The Standard Model Hypercharge: From Superconnection to Noncommutative Geometry

Chen Sun

Virginia Tech

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In collaboration with Ufuk Aydemir, Djordje Minic, Tatsu Takeuchi

Previous work:

arXiv:1409.7574 Ufuk Aydemir, Djordje Minic, CS, Tatsu Takeuchi.

and review paper on NCG in preparation.

Outline

What is superconnection formalism

- Extend the gauge connection from $su(2) + u(1)$ to $su(2|1)$, while leaving the gauge group still $SU(2) \times U(1)$, the even part of $SU(2|1)$.
- Has no relation to SUSY whatsoever.
- The two sheet picture: as the representation space, the LH fermions live on one sheet while the RH fermions live on the other.
- The Higgs is part of the extended gauge, the off diagonal part of the connection.
- With the two sheet picture, the Higgs field connects the fermions in the two sheets, while ordinary gauge field corresponds to the transformations in the continuous direction.
- Mass of fermions= VEV = (Separation of the two sheets)$^{-1}$.

$$J = J^a \lambda_a,$$

where $\lambda_a \in su(2|1)$, is the basis.

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$$\mathcal{J} = \mathcal{J}^a \lambda_a, \\ = i \begin{bmatrix} \mathcal{W} - \frac{1}{\sqrt{3}} B \cdot \mathbf{1}_{2\times 2} & \sqrt{2}\phi \\ \sqrt{2}\phi^\dagger & -\frac{2}{\sqrt{3}} B \end{bmatrix}, \qquad \Psi = \begin{bmatrix} \nu_L \\ e_L \\ e_R \end{bmatrix}.$$

$$\overline{\Psi}\mathcal{J}\Psi = \begin{bmatrix} \overline{\nu}_L & \overline{e}_L \end{bmatrix} \mathcal{W} \begin{bmatrix} \nu_L \\ e_L \end{bmatrix} - \frac{1}{\sqrt{3}} \overline{\nu}_L B \nu_L - \frac{1}{\sqrt{3}} \overline{e}_L B e_L - \frac{2}{\sqrt{3}} \overline{e}_R B e_R \\ + \sqrt{2} \overline{e}_R \phi^\dagger \begin{bmatrix} \nu_L \\ e_L \end{bmatrix} + \sqrt{2} \begin{bmatrix} \overline{\nu}_L & \overline{e}_L \end{bmatrix} \phi e_R . \qquad (1)$$

Problems:

- Conceptually, how can we interpret the mixing between differential forms of different degrees?
- Even we stand with that, the Higgs potential does not have the proper Mexican hat shape for SSB.

The Matrix Derivative

Introducing the matrix derivative to the total differential operator,

$$\mathrm{d}_S = \mathrm{d} + \mathrm{d}_M.$$

The matrix derivative transforms between odd and even part of $su(2|1)$

$$su(2/1)_0 \overset{d_M}{\longleftrightarrow} su(2/1)_1,$$

by

$$(d_M X) \;=\; i\,[\,\eta,\, X\,]_s\;.$$

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With super commutator,

$$\begin{aligned} [X, Y]_s =& [X_0 + X_1, Y_0 + Y_1]_s \\ =& [X_0, Y_0] + [X_0, Y_1] + [X_1, Y_0] + \{X_1, Y_1\} . \end{aligned}$$

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The gauge is

$$\mathcal{J} \sim \mathcal{W}_\mu \mathrm{d}x^\mu + B_\mu \mathrm{d}x^\mu + \phi \mathrm{d}_M,$$

where, $\mathrm{d}_M \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{aligned}\mathcal{F} &= \mathbf{d}\mathcal{J} + \mathbf{d}_M\mathcal{J} + \frac{1}{2}[\mathcal{J}, \mathcal{J}]_S \\ &= i\begin{bmatrix} F_W - \frac{1}{\sqrt{3}}F_B \cdot \mathbf{1}_{2\times 2} - 2\hat{\phi}\hat{\phi}^\dagger + \xi\xi^\dagger & \sqrt{2}\,D\hat{\phi} \\ \sqrt{2}(D\hat{\phi})^\dagger & -\frac{2}{\sqrt{3}}F_B - 2\hat{\phi}^\dagger\hat{\phi} + v^2 \end{bmatrix}\end{aligned} \tag{2}$$

In analogue to

$$\begin{aligned} F_G &= dG + \frac{1}{2}[G, G], \\ S_{\mathrm{QCD}} &= \frac{1}{4}\langle F_G, F_G\rangle = \frac{1}{4}\int \mathrm{Tr}\left[*F_G \wedge F_G\right] = -\frac{1}{4}\int G^a_{\mu\nu}G^{a\mu\nu}\, d^4x\,. \end{aligned} \tag{3}$$

Then we have

$$\begin{aligned} S_{YM+Higgs} &= \frac{1}{4}\langle\mathcal{F},\mathcal{F}\rangle_S = \frac{1}{4}\int \mathrm{Tr}\left[*\mathcal{F}\wedge\mathcal{F}\right] \\ &= -\frac{1}{4}\left(F^i_{W\mu\nu}F^{i\mu\nu}_W + F_{B\mu\nu}F^{\mu\nu}_B\right)d^4x \\ &\quad + (D_\mu\hat{\phi})^\dagger(D^\mu\hat{\phi})\,d^4x - 2\left(\hat{\phi}^\dagger\hat{\phi} - \frac{v^2}{2}\right)^2 d^4x\,, \end{aligned} \tag{4}$$

The Output

$$\mathcal{J} = i \begin{bmatrix} \mathcal{W} - \frac{1}{\sqrt{3}} B \cdot \mathbf{1}_{2\times 2} & \sqrt{2}\phi \\ \sqrt{2}\phi^\dagger & -\frac{2}{\sqrt{3}} B \end{bmatrix} .$$

- The Lagrangian for YM + Higgs, with Higgs potential.
- The coupling relation – a theory at a certain scale.
- The correct hypercharge from the algebraic structure.

But that is not the end of the story, while there are questions demanding an answer.

- How can we extend the gauge connection without introducing extra symmetry?
- How can we have d_M without a geometry?
- How about the hypercharge of the quarks?

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What NCG gives us

The NCG formalism gives us

- gauge-Higgs unification,
- the SM minimally coupled to gravity,
- gauge coupling relations,
- mass relation between (top) quark and gauge bosons,
- the Higgs mass,
- the correct SM hypercharge.

What is NCG

$(\mathcal{A}, \mathcal{H}, D)$

- Replace the algebra $\mathcal{A} = C^\infty(M)$ with $\mathcal{A} = C^\infty(M, M_N(\mathbb{C}))$.
- Generalize the Hilbert space $\mathcal{H} = L^2(S)$ to $\mathcal{H} = L^2(S) \otimes (\mathcal{H}^+ \oplus \mathcal{H}^-)$.
- Generalize the Dirac operator $\not{\partial}$ to $\not{\partial} + \gamma^5 \otimes D$, where $D = \begin{bmatrix} 0 & T \\ T^\dagger & 0 \end{bmatrix}$.

$\sim$ now plays the role of Differential operator, and the metric of NC space.

$\mathcal{A} = C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$

The gauge field

$$\begin{aligned} A &= \sum_i a_i[D, b_i] \\ &\sim \mathcal{W}_\mu \mathrm{d}x^\mu + B_\mu \mathrm{d}x^\mu + \phi \mathrm{d}_M, \end{aligned}$$

The symmetry,

$$SU(\mathcal{A}) = \{u \in \mathcal{A} | \det(a) = 1, aa^* = a^* a = 1\}.$$

- $A = A_\mu \mathrm{d}x^\mu = A^a_\mu T^a \mathrm{d}x^\mu$.
- The idea of superconnection formalism is changing T^a, to get a larger connection space.
- Instead, NCG changes $\mathrm{d}x^\mu$ instead, 'd' to be specific, to generate a larger connection.
- This partly fulfills our demand for an explanation in the superconnection formalism.

$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. This gives the flavor sector action

$$\mathbb{C} \oplus \mathbb{H} \sim U(1)_a \times SU(2)_L,$$

and the color sector action

$$\mathbb{C} \oplus M_3(\mathbb{C}) \sim U(1)_b \times U(3) \sim U(1)_b \times U(1)' \times SU(3)_C.$$

- $U(1)_a$ and $U(1)_b$ are identified naturally since they are both generated by the same $\mathbb{C}$.
- Requiring the symmetry to be $SU(\mathcal{A}_F)$ identifies $U(1)$ and $U(1)'$ in a certain way that the whole theory is anomaly free.
- $\det = 1$ guarantees there is no 'net' $U(1)$ charge left in the calculation of determinant, which means the two $U(1)$'s cancel each other.
- $U(1)_Y = U(1) + U(1)'$, the SM hypercharge is produced.

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	$\lvert\uparrow\rangle \otimes \mathbf{1}^0$	$\lvert\uparrow\rangle \otimes \mathbf{1}^0$	$\lvert\downarrow\rangle \otimes \mathbf{3}^0$	$\lvert\downarrow\rangle \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

The extension to left-right symmetric model

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}). \tag{5}$$

The $U(1)_{B-L}$ is reproduced:

	$\vert\uparrow\rangle \otimes \mathbf{1}^0$	$\vert\uparrow\rangle \otimes \mathbf{1}^0$	$\vert\downarrow\rangle \otimes \mathbf{3}^0$	$\vert\downarrow\rangle \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$

Outlook:

- the promotion of certain Dirac operator parameter to dynamical field,
- the breaking of LR symmetry,
- the energy scale.

Summary

- The superconnection formalism of $su(2|1)$ and the output.
  - The gauge-Higgs unification, together with Higgs potential for SSB.
  - The emergent scale.
  - The correct hypercharge from the algebraic structure.
- The conceptual and practical problems arise from it.
  - Extension of gauge connection without extra symmetry.
  - The geometry lurking behind d_M.
  - The hypercharge for quarks sector.
- Take NCG as a solution and extension to superconnection.
  - The geometry of d_M is studied.
  - The hypercharge for all SM particles.
  - Possible extension to LRSM.

Back up

$$\lambda_1^s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \lambda_2^s = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \lambda_3^s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda_4^s = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \lambda_5^s = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \ \lambda_6^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\lambda_7^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \ \lambda_8^s = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{6}$$

$$\begin{aligned} \frac{1}{i}\left[\lambda_i^s, \lambda_j^s\right] &= 2 f_{ijk}\lambda_k^s , \\ [\lambda_i^s, \lambda_8^s] &= 0 , \\ \frac{1}{i}[\lambda_i^s, \lambda_m^s] &= 2 f_{iml}\lambda_l^s , \\ \frac{1}{i}[\lambda_8^s, \lambda_m^s] &= \frac{2}{3} f_{8ml}\lambda_l^s , \\ \{\lambda_m^s, \lambda_n^s\} &= 2 d_{mnk}\lambda_k^s - \sqrt{3}\,\delta_{mn}\lambda_8^s , \end{aligned} \tag{7}$$

$$\text{Normalization: } \mathrm{Tr}(\lambda_a^s\lambda_b^s) = 2\delta_{ab} , \tag{8}$$