:
$$(G_i^{\text{Diag}})^2 = 1, \text{ for } i\text{=1, 2, and 3,}$$
$$G_i^{\text{Diag}} G_j^{\text{Diag}} = G_k^{\text{Diag}}, \text{ for } i \neq j \neq k$$

Sometimes called *SU, S,* and *U*

Therefore, these form a $Z_2 \times Z_2$ residual Klein symmetry!

In non-diagonal basis: $M_\nu = G_i^T M_\nu G_i$ with $G_i = U_\nu G_i^{\text{Diag}} U_\nu^\dagger$

How should we express U_ν to transform to the non-diagonal basis?

Parameterizing U_ν

Since we are bottom-up, we want to keep track of phases, so let

$$U_\nu = P R_x(\theta_{23}, \delta_x) R_y(\theta_{13}, \delta_y) R_z(\theta_{12}, \delta_z)$$

$$R_x(\theta_{23}, \delta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{-i\delta_x} \\ 0 & -s_{23}e^{i\delta_x} & c_{23} \end{pmatrix} \qquad R_y(\theta_{13}, \delta_y) = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_y} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_y} & 0 & c_{13} \end{pmatrix}$$

$$R_z(\theta_{12}, \delta_z) = \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_z} & 0 \\ -s_{12}e^{i\delta_z} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$s_{ij} = \sin(\theta_{ij}) \qquad c_{ij} = \cos(\theta_{ij})$$

$$U_\nu = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12}e^{-i\delta_z} & s_{13}e^{-i\delta_y} \\ -c_{23}s_{12}e^{-i\delta_z} - c_{12}s_{13}s_{23}e^{-i(\delta_x-\delta_y)} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{-i(\delta_x-\delta_y+\delta_z)} & c_{13}s_{23}e^{-i\delta_x} \\ c_{12}c_{23}s_{13}e^{i\delta_y} - s_{12}s_{23}e^{i(\delta_x+\delta_z)} & c_{23}s_{12}s_{13}e^{i(\delta_y-\delta_z)} + c_{12}s_{23}e^{i\delta_x} & -c_{13}c_{23} \end{pmatrix}$$

Anymore re-phasing freedom?

It's Looking More Familiar

Consider $P' = \mathrm{Diag}\left(1, \exp\left(-i\delta_z\right), \exp\left(-i(\delta_z + \delta_x)\right)\right)$

And identify Dirac CP-violating phase using Jarlskog Invariant. (C. Jarlskog (1985))

$$P' U_\nu(\theta_{23}, \theta_{13}, \theta_{12}, \delta) P'^* = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ -s_{12}s_{23} + c_{12}c_{23}s_{13}e^{i\delta} & c_{12}s_{23} + c_{23}s_{12}s_{13}e^{i\delta} & -c_{13}c_{23} \end{pmatrix}$$

$$\delta = \delta_y - \delta_x - \delta_z$$

Notice, if charged leptons are (assumed) diagonal U_e=1 and the above matrx is the MNSP matrix in the PDG convention up to left multiplication by P matrix.

Why express it like this?

Eigenvectors of the Klein Symmetry

Recall each nontrivial Klein element has one +1 eigenvalue.

The eigenvector associated with this eigenvalue will be one column of the MNSP matrix (in the diagonal charged lepton basis).
As an example consider tribimaximal mixing.

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} P$$

P. F. Harrison, D. H. Perkins, W. G. Scott (2002)
P. F. Harrison, W. G. Scott (2002)
Z. -z. Xing (2002)

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad U = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Now that we see the relevance our parameterization, let us find the most general Klein elements as functions of the mixing parameters.

Non-Diagonal Klein Elements

$$G_i = U_\nu G_i^{\mathrm{Diag}} U_\nu^\dagger$$

$$G_1 = \begin{pmatrix} (G_1)_{11} & (G_1)_{12} & (G_1)_{13} \\ (G_1)_{12}^* & (G_1)_{22} & (G_1)_{23} \\ (G_1)_{13}^* & (G_1)_{23}^* & (G_1)_{33} \end{pmatrix} \quad G_2 = \begin{pmatrix} (G_2)_{11} & (G_2)_{12} & (G_2)_{13} \\ (G_2)_{12}^* & (G_2)_{22} & (G_2)_{23} \\ (G_2)_{13}^* & (G_2)_{23}^* & (G_2)_{33} \end{pmatrix}$$

$$G_3 = \begin{pmatrix} -c'_{13} & e^{-i\delta} s_{23} s'_{13} & -e^{-i\delta} c_{23} s'_{13} \\ e^{i\delta} s_{23} s'_{13} & s_{23}^2 c'_{13} - c_{23}^2 & -c_{13}^2 s'_{23} \\ -e^{i\delta} c_{23} s'_{13} & -c_{13}^2 s'_{23} & c_{23}^2 c'_{13} - s_{23}^2 \end{pmatrix}$$

$$s_{ij} = \sin(\theta_{ij}) \quad c_{ij} = \cos(\theta_{ij}) \quad s'_{ij} = \sin(2\theta_{ij}) \quad c'_{ij} = \cos(2\theta_{ij})$$

Notice that in general the Klein elements are complex and Hermitian!
They cannot depend on Majorana phases because
$U_\nu \to U_\nu P_{\mathrm{Maj}}$ leaves transformation invariant.

Non-Diagonal Klein Elements (II)

$$(G_1)_{11} = c_{13}^2 c_{12}' - s_{13}^2 \quad (G_1)_{12} = -2c_{12}c_{13}\left(c_{23}s_{12} + e^{-i\delta}c_{12}s_{13}s_{23}\right)$$

$$(G_1)_{13} = 2c_{12}c_{13}\left(e^{-i\delta}c_{12}c_{23}s_{13} - s_{12}s_{23}\right)$$

$$(G_1)_{22} = -c_{23}^2 c_{12}' + s_{23}^2\left(s_{13}^2 c_{12}' - c_{13}^2\right) + \cos(\delta)s_{13}s_{12}'s_{23}'$$

$$(G_1)_{23} = c_{23}s_{23}c_{13}^2 + s_{13}\left(i\sin(\delta) - \cos(\delta)c_{23}'\right)s_{12}' + \frac{1}{4}c_{12}'\left(c_{13}' - 3\right)s_{23}'$$

$$(G_1)_{33} = \left(s_{13}^2 c_{12}' - c_{13}^2\right)c_{23}^2 - s_{23}^2 c_{12}' - \cos(\delta)s_{13}s_{12}'s_{23}'$$

$$(G_2)_{11} = -c_{12}'c_{13}^2 - s_{13}^2 \quad (G_2)_{12} = 2c_{13}s_{12}\left(c_{12}c_{23} - e^{-i\delta}s_{12}s_{13}s_{23}\right)$$

$$(G_2)_{13} = 2c_{13}s_{12}\left(e^{-i\delta}c_{23}s_{12}s_{13} + c_{12}s_{23}\right)$$

$$(G_2)_{22} = c_{12}'c_{23}^2 - s_{23}^2\left(c_{13}^2 + s_{13}^2 c_{12}'\right) - \cos(\delta)s_{13}s_{12}'s_{23}'$$

$$(G_2)_{23} = e^{-i\delta}s_{13}s_{12}'c_{23}^2 + \frac{1}{4}s_{23}'\left(2c_{13}^2 - c_{12}'\left(c_{13}' - 3\right)\right) - e^{i\delta}s_{12}'s_{13}s_{23}^2$$

$$(G_2)_{33} = -c_{23}^2\left(c_{13}^2 + s_{13}^2 c_{12}'\right) + s_{23}^2 c_{12}' + \cos(\delta)s_{13}s_{12}'s_{23}'$$

There is a Klein symmetry for each choice of mixing angle and CP-violating phase, implying a mass matrix left invariant for each choice.

Invariant Mass Matrix

$$M_\nu = U_\nu^* M_\nu^{\mathrm{Diag}} U_\nu^\dagger$$

$$(M_\nu)_{11} = c_{13}^2 m_2 s_{12}^2 + c_{12}^2 c_{13}^2 m_1 + e^{2i\delta} m_3 s_{13}^2$$

$$(M_\nu)_{12} = c_{13}(c_{12} m_1(-c_{23} s_{12} - c_{12} e^{-i\delta} s_{13} s_{23}) + m_2 s_{12}(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23}) + \\ + e^{i\delta} m_3 s_{13} s_{23}),$$

$$(M_\nu)_{13} = c_{13}(-c_{23} m_3 s_{13} e^{i\delta} + m_2 s_{12}(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13}) + \\ + c_{12} m_1(-s_{12} s_{23} + c_{12} c_{23} e^{-i\delta} s_{13})),$$

$$(M_\nu)_{22} = m_1(c_{23} s_{12} + c_{12} e^{-i\delta} s_{13} s_{23})^2 + m_2(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23})^2 + c_{13}^2 m_3 s_{23}^2$$

$$(M_\nu)_{23} = m_1(s_{12} s_{23} - c_{12} c_{23} e^{-i\delta} s_{13})(c_{23} s_{12} + c_{12} e^{-i\delta} s_{13} s_{23}) + \\ + m_2(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13})(c_{12} c_{23} - e^{-i\delta} s_{12} s_{13} s_{23}) - c_{13}^2 c_{23} m_3 s_{23}$$

$$(M_\nu)_{33} = m_2(c_{12} s_{23} + c_{23} e^{-i\delta} s_{12} s_{13})^2 + m_1(-s_{12} s_{23} + c_{12} c_{23} e^{-i\delta} s_{13})^2 + c_{13}^2 c_{23}^2 m_3$$

Recall these masses are complex. How can we predict their phases?

Generalized CP Symmetries

G. Branco, L. Lavoura, M. Rebelo (1986)...

Superficially look similar to flavor symmetries:

$$X_\nu^T M_\nu X_\nu = M_\nu^* \quad X_e^\dagger M_e X_e = M_e^*$$

***X*=1 is 'traditional' CP**

Related to automorphism group of flavor symmetry (Holthausen et al. (2012))

Since they act in a similar fashion to flavor symmetries, these two symmetries should be related. (Feruglio et al (2012), Holthausen et al. (2012)):

$$X_\nu G_i^* - G_i X_\nu = 0$$

Can be used to make predictions concerning both Dirac and Majorana CP violating phases, e.g. $X=G_2$

Proceed analogously to flavor symmetry.

The Harbingers of Majorana Phases

(S.M. Bilenky, J. Hosek, S.T. Petcov(1980))

Work in diagonal basis. Then it is trivial to see $X = U_\nu X^{\text{Diag}} U_\nu^T$

with $$X^{\text{Diag}} = \begin{pmatrix} \pm e^{i\alpha_1} & 0 & 0 \\ 0 & \pm e^{i\alpha_2} & 0 \\ 0 & 0 & \pm e^{i\alpha_3} \end{pmatrix}$$

where α_i are Majorana phases.

Notice we have freedom to globally re-phase: $M_\nu \rightarrow e^{i\theta} M_\nu$

Such a re-phasing will not affect the mixing angles or observable phases.

Now can make the important observation

$$X_i^{\text{Diag}} = G_i^{\text{Diag}} \times \text{Diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$$

Therefore, the X_i represent a *complexification* of the Klein symmetry elements!

So, they must inherit an algebra from the Klein elements...

Generalized CP Relations

To eliminate phases, must have one *X* conjugated

$$X_0 X_i^* = G_i \text{ for } i = 1,\ 2,\ 3 \quad X_i X_j^* = G_k \text{ for } i \neq j \neq k \neq 0$$

$$X_i X_i^* = G_0 = 1 \text{ for } i = 0,\ 1,\ 2,\ 3$$

Clearly these imply:

$$(X_0 X_i^*)^2 = 1 \text{ for } i = 1, 2, 3 \qquad (X_i X_j^*)^2 = 1 \text{ for } i \neq j \neq 0$$

$$X_i X_i^* = 1 \text{ for } i = 0,\ 1,\ 2,\ 3$$

Note if $X_i X_j^* = G' \neq G_k$ flavor symmetry is enlarged leading to unphysical predictions because Klein symmetry is largest symmetry to completely fix mixing and masses.

$$X_j^\dagger X_i^T M_\nu X_i X_j^* = M_\nu$$

So what do these generalized CP elements look like in non-diagonal basis?

The Non-Diagonal General CP

$$X_{11} = (-1)^a e^{i\alpha_1} c_{12}^2 c_{13}^2 + (-1)^b e^{i\alpha_2} c_{13}^2 s_{12}^2 + (-1)^c s_{13}^2 e^{i(\alpha_3 - 2\delta)}$$

$$X_{12} = (-1)^{a+1} e^{i\alpha_1} c_{12} c_{13} (c_{23} s_{12} + c_{12} s_{13} s_{23} e^{i\delta}) + (-1)^b e^{i\alpha_2} c_{13} s_{12} (c_{12} c_{23} - \\ - s_{12} s_{13} s_{23} e^{i\delta}) + (-1)^c c_{13} s_{13} s_{23} e^{i(\alpha_3 - \delta)},$$

$$X_{13} = (-1)^{a+1} e^{i\alpha_1} c_{12} c_{13} (s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta}) + (-1)^b e^{i\alpha_2} c_{13} s_{12} (c_{12} s_{23} + \\ + c_{23} s_{12} s_{13} e^{i\delta}) + (-1)^{c+1} c_{13} c_{23} s_{13} e^{i(\alpha_3 - \delta)},$$

$$X_{22} = (-1)^a e^{i\alpha_1} (c_{23} s_{12} + c_{12} s_{13} s_{23} e^{i\delta})^2 + (-1)^b e^{i\alpha_2} (c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta})^2 + \\ + (-1)^c e^{i\alpha_3} c_{13}^2 s_{23}^2,$$

$$X_{23} = (-1)^a e^{i\alpha_1} (s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta})(c_{23} s_{12} + c_{12} s_{13} s_{23} e^{i\delta}) + \\ + (-1)^b e^{i\alpha_2} (c_{12} s_{23} + c_{23} s_{12} s_{13} e^{i\delta})(c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta}) + (-1)^{c+1} e^{i\alpha_3} c_{23} c_{13}^2 s_{23}$$

$$X_{33} = (-1)^a e^{i\alpha_1} (s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta})^2 + (-1)^b e^{i\alpha_2} (c_{12} s_{23} + c_{23} s_{12} s_{13} e^{i\delta})^2 + \\ + (-1)^c e^{i\alpha_3} c_{13}^2 c_{23}^2.$$

$(-1)^a = (G_i^{\mathrm{Diag}})_{11}$ $\quad$ $(-1)^b = (G_i^{\mathrm{Diag}})_{22}$ $\quad$ $(-1)^c = (G_i^{\mathrm{Diag}})_{33}$

Proofs by Construction

Can use explicit forms for G_j and X_i to easily show

$$X_i G_j^* - G_j X_i = 0 \text{ for } i,j = 0,1,2,3$$

Now when just the Dirac CP-violation is trivial, it is easy to see

$$[X_i, G_j]_{\delta=0,\pi} = 0 \text{ for } i,j = 0,1,2,3$$

Can easily be understood from the forms of G_i since $G_i = G_i^*$ implies a trivial Dirac phase.

If just Majorana phases are let to vanish, then

$$(X_i - G_i)_{mn} \propto (e^{2i\delta} - 1) \text{ for } i = 0,1,2,3$$

implying equality if Dirac vanishes as well. Therefore, if one wants commutation between flavor and CP, then this will *always* lead to a trivial Dirac phase. Furthermore, if they are equal then all phases must vanish (**Think** *M*=*M**).

What else can we use this for?

Tribimaximal Mixing (I)

P. F. Harrison, D. H. Perkins, W. G. Scott (2002); P. F. Harrison, W. G. Scott (2002); Z. -z. Xing (2002)

$$\theta_{12}^{\rm TBM} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \theta_{23}^{\rm TBM} = \frac{\pi}{4} \quad \theta_{13}^{\rm TBM} = 0 \quad \delta^{\rm TBM} = 0$$

Plugging these values into the previous results yield:

$$U^{\rm TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad S_4$$

$$G_1^{\rm TBM} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{pmatrix} \quad G_2^{\rm TBM} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad G_3^{\rm TBM} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_\nu^{\rm TBM} = \frac{1}{3}\begin{pmatrix} (2m_1+m_2) & (m_2-m_1) & (m_2-m_1) \\ (m_2-m_1) & \frac{1}{2}(m_1+2m_2+3m_3) & \frac{1}{2}(m_1+2m_2-3m_3) \\ (m_2-m_1) & \frac{1}{2}(m_1+2m_2-3m_3) & \frac{1}{2}(m_1+2m_2+3m_3) \end{pmatrix}$$

The well-known mass matrix and Klein elements of TBM.

Tribimaximal Mixing (II)

$$X_{11}^{\text{TBM}} = \frac{1}{3}\left(2(-1)^a e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\right) \qquad X_{12}^{\text{TBM}} = \frac{1}{3}\left((-1)^{a+1} e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\right)$$

$$X_{22}^{\text{TBM}} = \frac{1}{6}\left((-1)^a e^{i\alpha_1} + 2e^{i\alpha_2}(-1)^b + 3e^{i\alpha_3}(-1)^c\right)$$

$$X_{13}^{\text{TBM}} = \frac{1}{3}\left((-1)^{a+1} e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\right)$$

$$X_{23}^{\text{TBM}} = \frac{1}{6}\left((-1)^a e^{i\alpha_1} + 2e^{i\alpha_2}(-1)^b - 3e^{i\alpha_3}(-1)^c\right)$$

$$X_{33}^{\text{TBM}} = \frac{1}{6}\left((-1)^a e^{i\alpha_1} + 2e^{i\alpha_2}(-1)^b + 3e^{i\alpha_3}(-1)^c\right)$$

Any generalized CP symmetry consistent with the TBM Klein symmetry will be given by the above results even if TBM is not coming from S_4.

Notice vanishing Majorana phases gives TBM Klein symmetry back.

Bitrimaximal Mixing (I)

R. Toorop, F. Feruglio, C. Hagedorn (2011); G.J. Ding (2012); S. King, C. Luhn, AS(2013)

$$\theta_{12}^{\mathrm{BTM}} = \theta_{23}^{\mathrm{BTM}} = \tan^{-1}(\sqrt{3}-1) \quad \theta_{13}^{\mathrm{BTM}} = \sin^{-1}(\frac{1}{6}(3-\sqrt{3})) \quad \delta^{\mathrm{BTM}} = 0$$

Yielding

$$U^{\mathrm{BTM}} = \begin{pmatrix} \frac{1}{6}\left(3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(3-\sqrt{3}\right) \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{6}\left(-3+\sqrt{3}\right) & \frac{1}{\sqrt{3}} & \frac{1}{6}\left(-3-\sqrt{3}\right) \end{pmatrix} \qquad \Delta(96)$$

$$G_1^{\mathrm{BTM}} = \begin{pmatrix} \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} & -\frac{1}{3} \\ -\frac{1}{3}-\frac{1}{\sqrt{3}} & -\frac{1}{3} & \frac{1}{\sqrt{3}}-\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} \end{pmatrix} \quad G_2^{\mathrm{BTM}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad G_3^{\mathrm{BTM}} = \begin{pmatrix} -\frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{3}}-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & -\frac{1}{3}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}-\frac{1}{3} \end{pmatrix}$$

And a mass matrix given by

$$(M_\nu^{\mathrm{BTM}})_{11} = \frac{1}{6}((2+\sqrt{3})m_1 + 2m_2 - (-2+\sqrt{3})m_3) \qquad (M_\nu^{\mathrm{BTM}})_{22} = \frac{1}{3}(m_1+m_2+m_3)$$

$$(M_\nu^{\mathrm{BTM}})_{13} = \frac{1}{6}(-m_1 + 2m_2 - m_3) \qquad (M_\nu^{\mathrm{BTM}})_{12} = \frac{1}{6}(-(1+\sqrt{3})m_1 + 2m_2 + (-1+\sqrt{3})m_3)$$

$$(M_\nu^{BTM})_{33} = \frac{1}{6}(-(-2+\sqrt{3})m_1 + 2m_2 + (2+\sqrt{3})m_3) \quad (M_\nu^{\mathrm{BTM}})_{23} = \frac{1}{6}((-1+\sqrt{3})m_1 + 2m_2 - (1+\sqrt{3})m_3)$$

Bitrimaximal Mixing (II)

$$X_{11}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{c+1}e^{i\alpha_3}\left(-2+\sqrt{3}\right)+(-1)^{a}\left(2+\sqrt{3}\right)e^{i\alpha_1}+2(-1)^{b}e^{i\alpha_2}\right)$$

$$X_{12}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{c}e^{i\alpha_3}\left(-1+\sqrt{3}\right)+(-1)^{a+1}\left(1+\sqrt{3}\right)e^{i\alpha_1}+2(-1)^{b}e^{i\alpha_2}\right)$$

$$X_{13}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{a+1}e^{i\alpha_1}+2(-1)^{b}e^{i\alpha_2}+(-1)^{c+1}e^{i\alpha_3}\right)$$

$$X_{22}^{\mathrm{BTM}} = \frac{1}{3}\left((-1)^{a}e^{i\alpha_1}+(-1)^{b}e^{i\alpha_2}+(-1)^{c}e^{i\alpha_3}\right)$$

$$X_{23}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{a}e^{i\alpha_1}\left(-1+\sqrt{3}\right)+2(-1)^{b}e^{i\alpha_2}+(-1)^{c+1}\left(1+\sqrt{3}\right)e^{i\alpha_3}\right)$$

$$X_{33}^{\mathrm{BTM}} = \frac{1}{6}\left((-1)^{a+1}e^{i\alpha_1}\left(-2+\sqrt{3}\right)+2(-1)^{b}e^{i\alpha_2}+(-1)^{c}\left(2+\sqrt{3}\right)e^{i\alpha_3}\right)$$

Non-Trivial Check: $\alpha_1 = \alpha_3 = \frac{\pi}{6}\ \ \alpha_2 = -\frac{\pi}{3}\ \ a = 1, b = 0, c = 1$

Matches known order 4 Δ(96) automorphism group element when unphysical phases redefined.

S. King, T. Neder(2014)
S. King, G. J. Ding (2014)

So this framework can match known results, can it be predictive?

Golden Ratio Mixing (GR1) (I)

A. Datta, F. Ling, P. Ramond (2003); Y. Kajiyama, M Raidal, A. Strumia (2007); L. Everett, AS (2008)

$$\theta_{12}^{\mathrm{GR1}} = \tan^{-1}\left(\frac{1}{\phi}\right) \quad \theta_{23}^{\mathrm{GR1}} = \frac{\pi}{4} \quad \theta_{13}^{\mathrm{GR1}} = 0 \quad \delta^{\mathrm{GR1}} = 0$$

$$\phi = (1+\sqrt{5})/2 \quad U^{\mathrm{GR1}} = \begin{pmatrix} \sqrt{\frac{\phi}{\sqrt{5}}} & \sqrt{\frac{1}{\sqrt{5}\phi}} & 0 \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{5}\phi}} & \frac{1}{\sqrt{2}}\sqrt{\frac{\phi}{\sqrt{5}}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{5}\phi}} & \frac{1}{\sqrt{2}}\sqrt{\frac{\phi}{\sqrt{5}}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad A_5$$

$$G_1^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\phi & \phi-1 \\ -\sqrt{2} & \phi-1 & -\phi \end{pmatrix} \quad G_2^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1-\phi & \phi \\ \sqrt{2} & \phi & 1-\phi \end{pmatrix} \quad G_3^{\mathrm{GR1}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$M_\nu^{\mathrm{GR1}} = \frac{1}{\sqrt{5}}\begin{pmatrix} \frac{m_1\phi^2+m_2}{\phi} & \frac{m_2-m_1}{\sqrt{2}} & \frac{m_2-m_1}{\sqrt{2}} \\ \frac{m_2-m_1}{\sqrt{2}} & \frac{(m_2+m_3)\phi^2+m_1+m_3}{2\phi} & \frac{m_2\phi^2-\sqrt{5}m_3\phi+m_1}{2\phi} \\ \frac{m_2-m_1}{\sqrt{2}} & \frac{m_2\phi^2-\sqrt{5}m_3\phi+m_1}{2\phi} & \frac{(m_2+m_3)\phi^2+m_1+m_3}{2\phi} \end{pmatrix}$$

What about the generalized CP symmetries?

Golden Ratio Mixing (II)

$$X_{11}^{\mathrm{GR1}} = \frac{(-1)^a e^{i\alpha_1}\phi^2 + e^{i\alpha_2}(-1)^b}{\sqrt{5}\phi} \qquad X_{12}^{\mathrm{GR1}} = \frac{(-1)^{a+1} e^{i\alpha_1} + e^{i\alpha_2}(-1)^b}{\sqrt{10}}$$

$$X_{13}^{\mathrm{GR1}} = \frac{(-1)^{a+1} e^{i\alpha_1} + e^{i\alpha_2}(-1)^b}{\sqrt{10}} \quad X_{22}^{\mathrm{GR1}} = \frac{(-1)^a e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\phi^2 + \sqrt{5}e^{i\alpha_3}(-1)^c\phi}{2\sqrt{5}\phi}$$

$$X_{23}^{\mathrm{GR1}} = \frac{(-1)^a e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\phi^2 + \sqrt{5}e^{i\alpha_3}(-1)^{c+1}\phi}{2\sqrt{5}\phi}$$

$$X_{33}^{\mathrm{GR1}} = \frac{(-1)^a e^{i\alpha_1} + e^{i\alpha_2}(-1)^b\phi^2 + \sqrt{5}e^{i\alpha_3}(-1)^c\phi}{2\sqrt{5}\phi}$$

Becomes Golden Klein Symmetry when Majorana phases vanish.
Any 'golden' generalized CP symmetry will be given by the above results, even if it does not come from A_5.

Is it really this easy?

A Caveat

If low energy parameters are not taken as inputs for generating the possible predictions for the Klein symmetry elements, it is possible to generate them by breaking a flavor group G_f *to* $Z_2 \times Z_2$ in the neutrino sector and Z_m in the charged lepton sector, while also consistently breaking H_{CP} *to* X_i.

Then predictions for parameters can become subject to charged lepton (CL) corrections, renormalization group evolution (RGE), and canonical normalization (CN) considerations.

Although, can expect these corrections to be subleading as RGE and CN effects are expected to be small in realistic models with hierarchical neutrino masses, and CL corrections are typically at most Cabibbo-sized. (J. Casa, J. Espinosa, A Ibarra, I Navarro (2000); S. Antusch, J Kersten, M. Lindner, M. Ratz (2003); S. King I. Peddie (2004); S. Antusch, S. King, M. Malinsky (2009);)

Conclusion (I)

- If neutrinos are Majorana particles, the possibility exists that there is a high scale flavor symmetry spontaneously broken to a residual Klein symmetry in the neutrino sector at low energies.
- If such a Klein symmetry is preserved it completely determines the mixing angles of the neutrino sector and provides specific relations between the entries of the neutrino mass matrix, but it is unable to provide predictions for the Majorana phases. For this, generalized CP must be implemented.
- Yet in such an approach, the exact roles that the generalized CP symmetry and flavor symmetry play in predicting mixing parameters appear to be quite model-dependent.

Conclusion (II)

- In arXiv:1501.04336, we have constructed a bottom-up approach that clarifies such roles by expressing the residual, unbroken Klein and generalized CP symmetries in terms of the lepton mixing parameters.
- This approach has been shown to reproduce key results in the literature (TBM, BTM) as well as predict future results (GR1).
- This framework can be used as a tool to aid future model-building efforts concerning Dirac and Majorana CP violating phases.
- The experimental confirmation of a non-trivial Dirac CP phase in the neutrino sector will add again another facet to the flavor puzzle. Theorists are preparing for such a result.

It is an exciting time to be a particle physicist!