Dualising consistent truncations: IIA/IIB on S^3

Emanuel Malek

The Laboratory for Quantum Gravity & Strings,
Department of Mathematics and Applied Mathematics,
University of Cape Town

7th August 2015
Based on EM, Samtleben arXiv:1508.xxxx

Gauged SUGRAs

Maximal gauged 7-D SUGRAs:

- Gauge group $\subset E_4 = SL(5)$
- Classified by embedding tensor (maximal SUSY-preserving deformations of ungauged SUGRA)
  - SUSY $\Longleftrightarrow$ linear constraint: $\mathbf{10} \oplus \mathbf{15} \oplus \overline{\mathbf{40}}$
  - Group-theoretic constraint $\Longleftrightarrow$ quadratic constraint
- Some examples can be obtained by Scherk-Schwarz reduction of $D = 10$ and $D = 11$ maximal SUGRA
  - Reductions involve complicated non-linear Ansatz
  - Consistency difficult to prove using SUGRA methods deWit, Nicolai, Nastase, Vaman, Nieuwenhuizen, Cvetic, Lu, Pope ...
- Other maximal SUGRAs orphaned – until now?

Gauged SUGRAs and EFT

Useful relationship to EFT Berman, Musaev, . . . (DFT Aldazabal, Baron, Geissbuhler, Graña, Marquéz, Nuñez, . . .)

- Embedding tensor $\longleftrightarrow$ torsion of EFT
- Linear constraint $\longleftrightarrow$ automatic (up to trombone)
- Quadratic constraint $\longleftrightarrow$ section condition (+ other solutions? Lee, Strickland-Constable, Waldram 1506.03457)
- Action + scalar potential $\longleftrightarrow$ Full EFT action
- Consistency $\longleftrightarrow$ QC + constant flattened torsion
- Non-linear reduction Ansatz $\longleftrightarrow$ Generalised Scherk-Schwarz twist
- Internal space $\longleftrightarrow$ Twist matrix (generalised vielbein)
- Twist matrix has to be "guessed"

Outline

- Recent twist matrices for $CSO(p, q, r)$ gaugings to various dimensions. Lee, Strickland-Constable, Waldram 1401.3360, Hohm, Samtleben 1410.8145, Dall'Agata, Baron 1410.8823
- These give consistent sphere and hyperboloid truncations of M-theory and IIA/IIB theory to various maximal SUGRA.
- When do IIA truncations imply IIB truncations and how are they related?
- Caveat: All geometric, all standard SUGRA!

Outline:

- Review EFT.
- Review embedding tensor as EFT torsion.
- "Dualise" IIA truncations into IIB truncations.
- Use this to generate IIB hyperboloid truncations.
- Discuss new uplifts of gaugings from IIA and IIB:
  - **10**'s and **4**'s,
  - **6**'s – including trombone.

Exceptional field theory: Extended space

- Goal is to rewrite 11-D SUGRA with manifest U-duality. Focus on bosonic scalar sector. (Full theory: Hohm, Samtleben 2013, 2014 (bosonic), + Nicolai, Godazgar[2] 1406.3235 (fermions), Samtleben, Musaev 1412.7286)
- Consider $d = 4$ case, i.e. compactification to 7-D SUGRA. U-duality group $SL(5)$. Local symmetry group $SO(5)$.
- Indices: 4-d $i = 1, \ldots 4$, $SL(5)$ $a = 1, \ldots 5$.
- U-duality: 6 wrapping modes of M2-branes $\longleftrightarrow$ 4 momenta
- $\Rightarrow$ introduce 6 wrapping coordinates y_{ij} and form U-duality covariant coordinates Y^{ab} in **10** of $SL(5)$. Berman, Godazgar[2], Perry 1110.3930

$$Y^{ab} = \left(Y^{i5},\, Y^{ij}\right) = \left(y^i,\, \frac{1}{2}\eta^{ijkl} y_{kl}\right) , \qquad \eta^{ijkl} = \pm 1 . \tag{3.1}$$

EFT: Generalised metric and Lie derivative

- Truncation Ansatz:
$$ds_{11}^2 = e^{\phi} ds_7^2 + ds_4^2 \,. \qquad (3.2)$$
(General Ansatz Hohm, Samtleben 2013, 2014)
- Scalar sector parameterises the coset space $\frac{SL(5)}{SO(5)} \times \mathbb{R}^+$. $\mathbb{R}^+$ related to the trombone symmetry:
- i.e. ds_4^2, 3-form in internal directions and ϕ can be combined in a "generalised metric" Duff 1990, Hull 0701203, Berman, Perry 1008.1763,
$$m_{ab} \in \frac{SL(5)}{SO(5)} \times \mathbb{R}^+ \,. \qquad (3.3)$$
- Introduce $SL(5)$-compatible generalised Lie derivative, i.e. it must preserve ϵ_{abcde}:
$$\mathcal{L}_\xi V^a = \frac{1}{2}\xi^{bc}\partial_{bc}V^a + \frac{1}{4}V^a\partial_{bc}\xi^{bc} - V^b\partial_{bc}\xi^{ac} \,. \qquad (3.4)$$

Berman, Godazgar^2, Perry 1110.3930

EFT: Section condition

- Generalised Lie derivative combines diffeos and gauge transformations.
- Closure of algebra of generalised Lie derivatives

$$[\mathcal{L}_\xi, \mathcal{L}_\chi]\, V^a = \mathcal{L}_{[\xi,\chi]} V^a + \text{junk} \tag{3.5}$$

$\Rightarrow$ "section condition" to kill junk

$$\partial_{[ab}\partial_{cd]}\Phi(X) = 0\,, \quad \partial_{[ab}\Phi(X)\partial_{cd]}\Phi'(X) = 0\,, \tag{3.6}$$

for all fields $\Phi(X), \Phi'(X)$ of the theory. Berman, Godazgar, Perry 1110.3930

- i.e. fields can't depend on all 10 coordinates!
- Two solutions
  - Dependence only on 4 coordinates X^{i5}, $i = 1, \ldots, 4$ (11-D).
  - Dependence only on 3 coordinates $X^{\mu\nu}$, $\mu, \nu = 1 \ldots 3$ (10-D IIB). Blair, EM, Park 1311.5109 (other duality groups Hohm, Samtleben 2013, 2014)

EFT: Action

There is a *unique* action given in terms of two derivatives (∂_{ab}) of m_{ab} that is invariant under generalised Lie derivatives *modulo section condition* Berman, Perry 1008.1763:

$$S_4 = \int_\Sigma |m|^{-1} \Big(-\frac{1}{8} m^{ab} m^{a'b'} \partial_{aa'} m^{cd} \partial_{bb'} m_{cd} + \frac{1}{2} m^{ab} m^{a'b'} \partial_{aa'} m^{cd} \partial_{cb'} m_{bd}$$
$$+ \frac{1}{2} \partial_{aa'} m^{ab} \partial_{bb'} m^{a'b'} + \frac{3}{8} m^{ab} m^{a'b'} \partial_{aa'} \ln|m| \, \partial_{bb'} \ln|m|$$
$$-2 m^{a'b'} \partial_{aa'} m^{ab} \partial_{bb'} \ln|m| + m^{a'b'} \partial_{aa'} \partial_{bb'} m^{ab} - m^{ab} m^{a'b'} \partial_{aa'} \partial_{bb'} \ln|m| \Big) \tag{3.7}$$

where $|m| = |\det m_{ab}|$ and Σ is lower-dimensional section satisfying the section condition.

Upon choosing a solution to the section condition, this reduces to scalar potential of reductions from 11-D SUGRA or IIB SUGRA.

The full 10/11-D theory can be reformulated this way (Hohm, Samtleben 2013, 2014).

7-D embedding tensor as EFT torsion

To build tensors we can use the Weitzenböck connection as a connection in EFT:

$$\Gamma_{ab,c}{}^{d} = E_c{}^{\bar{c}} \partial_{ab} E_{\bar{c}}{}^{d} \,, \tag{3.8}$$

where $E_a{}^{\bar{a}}(Y) \in SL(5) \times \mathbb{R}^+$ is the vielbein: $m_{ab} = E_a{}^{\bar{a}} E_{b\bar{b}}$.
The embedding tensor is the flattened torsion of the Weitzenböck connection. Blair, EM 1412.0635

$$\left(\mathcal{L}^{\nabla}_U - \mathcal{L}^{\partial}_U \right) V^a = \frac{1}{2} \tau_{bc,d}{}^{a} U^{bc} V^d \,. \tag{3.9}$$

Explicitly, (we will use flattened torsion from now)

$$\tau_{\bar{a}\bar{b},\bar{c}}{}^{\bar{d}} = 3\Gamma_{[\bar{a}\bar{b},\bar{c}]}{}^{\bar{d}} - \Gamma_{\bar{e}[\bar{a},\bar{b}]}{}^{\bar{e}} \delta^{\bar{d}}_{\bar{c}} - 2\Gamma_{\bar{e}\bar{c},[\bar{a}}{}^{\bar{e}} \delta^{\bar{d}}_{\bar{b}]} \,. \tag{3.10}$$

Let x be 7-D, Y^{ab} be EFT (here 10-D), then we obtain gauged SUGRAs by the twist Ansatz, e.g.

$$g_7(x, Y) = |E(Y)|^{-4/5} \hat{g}_7(x) \,, \qquad m_{ab}(x, Y) = E_a{}^{\bar{a}}(Y) E_b{}^{\bar{b}}(Y) \hat{m}_{\bar{a}\bar{b}}(x) \,. \tag{3.11}$$

Berman, Musaev 1208.0020, Godazgar[2], Nicolai 1312.1061, Hohm, Samtleben 1410.8145

Irreps of the torsion and the scalar potential

The torsion decomposes into the irreps $\mathbf{15} \oplus \overline{\mathbf{40}} \oplus \mathbf{10}$:

$$\tau_{\bar{a}\bar{b},\bar{c}}{}^{\bar{d}} = \frac{1}{2}\epsilon_{\bar{a}\bar{b}\bar{c}\bar{e}\bar{f}} Z^{\bar{e}\bar{f},\bar{d}} + \frac{1}{2}\delta^{\bar{d}}_{[\bar{a}} S_{\bar{b}]\bar{c}} + \frac{2}{3}\delta^{\bar{d}}_{[\bar{a}}\tau_{\bar{b}]\bar{c}} + \frac{1}{3}\delta^{\bar{d}}_{\bar{c}}\tau_{\bar{a}\bar{b}} \,. \qquad (3.12)$$

It satisfies the linear constraint!
Scalar potential written in terms of torsion irreps (is fixed by local $SO(5)$ invariance)

$$\begin{aligned} S_4 = \int_\Sigma dY \frac{1}{16}\hat{m}^{\bar{a}\bar{b}}\hat{m}^{\bar{c}\bar{d}} S_{\bar{a}\bar{c}} S_{\bar{b}\bar{d}} - \frac{1}{32}(\hat{m}^{\bar{a}\bar{b}} S_{\bar{a}\bar{b}})^2 + \frac{5}{3}\hat{m}^{\bar{a}\bar{b}}\hat{m}^{\bar{c}\bar{d}}\tau_{\bar{a}\bar{c},\bar{e}}{}^{\bar{e}}\tau_{\bar{b}\bar{d},\bar{f}}{}^{\bar{f}} \\ + \frac{1}{4}\hat{m}_{\bar{a}\bar{b}}\hat{m}_{\bar{c}\bar{d}}\hat{m}_{\bar{e}\bar{f}} Z^{\bar{a},\bar{c}\bar{e}} Z^{\bar{b},\bar{d}\bar{f}} - \frac{3}{8}\hat{m}_{\bar{a}\bar{b}}\hat{m}_{\bar{c}\bar{d}}\hat{m}_{\bar{e}\bar{f}} Z^{\bar{a},[\bar{c}\bar{e}} Z^{\bar{b}],\bar{d}\bar{f}} \\ - 2\hat{m}^{\bar{a}\bar{b}}\hat{m}^{\bar{c}\bar{d}}\nabla_{\bar{a}\bar{c}}\tau_{\bar{b}\bar{d},\bar{e}}{}^{\bar{e}} \,, \end{aligned} \qquad (3.13)$$

agrees precisely with scalar potential of maximal 7-D gauged SUGRAs.
Scalar potential is not generalised scalar with non-zero trombone $\tau_{\bar{a}\bar{b}} \implies$ leads to 7-D gauged SUGRA with EoMs but no action.

Consistency conditions and symmetries

The twisted Ansatz

$$m_{ab}(x,Y) = E_a{}^{\bar{a}}(Y) E_b{}^{\bar{b}} \hat{m}_{\bar{a}\bar{b}}(x)\,, \tag{4.1}$$

leads to a consistent truncation, i.e. all Y-dependence drops out of EoMs (and action if $\tau_{\bar{a}\bar{b}} = 0$), if

$$\begin{aligned} S_{\bar{a}\bar{b}} &= 4\Gamma_{\bar{e}(\bar{a},\bar{b})}{}^{\bar{e}}\,, \qquad \tau_{\bar{a}\bar{b}} = \Gamma_{\bar{a}\bar{b},\bar{e}}{}^{\bar{e}} - \Gamma_{\bar{e}[\bar{a},\bar{b}]}{}^{\bar{e}}\,, \\ Z^{\bar{a}\bar{b},\bar{c}} &= \frac{1}{2}\epsilon^{\bar{a}\bar{b}\bar{d}\bar{e}\bar{f}}\Gamma_{\bar{d}\bar{e}\bar{f}}{}^{\bar{c}} - \frac{1}{3}\epsilon^{\bar{a}\bar{b}\bar{c}\bar{e}\bar{f}}\Gamma_{\bar{g}\bar{e},\bar{f}}{}^{\bar{g}} - \frac{1}{3!}\epsilon^{\bar{a}\bar{b}\bar{c}\bar{e}\bar{f}}\Gamma_{\bar{e}\bar{f},\bar{g}}{}^{\bar{g}}\,, \end{aligned} \tag{4.2}$$

are constants and quadratic constraint is satisfied (e.g. by imposing section).

The consistency conditions are invariant under

- global $SL(5)$ acting on curved indices (leaves embedding tensor invariant),
- global $SL(5)$ acting on flat indices (changes embedding tensor - but equivalent SUGRA),
- anything else? T-duality?

T-duality

- $S_{\bar{a}\bar{b}} \longrightarrow$ sphere / hyperboloid truncations of 11-D and IIA SUGRA.
  Hohm, Samtleben 1410.8145
- Some $Z^{\bar{a}\bar{b},\bar{c}} \longrightarrow$ sphere / hyperboloid truncations of IIB SUGRA?
- Is there a way to convert IIA truncations into IIB ones?
- This would have to swap the two inequivalent solutions of section condition. Something akin to T-duality.
- What happens in half-maximal gauged SUGRA (nicely fits into DFT)?
  - Internal NS-NS sector captured by DFT torsion ${\tau_{\bar{A}\bar{B}}}^{\bar{C}}$ and trombone $\tau_{\bar{A}}$.
  - $\bar{A}, \bar{B} = 1, \ldots 6$ are $O(3,3)$ indices.
  - Split $O(3,3) \longrightarrow SO(3,3)$ to discuss T-duality.

$$ {\tau_{\bar{A}\bar{B}}}^{\bar{C}} : \mathbf{20} \longrightarrow \mathbf{10} \oplus \mathbf{10}' \,, \qquad \tau_{\bar{A}} : \mathbf{6} \longrightarrow \mathbf{6} \,. \tag{4.3} $$

Dibitetto, Fernández-Melgarejo, Marqués, Roest 1203.6562, 1506.01294

- How does this lift to maximal gauged SUGRA / EFT?

“T-duality” in EFT

- Let us break $SL(5) \longrightarrow SL(4) \sim Spin(3,3)$.
- $\bar{a} = 1, \ldots, 5 = (\bar{\alpha}, \bar{5})$ with $\bar{\alpha} = 1, \ldots 4$ $SL(4)$ indices.

$$\begin{aligned} S_{\bar{a}\bar{b}} &\longrightarrow S_{\bar{\alpha}\bar{\beta}} \oplus S_{\bar{\alpha}\bar{5}} \oplus S_{\bar{5}\bar{5}} \,, \\ \mathbf{15} &\longrightarrow \mathbf{10} \oplus \mathbf{4} \oplus \mathbf{1} \,. \end{aligned} \tag{5.1}$$

$$\begin{aligned} Z^{\bar{a}\bar{b},\bar{c}} &\longrightarrow Z^{\bar{\alpha}\bar{\beta},\bar{\gamma}} \oplus Z^{\bar{5}(\bar{\alpha},\bar{\beta})} \oplus Z^{\bar{5}[\bar{\alpha},\bar{\beta}]} \oplus Z^{\bar{5}\bar{\alpha},\bar{5}} \,, \\ \overline{\mathbf{40}} &\longrightarrow \overline{\mathbf{20}} \oplus \overline{\mathbf{10}} \oplus \mathbf{6} \oplus \overline{\mathbf{4}} \,. \end{aligned} \tag{5.2}$$

$$\begin{aligned} \mathcal{T}_{\bar{a}\bar{b}} &\longrightarrow \mathcal{T}_{\bar{\alpha}\bar{\beta}} \oplus \mathcal{T}_{\bar{\alpha}\bar{5}} \,, \\ \mathbf{10} &\longrightarrow \mathbf{6} \oplus \mathbf{4} \,. \end{aligned} \tag{5.3}$$

- How can we exchange the **10**’s, **6**’s and **4**’s?

"T-duality" in EFT: outer automorphism

- Consider IIA / IIB by dimensional reduction $\partial_{\alpha 5} = 0$, i.e. only 6 coordinates $Y^{\alpha\beta}$. No section condition yet.
- Make the Ansatz

$$\begin{aligned} E_a{}^{\bar{a}} &= \omega^{1/2} U_a{}^{\bar{a}} \,, \qquad |U| = 1 \,, \\ U_a{}^{\bar{a}} &= \begin{pmatrix} \omega^{-1/2} V_\alpha{}^{\bar{\alpha}} & 0 \\ 0 & \omega^2 \end{pmatrix} \,. \end{aligned} \tag{5.4}$$

- Define $\partial^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}$.
- We find the only non-zero gaugings are the **10**'s and **6**'s:

$$\begin{aligned} S_{\bar{\alpha}\bar{\beta}} &= 4 V_{(\bar{\alpha}}{}^{\alpha} \partial_{|\alpha\beta|} V_{\bar{\beta})}{}^{\beta} \,, \qquad Z^{\bar{5}(\bar{\alpha},\bar{\beta})} = V_\alpha{}^{(\bar{\alpha}} \partial^{|\alpha\beta|} V_\beta{}^{\bar{\beta})} \,, \\ 2\tau_{\bar{\alpha}\bar{\beta}} &= -\left(\partial_{\alpha\beta} V_{\bar{\alpha}\bar{\beta}}{}^{\alpha\beta} - 5 V_{\bar{\alpha}\bar{\beta}}{}^{\alpha\beta} \partial_{\alpha\beta} \ln \omega \right) \,, \\ 6 Z^{\bar{5}[\bar{\alpha},\bar{\beta}]} &= \left(\partial^{\alpha\beta} V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}} - 5 V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}} \ln \omega \right) \,. \end{aligned} \tag{5.5}$$

$$
\begin{aligned}
S_{\bar\alpha\bar\beta} &= 4 V_{(\bar\alpha}{}^{\alpha}\partial_{|\alpha\beta|} V_{\bar\beta)}{}^{\beta}\,, \qquad Z^{\bar 5(\bar\alpha,\bar\beta)} = V_\alpha{}^{(\bar\alpha}\partial^{|\alpha\beta|} V_\beta{}^{\bar\beta)}\,, \\
2\tau_{\bar\alpha\bar\beta} &= -\left(\partial_{\alpha\beta} V_{\bar\alpha\bar\beta}{}^{\alpha\beta} - 5 V_{\bar\alpha\bar\beta}{}^{\alpha\beta}\partial_{\alpha\beta}\ln\omega\right)\,, \\
6Z^{\bar 5[\bar\alpha,\bar\beta]} &= \left(\partial^{\alpha\beta} V_{\alpha\beta}{}^{\bar\alpha\bar\beta} - 5 V_{\alpha\beta}{}^{\bar\alpha\bar\beta}\ln\omega\right)\,.
\end{aligned}
\tag{5.6}
$$

- We can exchange the **10**'s and **6**'s and IIA / IIB by applying the $\mathbb{Z}_2$ outer automorphism of $SL(4)$:

$$
V_\alpha{}^{\bar\alpha} \longleftrightarrow \left(V^{-T}\right)_{\bar\alpha}{}^{\alpha}\,, \qquad \partial_{\alpha\beta} \longleftrightarrow \partial^{\alpha\beta}\,. \tag{5.7}
$$

- IIA section $\partial_{\mu 4} \neq 0$, exchanged with IIB section $\partial_{\mu\nu} \neq 0$, $\mu,\nu = 1,2,3$.
- NS-NS fields remain invariant!
- Example: Swapping spheres in IIA and IIB.

$CSO(p,q,r)$ gaugings from IIA

IIA twists of $CSO(p,q,r)$ gaugings are known. These gauge the $\mathbf{10} \subset \mathbf{15}$.
Let $S_{\bar{\alpha}\bar{\beta}}$ be diagonal and $\bar{\mu},\bar{\nu} = 1,\ldots,4-r$, $\bar{i} = 5-r,\ldots,4$ so that $S_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}}$, $S_{\bar{4}\bar{4}} = 1$, $S_{\bar{i}\bar{j}} = 0$.
Twist is:

$$V_\alpha{}^{\bar{\alpha}} = \begin{pmatrix} (1-v)^{1/4}\,\delta_\mu{}^{\bar{\mu}} & (1-v)^{-1/4}\,K\eta_{\mu\nu}y^\nu & 0 \\ (1-v)^{-1/4}\,\eta^{\bar{\mu}\bar{\nu}}y_{\bar{\nu}} & (1-v)^{-3/4}\,(1+Kv) & 0 \\ 0 & 0 & (1-v)^{1/4}\,\mathbb{1}_r \end{pmatrix}, \tag{6.1}$$

$$\omega = (1-v)^{1/10}\,.$$

- Coordinates $y^\mu = Y^{\mu 4}$.
- $v = \eta_{\mu\nu}y^\mu y^\nu$, $u = y^\mu y_\mu$.
- $K(u,v)$ satisfies PDE:

$$2\,(1-v)\,(u\partial_v K + v\partial_u K) = ((1+q-p)\,(1-v)-u)\,K - 1\,. \tag{6.2}$$

- Indices raised/lowered with $\delta_{\mu\nu}$.

Hohm, Samtleben 1410.8145

$SO(p, q)$ gaugings from IIA

The internal space of the truncation can be read off from the generalised metric m_{ab}. One finds the non-linear reduction Ansatz similar to Hull, Warner 1988 :

$$
\begin{aligned}
ds^2 &= (1+u-v)^{-1}\left[\left(\delta_{\mu\nu} + \frac{\eta_{\mu\rho}y^\rho\eta_{\nu\sigma}y^\sigma}{1-v}\right)dy^\mu dy^\nu + \delta_{ij}dy^i dy^j\right] \\
&\quad + ds_7^2\,, \\
B_{AB} &= -\left(1+u-v\right)^{-1/2}\left(1-v\right)^{-1/2}\left[1+K\left(1+u-v\right)\right]\epsilon_{ABC}\eta^{CD}y_D\,, \\
e^\psi &= (1+u-v)^{1/2}\,.
\end{aligned}
\tag{6.3}
$$

Here $A, B = (1, 2, 3)$.
This is a warped $H^{p,q} \times \mathbb{R}^r$: surface of $\eta^{\mu\nu}y_\mu y_\nu + z^2 =$ constant and r flat directions.

Example: $CSO(p,q,r)$ gaugings from IIB

Apply outer automorphism: $V \longleftrightarrow V^{-T}$ and $\partial_{\alpha\beta} \longleftrightarrow \partial^{\alpha\beta}$. Coordinates, $Y^{\mu\nu} = \epsilon^{\mu\nu\rho}\tilde{y}_\rho$, come from IIB theory.
Twist is

$$V_\alpha{}^{\bar{\alpha}} = \begin{pmatrix} (1-v)^{-1/4} A_\mu{}^{\bar{\mu}} & -(1-v)^{1/4} K\eta_{\mu\nu}\tilde{y}^\nu & 0 \\ -(1-v)^{1/4}\eta^{\bar{\mu}\bar{\nu}}\tilde{y}_{\bar{\nu}} & (1-v)^{3/4} & 0 \\ 0 & 0 & (1-v)^{-1/4}\mathbb{1}_r \end{pmatrix} \quad (6.4)$$

where $A_\mu{}^{\bar{\mu}} = \delta_\mu{}^{\bar{\mu}} + K\tilde{y}_\mu\tilde{y}^{\bar{\mu}}$. The resulting internal space was expected to be different from the IIA case but is the **same**:

$$\begin{aligned} ds^2 &= (1+u-v)^{-3/4}\left[\left(\delta^{\mu\nu} + \frac{\eta^{\mu\rho}\tilde{y}_\rho\eta^{\nu\sigma}\tilde{y}_\sigma}{1-v}\right)d\tilde{y}_\mu d\tilde{y}_\nu + \delta^{ij}d\tilde{y}_i d\tilde{y}_j\right] \\ &\quad + (1+u-v)^{1/4}\, ds_7^2\,, \\ B^{AB,4} &= -(1+u-v)^{-1/2}(1-v)^{-1/2}(1+K(1+u-v))\,\epsilon^{ABC}\eta_{CD}\tilde{y}^D\,, \\ e^\psi &= (1+u-v)^{1/2}\,. \end{aligned} \quad (6.5)$$

A no-go theorem

Can we obtain $CSO(p,q,r)$ gaugings in $S_{\bar{\alpha}\bar{\beta}}$ from IIB? $Z^{\bar{5}(\bar{\alpha},\bar{\beta})}$ from IIA?
This question only makes sense when we have dependence on 3 coordinates. Otherwise, IIA / IIB are the same.
$\implies$ non-degenerate $S_{\bar{\alpha}\bar{\beta}}$.
Write consistency conditions with **10** indices.

$$\tau_{\bar{a}\bar{b},\bar{c}\bar{d}}{}^{\bar{e}\bar{f}} = 4\tau_{\bar{a}\bar{b},[\bar{c}}{}^{[\bar{e}}\delta_{\bar{d}]}{}^{\bar{f}]} \,. \tag{6.6}$$

With IIA section $(Y^{\mu 4})$ we find:

$$\begin{aligned} -\tau_{\bar{a}\bar{b},\bar{c}\bar{d}}{}^{\bar{e}\bar{f}} E_{\bar{e}\bar{f}}{}^{54} &= E_{\bar{a}\bar{b}}{}^{4\mu}\partial_\mu E_{\bar{c}\bar{d}}{}^{54} - E_{\bar{c}\bar{d}}{}^{4\mu}\partial_\mu E_{\bar{a}\bar{b}}{}^{54} \,, \\ -\tau_{\bar{a}\bar{b},\bar{c}\bar{d}}{}^{\bar{e}\bar{f}} E_{\bar{e}\bar{f}}{}^{4\mu} &= E_{\bar{a}\bar{b}}{}^{4\nu}\partial_\nu E_{\bar{c}\bar{d}}{}^{4\mu} - E_{\bar{c}\bar{d}}{}^{4\nu}\partial_\nu E_{\bar{a}\bar{b}}{}^{4\mu} \,. \end{aligned} \tag{6.7}$$

The RHS is anti-symmetric in $\left(\bar{a}\bar{b}\right) \leftrightarrow \left(\bar{c}\bar{d}\right)$. For $Z^{\bar{5}(\bar{\alpha},\bar{\beta})} = \eta^{\bar{\alpha}\bar{\beta}}$, the LHS is not antisymmetric.
$Z^{\bar{5}(\bar{\alpha},\bar{\beta})}$ only comes from IIB! By the "duality", $S_{\bar{\alpha}\bar{\beta}}$ only comes from IIA.

Gauging the **4**'s

The **4** gaugings are automatically zero with the previous Ansatz. Consider instead

$$E_a{}^{\bar{a}} = \omega^{1/2} U_a{}^{\bar{a}} \,, \qquad U_a{}^{\bar{a}} = \begin{pmatrix} \omega^{-1/2} V_\alpha{}^{\bar{\alpha}} & \omega^{-1/2} A_\alpha \\ 0 & \omega^2 \end{pmatrix} . \tag{7.1}$$

The consistency equations for **10**'s and **6**'s are unchanged. Additionally we have:

$$Z^{\bar{\alpha}\bar{5},\bar{5}} = V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}} \partial^{\alpha\beta} A_{\bar{\beta}} + \left(Z^{\bar{5}(\bar{\alpha},\bar{\beta})} + 3 Z^{\bar{5}[\bar{\alpha},\bar{\beta}]} \right) A_{\bar{\beta}} \,, \tag{7.2}$$

where $A_{\bar{\alpha}} = V_{\bar{\alpha}}{}^{\alpha} A_\alpha$.

Gauging the **4**'s

If instead, we consider

$$U_a{}^{\bar{a}} = \begin{pmatrix} \omega^{-1/2} V_\alpha{}^{\bar{\alpha}} & 0 \\ \omega^2 B^{\bar{\alpha}} & \omega^2 \end{pmatrix} \,, \tag{7.3}$$

we get the same **10**'s and **6**'s but additionally:

$$\begin{aligned} \tau_{\bar{\alpha}\bar{5}} &= -\frac{1}{2} V_{\bar{\alpha}\bar{\beta}}{}^{\alpha\beta} \partial_{\alpha\beta} B^{\bar{\beta}} - B^{\bar{\beta}} \tau_{\bar{\alpha}\bar{\beta}} \,, \\ Z^{\bar{\alpha}\bar{\beta}\bar{\gamma}} &= V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}} \partial^{\alpha\beta} B^{\bar{\gamma}} + 2 B^{[\bar{\alpha}} \left(Z^{|\bar{5}|(\bar{\beta}],\bar{\gamma})} + 3 Z^{|\bar{5}|[\bar{\beta}],\bar{\gamma}]} \right) - ([\bar{\alpha}\bar{\beta}\bar{\gamma}]) \,, \\ S_{\bar{\alpha}\bar{5}} &= -2 V_{\bar{\alpha}\bar{\beta}}{}^{\alpha\beta} \partial_{\alpha\beta} B^{\bar{\beta}} - B^{\bar{\beta}} S_{\bar{\alpha}\bar{\beta}} \,, \\ S_{\bar{5}\bar{5}} &= -B^{\bar{\alpha}} \left(2 S_{\bar{\alpha}\bar{5}} + B^{\bar{\beta}} S_{\bar{\alpha}\bar{\beta}} \right) \,. \end{aligned} \tag{7.4}$$

Swapping the **4**'s

Can we extend our swap to the **4**'s?
Take the previous automorphism

$$V \longleftrightarrow V^{-T}, \qquad \partial_{\alpha\beta} \longleftrightarrow \partial^{\alpha\beta}, \tag{7.5}$$

and supplement with

$$A_\alpha \longleftrightarrow B^\alpha. \tag{7.6}$$

It is easy to show that:

$$\begin{aligned} \mathbf{4} \subset \mathbf{15} &\longrightarrow \mathbf{4} \subset \mathbf{40}, \\ \mathbf{4} \subset \mathbf{40} &\not\longrightarrow \mathbf{4} \subset \mathbf{15}. \end{aligned} \tag{7.7}$$

The two **4**'s are not on an equal footing!

Example: **10**'s and **4**'s

- The previous hyperboloid solutions completely describe the gaugings of the **10**'s (up to $SL(5)$ rotations).
- The $S_{\bar{\alpha}\bar{\beta}}$ gaugings can be deformed by introducing the **4**, $Z^{\bar{\alpha}\bar{5},\bar{5}} = c^{\bar{\alpha}}$.
- Quadratic constraint:

$$S_{\bar{\alpha}\bar{\beta}} Z^{\bar{\beta}\bar{5},\bar{5}} = 0 \,, \tag{7.8}$$

  implies there are only solutions when $S_{\bar{\alpha}\bar{\beta}}$ is degenerate.
- Find the twist matrix that gives this gauging.
- Consider $S_{\bar{\alpha}\bar{\beta}}$ with rank 3 and 2 separately.

Rank 3

Let

$$S_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}}\,, \qquad \eta_{\bar{4}\bar{4}} = 1\,, \qquad \text{with } \bar{\mu} = 1, 2\,. \tag{7.9}$$

Quadratic Constraint

$$\implies Z^{\bar{\alpha}\bar{5},\bar{5}} = c\delta_{\bar{3}}{}^{\bar{\alpha}}\,. \tag{7.10}$$

This cannot be solved with a IIA compactification. Instead, we can solve the consistency equations with a IIB section using a third coordinate

$$y_3 = \frac{1}{2}\epsilon_{\mu\nu}Y^{\mu\nu}\,. \tag{7.11}$$

Consistency equations:

$$V_{\alpha\beta}{}^{\bar{3}\bar{\beta}}\partial^{\alpha\beta}A_{\bar{\beta}} = c\,, \qquad V_{\alpha\beta}{}^{\bar{\mu}\bar{\beta}}\partial^{\alpha\beta}A_{\bar{\beta}} = V_{\alpha\beta}{}^{\bar{4}\bar{\beta}}\partial^{\alpha\beta}A_{\bar{\beta}} = 0\,, \tag{7.12}$$

are solved by

$$A_4 = -cy_3\left(1-v\right)^{-1/4}\,, \qquad A_\mu = A_3 = 0\,. \tag{7.13}$$

Rank 3 – internal space

The internal space is

$$
\begin{aligned}
ds^2 = ds_7^2 &+ (1-v)^{-1}\left[\delta^{\mu\nu}dy_\mu dy_\nu - \frac{\left(\eta^{\mu\nu}y_\mu dy_\nu\right)^2}{1+u-v}\right] \\
&+ (1+u-v)\left[dy_3 + (1-v)^{-1/2}\,\frac{1+K\,(1+u-v)}{1+u-v}\eta^{\mu\nu}y_\mu dy_\nu\right]^2 \,, \\
B^{\mu\nu,5} &= -cy_3\,(1-v)^{-1/2}\,\epsilon^{\mu\nu}\,.
\end{aligned}
\tag{7.14}
$$

For $c=0$, this is the internal space corresponding to the T-dual of the $H^{p,q}$ solution.

We can see that the **4** gauging turns on Ramond-Ramond flux and leaves NS sector invariant!

Rank 2 – internal space

Now we have

$$S_{\bar{1}\bar{1}} = \pm 1 \,, \qquad S_{\bar{4}\bar{4}} = 1 \,, \tag{7.15}$$

with all others zero. Quadratic constraint implies

$$Z^{\bar{2}\bar{5},\bar{5}} = c \,, \qquad Z^{\bar{3}\bar{5},\bar{5}} = d \,. \tag{7.16}$$

${V_\alpha}^{\bar{\alpha}}$ depends only on one coordinate $y_1 = Y^{14}$. We can solve the consistency equations with

$$A_1 = (1-v)^{-1/4}\left(-dy_2 + cy_3\right) \,, \qquad A_2 = A_3 = A_4 = 0 \,. \tag{7.17}$$

Here $y_2 = Y^{24}$ and $y_3 = Y^{34} \implies$ IIA section!

Rank 2 – internal space

The internal space is now a circle / hyperbola with R-R 1-form $\mathcal{A}_1$ such that

$$\begin{aligned} ds_{11}^2 &= (1+u-v)^{-2/3}\left[dy_2^2 + dy_3^2 + (dz + \mathcal{A}_1 dy_1)^2\right] \\ &\quad + (1+u-v)^{1/3}\,\frac{dy_1^2}{1-v} + (1+u-v)^{-2/3}\,ds_7^2\,, \\ C_{23z} &= -\,(1+u-v)^{-1}\,(1-v)^{-1/2}\,[1 + K\,(1+u-v)]\,\eta y_1\,, \\ \mathcal{A}_1 &= (1-v)^{-1/2}\,(cy_3 - dy_2)\,. \end{aligned} \tag{7.18}$$

Not swapping the **4**'s!

Can we "dualise" these twist matrices to obtain more gaugings?
If we apply the swap $A_\alpha \leftrightarrow B^\alpha$, $V \leftrightarrow V^{-T}$ and $\partial_{\alpha\beta} \leftrightarrow \partial^{\alpha\beta}$, we get

- constant $\mathbf{10} \subset \mathbf{40}$,
- constant $\mathbf{4} \subset \mathbf{15}$,
- constant $\mathbf{4} \subset \mathbf{10}$,
- non-constant $\mathbf{20} \subset \mathbf{40}$.

In fact, one can show that it is impossible to have a gauging of the above irreps in IIA!

$$\begin{aligned} -\tau_{\bar{a}\bar{b},\bar{c}\bar{d}}{}^{\bar{e}\bar{f}} E_{\bar{e}\bar{f}}{}^{54} &= E_{\bar{a}\bar{b}}{}^{4\mu} \partial_\mu E_{\bar{c}\bar{d}}{}^{54} - E_{\bar{a}\bar{b}}{}^{4\mu} \partial_\mu E_{\bar{c}\bar{d}}{}^{54} \,, \\ -\tau_{\bar{a}\bar{b},\bar{c}\bar{d}}{}^{\bar{e}\bar{f}} E_{\bar{e}\bar{f}}{}^{4\mu} &= E_{\bar{a}\bar{b}}{}^{4\nu} \partial_\nu E_{\bar{c}\bar{d}}{}^{4\mu} - E_{\bar{c}\bar{d}}{}^{4\nu} \partial_\nu E_{\bar{a}\bar{b}}{}^{4\mu} \,. \end{aligned} \tag{7.19}$$

RHS is antisymmetric but LHS is not.

Gauging the **6**'s

Let us try and gauge the **6**'s now. Let us consider a more general block-diagonal Ansatz than for the **10**'s:

$$E_a{}^{\bar{a}} = \rho^{1/2} U_a{}^{\bar{a}} \,, \qquad U_a{}^{\bar{a}} = \begin{pmatrix} \omega^{-1/2} V_\alpha{}^{\bar{\alpha}} & 0 \\ 0 & \omega^2 \end{pmatrix} \,. \tag{8.1}$$

So far we have taken $\rho = \omega$. Now we have the consistency conditions:

$$\begin{aligned} S_{\bar{\alpha}\bar{\beta}} &= 4\rho^{-1}\omega V_{(\bar{\alpha}}{}^{\alpha} \partial_{|\alpha\beta|} V_{\bar{\beta})}{}^{\beta} \,, \qquad Z^{\bar{5}(\bar{\alpha},\bar{\beta})} = \rho^{-1}\omega V_\alpha{}^{(\bar{\alpha}} \partial^{|\alpha\beta|} V_\beta{}^{\bar{\beta})} \,, \\ 2\tau_{\bar{\alpha}\bar{\beta}} &= -\rho^{-1}\omega \left(\partial_{\alpha\beta} V_{\bar{\alpha}\bar{\beta}}{}^{\alpha\beta} - 5 V_{\bar{\alpha}\bar{\beta}}{}^{\alpha\beta} \partial_{\alpha\beta} \ln\omega + 6 V_{\bar{\alpha}\bar{\beta}}{}^{\alpha\beta} \partial_{\alpha\beta} \ln\left(\rho^{-1}\omega\right) \right) \,, \\ 6 Z^{\bar{5}[\bar{\alpha},\bar{\beta}]} &= \rho^{-1}\omega \left(\partial^{\alpha\beta} V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}} - 5 V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}} \ln\omega \right) \,. \end{aligned} \tag{8.2}$$

Dualise $\tau_{\bar{\alpha}\bar{\beta}} \longrightarrow (\star\tau)^{\bar{\alpha}\bar{\beta}} = \frac{1}{2}\epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \tau_{\bar{\gamma}\bar{\delta}}$:

$$(\star\tau)^{\bar{\alpha}\bar{\beta}} = 3 Z^{\bar{5}[\bar{\alpha},\bar{\beta}]} + 3 V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}} \partial^{\alpha\beta} \left(\rho^{-1}\omega\right) \,. \tag{8.3}$$

Gauging the **6**'s

$$(\star\tau)^{\bar{\alpha}\bar{\beta}} = 3Z^{\bar{5}[\bar{\alpha},\bar{\beta}]} + 3V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}}\partial^{\alpha\beta}\left(\rho^{-1}\omega\right) . \tag{8.4}$$

This implies

$$V_{\alpha\beta}{}^{\bar{\alpha}\bar{\beta}}\partial^{\alpha\beta}\left(\rho^{-1}\omega\right) = \text{constant} . \tag{8.5}$$

- So far, $\rho = \omega$ which makes $(\star\tau)^{\bar{\alpha}\bar{\beta}}$ and $Z^{\bar{5}[\bar{\alpha},\bar{\beta}]}$ the same.
- The condition above fixes $\rho^{-1}\omega$ but then ρ can be used to tune between trombone and $\mathbf{6} \subset \mathbf{40}$.
- Recall: trombone gaugings $\implies$ no action principle. $\mathbf{6} \subset \mathbf{40}$ has an action principle.
- Does the "duality" relate a theory with an action principle to one without action principle?

NO! $V \longleftrightarrow V^{-T}$ and $\partial_{\alpha\beta} \longleftrightarrow \partial^{\alpha\beta}$ exchanges

$$\tau_{\bar{\alpha}\bar{\beta}} \longleftrightarrow (\star\tau)^{\bar{\alpha}\bar{\beta}} , \qquad Z^{\bar{5}[\bar{\alpha},\bar{\beta}]} \longleftrightarrow \frac{1}{2}\epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}Z^{\bar{5}[\bar{\gamma},\bar{\delta}]} . \tag{8.6}$$

It does not exchange the **6**'s amongst themselves!

Example: Gauging of **6** $\subset$ **40**

Consider twist with $V_\alpha{}^{\bar\alpha} = \delta_\alpha{}^{\bar\alpha}$. This fixes $\rho^{-1}\omega$. We can use ρ to dial between the two **6**'s.
Take $\tau_{\bar\alpha\bar\beta} = 0$. This requires

$$\rho = \omega^{1/6}\,. \tag{8.7}$$

We can then solve

$$\omega = \left(1 - \frac{1}{4} Z^{\bar5[\bar\alpha,\bar\beta]}\epsilon_{\bar\alpha\bar\beta\bar\gamma\bar\delta}Y^{\bar\gamma\bar\delta}\right)^{6/5}\,, \tag{8.8}$$

Now let $\bar\mu,\bar\nu = 1,\ldots,3$ and

$$Z^{\bar5[\bar\alpha,\bar\beta]} = \begin{cases} Z^{\bar5[\bar\mu,\bar\nu]} = \epsilon^{\bar\mu\bar\nu\bar\rho}a_{\bar\rho} \\ Z^{\bar5[\bar\mu,\bar4]} = b^{\bar\mu} \end{cases}\,. \tag{8.9}$$

Then we have

$$\omega = (1 - a_\mu y^\mu - b^\mu\tilde y_\mu)^{6/5}\,. \tag{8.10}$$

Quadratic constraint: $a_\mu b^\mu = 0$. Reduces exactly to a IIA or IIB section.

Internal space

The internal space for IIA ($b^{\bar{\mu}} = 0$) is

$$ds^2_{S,10} = dy_\mu dy^\mu + (1 - a \cdot y)^2 \, ds^2_7 \,, \qquad e^\psi = (1 - a \cdot y)^3 \,. \tag{8.11}$$

For IIB ($a_{\bar{\mu}} = 0$), it is

$$ds^2_{E,10} = (1 - b \cdot \tilde{y})^{-3/2} \, d\tilde{y}_\mu d\tilde{y}^\mu + (1 - b \cdot \tilde{y})^{1/2} \, ds^2_7 \,, \qquad e^\psi = (1 - b \cdot y)^3 \,. \tag{8.12}$$

These two truncations (with $a_{\bar{\mu}} = b^{\bar{\mu}}$) are “dualised” into each other by the outer automorphism.
Note that we have the same internal space but one is expressed in string frame and one in Einstein frame.

Example: Gauging the trombone

Let us now consider having both $\tau_{\bar{\alpha}\bar{\beta}}$ and $Z^{\bar{5}[\bar{\alpha},\bar{\beta}]}$ by taking

$$\omega = (1 - a \cdot y)^n \,, \qquad \rho = (1 - a \cdot y)^{n-1} \,. \tag{8.13}$$

If y are IIA coordinates, we get the gaugings

$$Z^{\bar{5}[\bar{\mu},\bar{\nu}]} = \frac{5}{6} n \epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}} a_{\bar{\rho}} \,, \qquad \tau_{\bar{\mu}\bar{4}} = \frac{5n-6}{2} a_{\bar{\mu}} \,, \tag{8.14}$$

while if they are IIB coordinates, we get

$$Z^{\bar{5}[\bar{\mu},\bar{4}]} = \frac{5}{6} n a^{\bar{\mu}} \,, \qquad \tau_{\bar{\mu}\bar{\nu}} = \frac{5n-6}{2} \epsilon_{\bar{\mu}\bar{\nu}\bar{\rho}} a^{\bar{\rho}} \,. \tag{8.15}$$

In both cases, the internal space in string frame is

$$ds^2 = dy_\mu dy^\mu + (1 - a \cdot y)^2 \, ds_7^2 \,, \qquad e^\psi = (1 - a \cdot y)^{5n/2} \,. \tag{8.16}$$

We can see that the metric is independent of n and we are tuning the dilaton to obtain different gaugings.

These reductions have no action principle but do have equations of motion.

Conclusions and further work

- We have shown that there is a “duality” relating IIA and IIB truncations in maximal 7-D SUGRA.
- The duality swaps the **10**’s, and dualises the **6**’s.
- The NS-NS sector remains invariant!
- Gaugings with **4**’s cannot in general be dualised.
- We obtained IIB hyperboloid truncations with the same reduction as for IIA.
- We found geometric uplifts for new gaugings
  - Hyperboloids with R-R fields.
  - Gaugings of the **6** in IIA and IIB.
  - Gaugings of the trombone.

Further work:

- Find truncations corresponding to known solutions of 7-D QC.
- Full classification of solutions of 7-D QC.
- Section-violating solutions? Lee, Strickland-Constable, Waldram 1506.03457
- Extend to E_n.