DOUBLED-YET-GAUGED, SEMI-COVARIANCE & TWOFOLD SPIN

JEONG-HYUCK PARK

Sogang University, Seoul

CERN-CKC TH Institute on Duality Symmetries in String and M-Theories

10th August 2015

Three themes in this talk:

- **Doubled-yet-gauged coordinate system**
- **Semi-covariant formulation of DFT/SDFT**
- **Twofold spin and Standard Model**

To summarize collaborated works with Imtak Jeon (8), Kanghoon Lee (8), Yoonji Suh (3), Chris Blair (1), Emanuel Malek (1), Wonyoung Cho (1), Jose Fernández-Melgarejo (1), Kang-Sin Choi (1: *Phenomenologist*), Soo-Jong Rey (1), Woohyun Rim (1), Yuho Sakatani (1), Sung Moon Ko (1), Charles Melby-Thompson (1), Rene Meyér (1).

- *Differential geometry with a projection: Application to double field theory* 1011.1324
- *Stringy differential geometry, beyond Riemann* 1105.6294
- *Incorporation of fermions into double field theory* 1109.2035
- *Ramond-Ramond Cohomology and O(D,D) T-duality* 1206.3478
- *Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity* 1112.0069
- *Stringy Unification of IIA and IIB Supergravities under* $\mathcal{N}=2$ $D=10$ *Supersymmetric Double Field Theory* 1210.5078
- *Supersymmetric gauged Double Field Theory: Systematic derivation by virtue of 'Twist'* 1505.01301 *c.f. Jose F-M's talk*
- *Comments on double field theory and diffeomorphisms* 1304.5946
- *Covariant action for a string in doubled yet gauged spacetime* 1307.8377
- *Double field formulation of Yang-Mills theory* $\Rightarrow$ *Standard Model Double Field Theory* 1102.0419/1506.05277
- **O**(D,D) *Covariant Noether Currents and Global Charges in Double Field Theory* 1507.07545 *c.f. Chris Blair's talk*
- *Dynamics of Perturbations in Double Field Theory & Non-Relativistic String Theory* 1508.01121
- *U-geometry: SL(5)* $\Rightarrow$ *U-gravity: SL(N)* 1302.1652/1402.5027 *c.f. Yoonji Suh's talk*
- *M-theory and F-theory from a Duality Manifest Action* 1311.5109

Notation 1/2

- Capital Latin letters denote the $\mathbf{O}(D,D)$ vector indices,

$$A,B,C,\cdots,L,M,N,\cdots=1,2,\cdots,D{+}D\,.$$

- They can be freely raised or lowered by the $\mathbf{O}(D,D)$ invariant constant metric,

$$\mathcal{J}_{AB}=\left(\begin{array}{cc}0&1\\1&0\end{array}\right).$$

Doubled-yet-gauged coordinate system

1304.5946/1307.8377

Section condition in DFT

- The section condition in DFT,

$$\partial_M \partial^M = 0\,,$$

implies that, arbitrary functions and their arbitrary derivatives, collectively Φ, are invariant under translations generated by a **derivative-index-valued vector,**

$$\Phi_0(x+\Delta) = \Phi_0(x)\,, \qquad \Delta^M = \Phi_1 \partial^M \Phi_2\,.$$

- In fact, the converse is also true. *c.f.* 1307.8377

Doubled-yet-gauged coordinate system

- Start with any D-dimensional coordinate system, x^{μ}, *e.g.* (t, x, y, z) *or* (t, r, θ, ϕ) *etc.*
- The doubled coordinates,

$$x^{M} = (\, \tilde{x}_{\mu} \,,\, x^{\nu} \,) \,,$$

are then required **to be gauged**: they are subject to an **equivalence relation**,

$$x^{M} \;\sim\; x^{M} + \Phi_{1}\partial^{M}\Phi_{2} \,,$$

which we call **coordinate gauge symmetry**.

- Each equivalence class, or gauge orbit, represents a single physical point.
- (Strongly constrained) DFT employs such a **doubled-yet-gagued coordinate system.**
- Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the 'gauge orbits', rather than 'points' in the doubled coordinate space:
  - Hohm-Zwiebach ansatz for finite transformations $\equiv \mathbf{exp}(\hat{\mathcal{L}}_{X})$.

    c.f. Berman-Cederwall-Perry, Hull
  - Alternative approach to the cohomological triviality *à la* Papadopoulos

Doubled-yet-gauged coordinate system

- Start with any D-dimensional coordinate system, x^{μ}, *e.g.* (t, x, y, z) *or* (t, r, θ, ϕ) *etc.*
- The doubled coordinates,

$$x^{M} = (\,\tilde{x}_{\mu}\,,\, x^{\nu}\,)\,,$$

are then required **to be gauged**: they are subject to an **equivalence relation**,

$$x^{M} \;\sim\; x^{M} + \Phi_{1}\partial^{M}\Phi_{2}\,,$$

which we call **coordinate gauge symmetry**.

- Each equivalence class, or gauge orbit, represents a single physical point.
- (Strongly constrained) DFT employs such a **doubled-yet-gagued coordinate system.**
- Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the 'gauge orbits', rather than 'points' in the doubled coordinate space:
  - Hohm-Zwiebach ansatz for finite transformations $\equiv \exp(\hat{\mathcal{L}}_{X})$.

    c.f. Berman-Cederwall-Perry, Hull
  - Alternative approach to the cohomological triviality *à la* Papadopoulos

Doubled-yet-gauged coordinate system

- Start with any D-dimensional coordinate system, x^{μ}, *e.g.* (t,x,y,z) *or* (t,r,θ,ϕ) *etc.*
- The doubled coordinates,

$$x^{M} = (\,\tilde{x}_{\mu}\,,\,x^{\nu}\,)\;,$$

  are then required **to be gauged**: they are subject to an **equivalence relation**,

$$x^{M}\;\sim\;x^{M}+\Phi_{1}\partial^{M}\Phi_{2}\,,$$

  which we call **coordinate gauge symmetry**.

- Each equivalence class, or gauge orbit, represents a single physical point.
- (Strongly constrained) DFT employs such a **doubled-yet-gagued coordinate system.**
- Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the 'gauge orbits', rather than 'points' in the doubled coordinate space:
  - Hohm-Zwiebach ansatz for finite transformations $\equiv \exp(\hat{\mathcal{L}}_{X})$.

    c.f. Berman-Cederwall-Perry, Hull
  - Alternative approach to the cohomological triviality *à la* Papadopoulos

Doubled-yet-gauged coordinate system

- Start with any D-dimensional coordinate system, x^{μ}, *e.g.* (t,x,y,z) *or* (t,r,θ,ϕ) *etc.*
- The doubled coordinates,
$$x^{M}=(\,\tilde{x}_{\mu}\,,\,x^{\nu}\,)\,,$$
are then required **to be gauged**: they are subject to an **equivalence relation**,
$$x^{M}\;\sim\;x^{M}+\Phi_{1}\partial^{M}\Phi_{2}\,,$$
which we call **coordinate gauge symmetry**.
- Each equivalence class, or gauge orbit, represents a single physical point.
- (Strongly constrained) DFT employs such a **doubled-yet-gagued coordinate system.**
- Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the 'gauge orbits', rather than 'points' in the doubled coordinate space:
  - Hohm-Zwiebach ansatz for finite transformations $\equiv\mathbf{exp}(\hat{\mathcal{L}}_{X})$.

    c.f. Berman-Cederwall-Perry, Hull
  - Alternative approach to the cohomological triviality *à la* Papadopoulos

Doubled-yet-gauged coordinate system

- Start with any D-dimensional coordinate system, x^{μ}, *e.g.* (t,x,y,z) *or* (t,r,θ,ϕ) *etc.*
- The doubled coordinates,

$$x^{M} = (\,\tilde{x}_{\mu}\,,\,x^{\nu}\,)\;,$$

  are then required **to be gauged**: they are subject to an **equivalence relation**,

$$x^{M}\;\sim\;x^{M}+\Phi_{1}\partial^{M}\Phi_{2}\,,$$

  which we call **coordinate gauge symmetry**.

- Each equivalence class, or gauge orbit, represents a single physical point.
- (Strongly constrained) DFT employs such a **doubled-yet-gagued coordinate system.**
- Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the 'gauge orbits', rather than 'points' in the doubled coordinate space:
  - Hohm-Zwiebach ansatz for finite transformations $\equiv \mathbf{exp}(\hat{\mathcal{L}}_{X})$.

    c.f. Berman-Cederwall-Perry, Hull
  - Alternative approach to the cohomological triviality *à la* Papadopoulos

Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- The usual infinitesimal one-form, $\mathrm{d}x^M$, is NOT a covariant vector in DFT: it does not transform covariantly under DFT diffeomorphisms, obeying the way the 'generalized Lie derivative' would dictate.
- Hence, $\mathrm{d}x^M\mathrm{d}x^N\mathcal{H}_{MN}$ can NOT give a 'proper length' in DFT.
- Further, it is NOT coordinate gauge symmetry invariant,

$$\mathrm{d}x^M \quad\longrightarrow\quad \mathrm{d}(x^M+\Phi_1\partial^M\Phi_2)\;\neq\;\mathrm{d}x^M\,.$$

- These can be all cured by introducing a **gauged infinitesimal one-form**,

$$Dx^M:=\mathrm{d}x^M-\mathcal{A}^M\,,$$

where $\mathcal{A}^M$ is the 'coordinate gauge potential'. Being a derivative-index-valued vector, it satisfies $\mathcal{A}^M\partial_M=0$, $\mathcal{A}_M\mathcal{A}^M=0$, or suggestively the 'gauged section condition',

$$(\partial_M+\mathcal{A}_M)(\partial^M+\mathcal{A}^M)=0\,.$$

Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- The usual infinitesimal one-form, $\mathrm{d}x^M$, is NOT a covariant vector in DFT: it does not transform covariantly under DFT diffeomorphisms, obeying the way the 'generalized Lie derivative' would dictate.
- Hence, $\mathrm{d}x^M \mathrm{d}x^N \mathcal{H}_{MN}$ can NOT give a 'proper length' in DFT.
- Further, it is NOT coordinate gauge symmetry invariant,

$$\mathrm{d}x^M \quad \longrightarrow \quad \mathrm{d}(x^M + \Phi_1 \partial^M \Phi_2) \neq \mathrm{d}x^M .$$

- These can be all cured by introducing a **gauged infinitesimal one-form**,

$$Dx^M := \mathrm{d}x^M - \mathcal{A}^M ,$$

where $\mathcal{A}^M$ is the 'coordinate gauge potential'. Being a derivative-index-valued vector, it satisfies $\mathcal{A}^M \partial_M = 0$, $\mathcal{A}_M \mathcal{A}^M = 0$, or suggestively the 'gauged section condition',

$$(\partial_M + \mathcal{A}_M)(\partial^M + \mathcal{A}^M) = 0 .$$

Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- Under coordinate gauge symmetry, we have the invariance of Dx^M,

$$x^M \longrightarrow x'^M = x^M + \Phi_1\partial^M\Phi_2\,,$$

$$\mathcal{A}^M \longrightarrow \mathcal{A}'^M = \mathcal{A}^M + \mathrm{d}(\Phi_1\partial^M\Phi_2) \qquad : \quad \mathcal{A}'^M\partial'_M \equiv 0\,,$$

$$Dx^M \longrightarrow D'x'^M = Dx^M = \mathrm{d}x^M - \mathcal{A}^M\,.$$

- Similarly, under (finite) DFT diffeomorphisms *à la* Hohm-Zwiebach

$$L_M{}^N := \partial_M x'^N\,, \qquad \bar{L} := \mathcal{J}L^t\mathcal{J}^{-1}\,,$$

$$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right), \qquad \bar{F} := \mathcal{J}F^t\mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1}\,,$$

we have the covariance,

$$x^M \longrightarrow x'^M(x)\,,$$

$$\mathcal{H}_{MN}(x) \longrightarrow \mathcal{H}'_{MN}(x') = \bar{F}_M{}^K\bar{F}_N{}^L\mathcal{H}_{KL}(x)\,,$$

$$\mathcal{A}^M \longrightarrow \mathcal{A}'^M = \mathcal{A}^N F_N{}^M + \mathrm{d}X^N(L-F)_N{}^M \qquad : \quad \mathcal{A}'^M\partial'_M \equiv 0\,,$$

$$Dx^M \longrightarrow D'x'^M = Dx^N F_N{}^M\,.$$

Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- Under coordinate gauge symmetry, we have the invariance of Dx^M,

$$x^M \longrightarrow x'^M = x^M + \Phi_1\partial^M\Phi_2\,,$$
$$\mathcal{A}^M \longrightarrow \mathcal{A}'^M = \mathcal{A}^M + \mathrm{d}(\Phi_1\partial^M\Phi_2) \qquad : \quad \mathcal{A}'^M\partial'_M \equiv 0\,,$$
$$Dx^M \longrightarrow D'x'^M = Dx^M = \mathrm{d}x^M - \mathcal{A}^M\,.$$

- Similarly, under (finite) DFT diffeomorphisms *à la* Hohm-Zwiebach

$$L_M{}^N := \partial_M x'^N\,, \qquad \bar{L} := \mathcal{J}L^t\mathcal{J}^{-1}\,,$$
$$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right)\,, \qquad \bar{F} := \mathcal{J}F^t\mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1}\,,$$

we have the covariance,

$$x^M \longrightarrow x'^M(x)\,,$$
$$\mathcal{H}_{MN}(x) \longrightarrow \mathcal{H}'_{MN}(x') = \bar{F}_M{}^K\bar{F}_N{}^L\mathcal{H}_{KL}(x)\,,$$
$$\mathcal{A}^M \longrightarrow \mathcal{A}'^M = \mathcal{A}^N F_N{}^M + \mathrm{d}X^N(L-F)_N{}^M \qquad : \quad \mathcal{A}'^M\partial'_M \equiv 0\,,$$
$$Dx^M \longrightarrow D'x'^M = Dx^N F_N{}^M\,.$$

- In DFT –unlike EFT or U-gravity– the solution of the section condition, *i.e.* the section is unique up to the duality rotations,

$$\frac{\partial}{\partial x^M} = \left(\frac{\partial}{\partial \tilde{x}_\mu}\,,\,\frac{\partial}{\partial x^\nu}\right) \equiv \left(0\,,\,\frac{\partial}{\partial x^\nu}\right) \qquad \text{'Canonical choice of the section'}$$

- Then, the 'coordinate gauge symmetry' reads

$$(\;\tilde{x}_\mu\;\;,\;\;x^\nu\;)\;\sim\;(\;\tilde{x}_\mu+\Phi_1\partial_\mu\Phi_2\;\;,\;\;x^\nu\;)\,.$$

- The coordinate gauge potential and the gauged infinitesimal one-form become

$$\mathcal{A}^M = A_\lambda\partial^M x^\lambda = (\;A_\mu\;\;,\;\;0\;)\,, \qquad\qquad Dx^M = (\;\mathrm{d}\tilde{x}_\mu - A_\mu\;\;,\;\;\mathrm{d}x^\nu\;)\,.$$

Fixing the coordinate gauge symmetry : canonical choice of the section

- In DFT –unlike EFT or U-gravity– the solution of the section condition, *i.e.* the section is unique up to the duality rotations,

$$\frac{\partial}{\partial x^M} = \left(\frac{\partial}{\partial \tilde{x}_\mu}, \frac{\partial}{\partial x^\nu}\right) \equiv \left(0, \frac{\partial}{\partial x^\nu}\right) \qquad \text{'Canonical choice of the section'}$$

- Then, the 'coordinate gauge symmetry' reads

$$(\ \tilde{x}_\mu\ ,\ x^\nu\ )\ \sim\ (\ \tilde{x}_\mu + \Phi_1 \partial_\mu \Phi_2\ ,\ x^\nu\ )\,.$$

- The coordinate gauge potential and the gauged infinitesimal one-form become

$$\mathcal{A}^M = A_\lambda \partial^M x^\lambda = (\ A_\mu\ ,\ 0\ )\,, \qquad Dx^M = (\ \mathrm{d}\tilde{x}_\mu - A_\mu\ ,\ \mathrm{d}x^\nu\ )\,.$$

Newton mechanics with doubled-yet-gauged coordinate system

- The doubled-yet-gauged coordinates can be applied to any physical system, not exclusively to DFT.

- Newton mechanics can be formulated on the doubled-yet-gauged space, $x^I = (\tilde{x}_j, x^k)$,

$$\mathcal{L}_{\rm Newton} = \tfrac{1}{2} m \, D_t x^I D_t x^J \, \delta_{IJ} \; - \; V(x) \, ,$$

where $I, J = 1, 2, \cdots, 6$ and the potential, $V(x)$, satisfies the section condition.

- With the canonical choice of the section, we get

$$\mathcal{L}_{\rm Newton} = \tfrac{1}{2} m \, \dot{x}^j \dot{x}^k \, \delta_{jk} \; - \; V(x) \; + \; \tfrac{1}{2} m \left(\dot{\tilde{x}}_j - A_j \right) \left(\dot{\tilde{x}}_k - A_k \right) \delta^{jk} \, .$$

Hence, after integrating out A_j, we recover the conventional formulation.

Newton mechanics with doubled-yet-gauged coordinate system

- The doubled-yet-gauged coordinates can be applied to any physical system, not exclusively to DFT.

- Newton mechanics can be formulated on the doubled-yet-gauged space, $x^I = (\tilde{x}_j, x^k)$,

$$\mathcal{L}_{\rm Newton} = \tfrac{1}{2} m\, D_t x^I D_t x^J \,\delta_{IJ} \;-\; V(x)\,,$$

where $I, J = 1, 2, \cdots, 6$ and the potential, $V(x)$, satisfies the section condition.

- With the canonical choice of the section, we get

$$\mathcal{L}_{\rm Newton} = \tfrac{1}{2} m\, \dot{x}^j \dot{x}^k \,\delta_{jk} \;-\; V(x) \;+\; \tfrac{1}{2} m \left(\dot{\tilde{x}}_j - A_j\right) \left(\dot{\tilde{x}}_k - A_k\right) \delta^{jk}\,.$$

Hence, after integrating out A_j, we recover the conventional formulation.

String probes the doubled-yet-gauged spacetime

- **DFT string action** is with $D_iX^M = \partial_i X^M - \mathcal{A}_i^M$, JHP-Lee 2013

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \; \mathcal{L}_{\mathrm{string}} \,, \qquad \mathcal{L}_{\mathrm{string}} = -\tfrac{1}{2}\sqrt{-h}\, h^{ij} D_iX^M D_jX^N \mathcal{H}_{MN}(X) - \epsilon^{ij} D_iX^M \mathcal{A}_{jM} \,,$$

- The action is **fully symmetric**, essentially due to the auxiliary gauge field, $\mathcal{A}_i^M$, under
  - String worldsheet diffeomorphisms plus Weyl symmetry (as usual)
  - $\mathbf{O}(D,D)$ T-duality
  - Target spacetime DFT diffeomorphisms
  - The coordinate gauge symmetry

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

String probes the doubled-yet-gauged spacetime

- **DFT string action** is with $D_iX^M = \partial_i X^M - \mathcal{A}_i^M$, JHP-Lee 2013

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\mathrm{string}}\,, \qquad \mathcal{L}_{\mathrm{string}} = -\tfrac{1}{2}\sqrt{-h}\,h^{ij}D_iX^MD_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

- $\mathcal{H}_{AB}(x)$ is the "generalized metric" which can be defined as a symmetric $\mathbf{O}(D,D)$ element,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}\,, \qquad \mathcal{H}_A{}^C\mathcal{H}_B{}^D\mathcal{J}_{CD} = \mathcal{J}_{AB}\,,$$

and satisfies the section condition.

- There are **two types** of "generalized metric" : **Riemannian vs. non-Riemannian**.

String probes the doubled-yet-gauged spacetime

- **DFT string action** is with $D_iX^M = \partial_i X^M - \mathcal{A}_i^M$, JHP-Lee 2013

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\rm string}\,, \qquad \mathcal{L}_{\rm string} = -\tfrac{1}{2}\sqrt{-h}\,h^{ij}D_iX^M D_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

- $\mathcal{H}_{AB}(x)$ is the "generalized metric" which can be defined as a symmetric $\mathbf{O}(D,D)$ element,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}\,, \qquad \mathcal{H}_A{}^C\mathcal{H}_B{}^D\mathcal{J}_{CD} = \mathcal{J}_{AB}\,,$$

and satisfies the section condition.

- There are **two types** of "generalized metric" : **Riemannian vs. non-Riemannian.**

String probes the doubled-yet-gauged spacetime

- **DFT string action** is with $D_i X^M = \partial_i X^M - \mathcal{A}_i^M$, JHP-Lee 2013

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\rm string}\,, \qquad \mathcal{L}_{\rm string} = -\tfrac{1}{2}\sqrt{-h}\,h^{ij}D_iX^MD_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

- $\mathcal{H}_{AB}(x)$ is the "generalized metric" which can be defined as a symmetric $\mathbf{O}(D,D)$ element,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}\,, \qquad \mathcal{H}_A{}^C\mathcal{H}_B{}^D\mathcal{J}_{CD} = \mathcal{J}_{AB}\,,$$

  and satisfies the section condition.

- There are **two types** of "generalized metric" : **Riemannian vs. non-Riemannian**.

DFT backgrounds : Riemannian vs. non-Riemannian

- W.r.t. the canonical choice of the section, $\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0$, **Riemannian generalized metric** assumes the well-known form,

$$\mathcal{H}_{AB} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} .$$

Up to field redefinition (e.g. β-gravity Andriot-Betz) this is the most general form of a symmetric $\mathbf{O}(D, D)$ element **if the upper left $D \times D$ block is 'non-degenerate'.**

- The DFT sigma model then reduces to the standard string action,

$$\frac{1}{4\pi\alpha'}\mathcal{L}_{\rm string} \equiv \frac{1}{2\pi\alpha'}\left[-\frac{1}{2}\sqrt{-h}h^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu}(X) + \frac{1}{2}\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu}(X) + \frac{1}{2}\epsilon^{ij}\partial_i \tilde{X}_\mu \partial_j X^\mu\right] ,$$

with the bonus of the topological term introduced by Giveon-Rocek; Hull.

- The EOM of $\mathcal{A}_i^M$ implies **self-duality** on the full doubled spacetime,

$$\mathcal{H}^M{}_N D^i X^N + \frac{1}{\sqrt{-h}}\epsilon^{ij} D_j X^M = 0 .$$

DFT backgrounds : Riemannian vs. non-Riemannian

- W.r.t. the canonical choice of the section, $\frac{\partial}{\partial\tilde{x}_\mu}\equiv 0$, **Riemannian generalized metric** assumes the well-known form,

$$\mathcal{H}_{AB}=\begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G-BG^{-1}B \end{pmatrix}.$$

Up to field redefinition (e.g. β-gravity Andriot-Betz) this is the most general form of a symmetric $\mathbf{O}(D,D)$ element **if the upper left $D\times D$ block is 'non-degenerate'.**

- The DFT sigma model then reduces to the standard string action,

$$\frac{1}{4\pi\alpha'}\mathcal{L}_{\rm string}\equiv\frac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_iX^\mu\partial_jX^\nu G_{\mu\nu}(X)+\tfrac{1}{2}\epsilon^{ij}\partial_iX^\mu\partial_jX^\nu B_{\mu\nu}(X)+\tfrac{1}{2}\epsilon^{ij}\partial_i\tilde{X}_\mu\partial_jX^\mu\right],$$

with the bonus of the topological term introduced by Giveon-Rocek; Hull.

- The EOM of $\mathcal{A}_i^M$ implies **self-duality** on the full doubled spacetime,

$$\mathcal{H}^M{}_N D^iX^N+\frac{1}{\sqrt{-h}}\epsilon^{ij}D_jX^M=0\,.$$

DFT backgrounds : Riemannian vs. non-Riemannian

- W.r.t. the canonical choice of the section, $\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0$, **Riemannian generalized metric** assumes the well-known form,

$$\mathcal{H}_{AB} = \left(\begin{array}{cc} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{array}\right).$$

Up to field redefinition (e.g. β-gravity Andriot-Betz) this is the most general form of a symmetric $\mathbf{O}(D,D)$ element **if the upper left $D \times D$ block is 'non-degenerate'.**

- The DFT sigma model then reduces to the standard string action,

$$\frac{1}{4\pi\alpha'}\mathcal{L}_{\text{string}} \equiv \frac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_i \tilde{X}_\mu \partial_j X^\mu\right],$$

with the bonus of the topological term introduced by Giveon-Rocek; Hull.

- The EOM of $\mathcal{A}_i^M$ implies **self-duality** on the full doubled spacetime,

$$\mathcal{H}^M{}_N D^i X^N + \tfrac{1}{\sqrt{-h}}\epsilon^{ij} D_j X^M = 0\,.$$

DFT backgrounds : Riemannian vs. non-Riemannian

- W.r.t. $\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0$ again, the **non-Riemannian DFT background** is then characterized by the degenerate upper left $D \times D$ block, such that it does not admit any Riemannian interpretation even locally. For example, with the decomposition, $D = 10 = 2 + 8$,

$$\mathcal{H}_{MN} = \begin{pmatrix} 0 & 0 & e^{\alpha}{}_{\beta} & 0 \\ 0 & \delta^{ij} & 0 & 0 \\ -e_{\alpha}{}^{\beta} & 0 & f\eta_{\alpha\beta} & 0 \\ 0 & 0 & 0 & \delta_{ij} \end{pmatrix}, \qquad f = 1 + \tfrac{Q}{r^6}\,, \qquad r^2 = \textstyle\sum_{i=2}^{9}(x^i)^2\,.$$

  This is "doublly T-dual", $(t, x^1) \Leftrightarrow (\tilde{t}, \tilde{x}_1)$, DFT background to F1 *à la* Dabholkar-Gibbons-Harvey-Ruiz 1990, *c.f.* 2*D* null-wave *à la* Berkeley-Berman-Rudolf

- DFT as well as the DFT sigma model is well-defined even for such a non-Riemannian background.

- In particular, the Gomis-Ooguri 'non-relativistic' string theory can be identified precisely as the DFT sigma model on the above non-Riemannian background. Further, the sigma model spectrum matches with the pertubations of DFT around the non-Riemannian background. Ko-Meyer-Melby-Thompson-JHEP (last week)

DFT backgrounds : Riemannian vs. non-Riemannian

- W.r.t. $\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0$ again, the **non-Riemannian DFT background** is then characterized by the degenerate upper left $D \times D$ block, such that it does not admit any Riemannian interpretation even locally. For example, with the decomposition, $D = 10 = 2 + 8$,

$$\mathcal{H}_{MN} = \begin{pmatrix} 0 & 0 & e^{\alpha}{}_{\beta} & 0 \\ 0 & \delta^{ij} & 0 & 0 \\ -e_{\alpha}{}^{\beta} & 0 & f\eta_{\alpha\beta} & 0 \\ 0 & 0 & 0 & \delta_{ij} \end{pmatrix}, \qquad f = 1 + \frac{Q}{r^6}\,, \qquad r^2 = \sum_{i=2}^{9} (x^i)^2\,.$$

  This is "doublly T-dual", $(t, x^1) \Leftrightarrow (\tilde{t}, \tilde{x}_1)$, DFT background to F1 *à la* Dabholkar-Gibbons-Harvey-Ruiz 1990, *c.f.* 2*D* null-wave *à la* Berkeley-Berman-Rudolf

- DFT as well as the DFT sigma model is well-defined even for such a non-Riemannian background.

- In particular, the Gomis-Ooguri 'non-relativistic' string theory can be identified precisely as the DFT sigma model on the above non-Riemannian background. Further, the sigma model spectrum matches with the pertubations of DFT around the non-Riemannian background. Ko-Meyer-Melby-Thompson-JHP (last week)

Semi-covariant formulation of DFT/SDFT:

Gravity on doubled-yet-gauged spacetime

1011.1324/1105.6294/$\cdots$

- **Contrary to what it may sound like, the semi-covariant formalism is a completely covariant approach to DFT, as it manifests simultaneously**

  - **O**(D,D) **T-duality**

  - **DFT-diffeomorphisms (generalized Lie derivative)**

  - **A *pair* of local Lorentz symmetries, Spin**$(1,D{-}1)_L\times$ **Spin**$(D{-}1,1)_R$

In particular, it makes each term in $D=10$ Maximal SDFT completely covariant:

$$\mathcal{L}_{\rm Type\,II}=e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho-i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}}-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho'+i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]$$

Jeon-Lee-Suh-JHP 2012

- **It also works for SL**(N) **duality group,** $N\neq 4$ *c.f.* **Yoonji Suh's talk on U-gravity**

Notation 2/2

Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots$	$\mathbf{O}(D,D)$ & DFT-diffeom. vector	$\mathcal{J}_{AB}$
$p, q, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ vector	$\eta_{pq} = \mathrm{diag}(-++\cdots+)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ spinor	$C_{+\alpha\beta}, \qquad (\gamma^p)^T = C_+\gamma^p C_+^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(D{-}1, 1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+--\cdots-)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(D{-}1, 1)_R$ spinor	$\bar{C}_{+\bar{\alpha}\bar{\beta}}, \qquad (\bar{\gamma}^{\bar{p}})^T = \bar{C}_+\bar{\gamma}^{\bar{p}}\bar{C}_+^{-1}$

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector**
    - **DFT-dilaton:** d
    - **DFT-vielbeins:** $V_{Ap}\,, \quad \bar{V}_{A\bar{p}}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions (Majorana-Weyl)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

R-R potential and Fermions carry NOT $(D+D)$-dimensional BUT undoubled D-dimensional indices.

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector**
    - **DFT-dilaton:** d
    - **DFT-vielbeins:** $V_{Ap}\,,\quad \bar{V}_{A\bar{p}}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions (Majorana-Weyl)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,,\quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,,\quad \psi'^{\bar{\alpha}}_{p}$

A priori, **O**(D,D) **rotates only the O**(D,D) **vector indices (capital Roman), and the R-R sector and all the fermions are O**(D,D) **T-duality singlet.**

The usual IIA $\Leftrightarrow$ IIB exchange will follow only after the diagonal gauge fixing of the twofold local Lorentz symmetries.

- The DFT-dilaton gives rise to a scalar density with weight one,

$$e^{-2d}\,.$$

- The DFT-vielbeins satisfy four 'defining' properties:

$$V_{Ap}V^{A}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Ap}\bar{V}^{A}{}_{\bar{q}}=0\,,\qquad V_{Ap}V_{B}{}^{p}+\bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}}=\mathcal{J}_{AB}\,.$$

- And they generate a pair of two-index 'projectors',

$$P_{AB}:=V_{A}{}^{p}V_{Bp}\,,\qquad P_{A}{}^{B}P_{B}{}^{C}=P_{A}{}^{C}\,,\qquad \bar{P}_{AB}:=\bar{V}_{A}{}^{\bar{p}}\bar{V}_{B\bar{p}}\,,\qquad \bar{P}_{A}{}^{B}\bar{P}_{B}{}^{C}=\bar{P}_{A}{}^{C}\,,$$

which are symmetric, orthogonal and complementary to each other,

$$P_{AB}=P_{BA}\,,\qquad \bar{P}_{AB}=\bar{P}_{BA}\,,\qquad P_{A}{}^{B}\bar{P}_{B}{}^{C}=0\,,\qquad P_{A}{}^{B}+\bar{P}_{A}{}^{B}=\delta_{A}{}^{B}\,.$$

- Some further projection properties follow

$$P_{A}{}^{B}V_{Bp}=V_{Ap}\,,\qquad \bar{P}_{A}{}^{B}\bar{V}_{B\bar{p}}=\bar{V}_{A\bar{p}}\,,\qquad \bar{P}_{A}{}^{B}V_{Bp}=0\,,\qquad P_{A}{}^{B}\bar{V}_{B\bar{p}}=0\,.$$

- Note also $\mathcal{H}_{AB}=P_{AB}-\bar{P}_{AB}$. However, our emphasis lies on the 'projectors' rather than the "generalized metric".

- The DFT-dilaton gives rise to a scalar density with weight one,

$$e^{-2d}\,.$$

- The DFT-vielbeins satisfy four ‘defining’ properties:

$$V_{Ap}V^{A}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Ap}\bar{V}^{A}{}_{\bar{q}}=0\,,\qquad V_{Ap}V_{B}{}^{p}+\bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}}=\mathcal{J}_{AB}\,.$$

- And they generate a pair of two-index ‘projectors’,

$$P_{AB}:=V_{A}{}^{p}V_{Bp}\,,\qquad P_{A}{}^{B}P_{B}{}^{C}=P_{A}{}^{C}\,,\qquad \bar{P}_{AB}:=\bar{V}_{A}{}^{\bar{p}}\bar{V}_{B\bar{p}}\,,\qquad \bar{P}_{A}{}^{B}\bar{P}_{B}{}^{C}=\bar{P}_{A}{}^{C}\,,$$

which are symmetric, orthogonal and complementary to each other,

$$P_{AB}=P_{BA}\,,\qquad \bar{P}_{AB}=\bar{P}_{BA}\,,\qquad P_{A}{}^{B}\bar{P}_{B}{}^{C}=0\,,\qquad P_{A}{}^{B}+\bar{P}_{A}{}^{B}=\delta_{A}{}^{B}\,.$$

- Some further projection properties follow

$$P_{A}{}^{B}V_{Bp}=V_{Ap}\,,\qquad \bar{P}_{A}{}^{B}\bar{V}_{B\bar{p}}=\bar{V}_{A\bar{p}}\,,\qquad \bar{P}_{A}{}^{B}V_{Bp}=0\,,\qquad P_{A}{}^{B}\bar{V}_{B\bar{p}}=0\,.$$

- Note also $\mathcal{H}_{AB}=P_{AB}-\bar{P}_{AB}$. However, our emphasis lies on the ‘projectors’ rather than the “generalized metric".

- We continue to define a pair of six-index projectors,

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D}\,, \qquad \mathcal{P}_{CAB}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{CAB}{}^{GHI}\,,$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D}\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF} \bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{CAB}{}^{GHI}\,,$$

which are symmetric and traceless,

$$\mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}\,, \qquad \bar{\mathcal{P}}_{CABDEF} = \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]}\,,$$

$$\mathcal{P}^A{}_{ABDEF} = 0\,, \quad P^{AB} \mathcal{P}_{ABCDEF} = 0\,, \qquad \bar{\mathcal{P}}^A{}_{ABDEF} = 0\,, \quad \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} = 0\,.$$

- As we shall see shorlty, these projectors govern the DFT-diffeomorphic anomaly in the semi-covariant formalism, which can be then easily projected out.

- **Having all the 'right' field-variables prepared, we now discuss their derivatives or what we call, 'semi-covariant derivative'.**

- The meaning of 'semi-covariane' will be clear later.

- **Having all the 'right' field-variables prepared, we now discuss their derivatives or what we call, 'semi-covariant derivative'.**

- The meaning of 'semi-covariane' will be clear later.

Semi-covariant derivatives

- For each gauge symmetry we assign a corresponding connection,
  - Γ_A for the DFT-diffeomorphism (generalized Lie derivative),
  - Φ_A for the 'unbarred' local Lorentz symmetry, $\mathbf{Spin}(1, D-1)_L$,
  - $\bar{\Phi}_A$ for the 'barred' local Lorentz symmetry, $\mathbf{Spin}(D-1, 1)_R$.
- Combining all of them, we introduce **master 'semi-covariant' derivative,**

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A \,.$$

- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A \,, \qquad\qquad D_A = \partial_A + \Phi_A + \bar{\Phi}_A \,.$$

- The former is the 'semi-covariant' derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,.$$

- And the latter is the covariant derivative for the twofold local Lorenz symmetries.

- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A \,, \qquad\qquad D_A = \partial_A + \Phi_A + \bar{\Phi}_A \,.$$

- The former is the 'semi-covariant' derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,.$$

- And the latter is the covariant derivative for the twofold local Lorenz symmetries.

- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A\,, \qquad\qquad D_A = \partial_A + \Phi_A + \bar{\Phi}_A\,.$$

- The former is the 'semi-covariant' derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

$$\nabla_C T_{A_1 A_2\cdots A_n} := \partial_C T_{A_1 A_2\cdots A_n} - \omega\Gamma^B{}_{BC} T_{A_1 A_2\cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1\cdots A_{i-1} B A_{i+1}\cdots A_n}\,.$$

- And the latter is the covariant derivative for the twofold local Lorenz symmetries.

- **By definition, the master derivative annihilates all the 'constants',**

$$\mathcal{D}_A\mathcal{J}_{BC} = \nabla_A\mathcal{J}_{BC} = \Gamma_{AB}{}^D\mathcal{J}_{DC} + \Gamma_{AC}{}^D\mathcal{J}_{BD} = 0\,,$$

$$\mathcal{D}_A\eta_{pq} = D_A\eta_{pq} = \Phi_{Ap}{}^r\eta_{rq} + \Phi_{Aq}{}^r\eta_{pr} = 0\,,$$

$$\mathcal{D}_A\bar{\eta}_{\bar{p}\bar{q}} = D_A\bar{\eta}_{\bar{p}\bar{q}} = \bar{\Phi}_{A\bar{p}}{}^{\bar{r}}\bar{\eta}_{\bar{r}\bar{q}} + \bar{\Phi}_{A\bar{q}}{}^{\bar{r}}\bar{\eta}_{\bar{p}\bar{r}} = 0\,,$$

$$\mathcal{D}_A C_{+\alpha\beta} = D_A C_{+\alpha\beta} = \Phi_{A\alpha}{}^{\delta}C_{+\delta\beta} + \Phi_{A\beta}{}^{\delta}C_{+\alpha\delta} = 0\,,$$

$$\mathcal{D}_A\bar{C}_{+\bar{\alpha}\bar{\beta}} = D_A\bar{C}_{+\bar{\alpha}\bar{\beta}} = \bar{\Phi}_{A\bar{\alpha}}{}^{\bar{\delta}}\bar{C}_{+\bar{\delta}\bar{\beta}} + \bar{\Phi}_{A\bar{\beta}}{}^{\bar{\delta}}\bar{C}_{+\bar{\alpha}\bar{\delta}} = 0\,,$$

including the gamma matrices,

$$\mathcal{D}_A(\gamma^p)^\alpha{}_\beta = D_A(\gamma^p)^\alpha{}_\beta = \Phi_A{}^p{}_q(\gamma^q)^\alpha{}_\beta + \Phi_A{}^\alpha{}_\delta(\gamma^p)^\delta{}_\beta - (\gamma^p)^\alpha{}_\delta\Phi_A{}^\delta{}_\beta = 0\,,$$

$$\mathcal{D}_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}} = D_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}} = \bar{\Phi}_A{}^{\bar{p}}{}_{\bar{q}}(\bar{\gamma}^{\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}} + \bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\delta}}(\bar{\gamma}^{\bar{p}})^{\bar{\delta}}{}_{\bar{\beta}} - (\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\delta}}\bar{\Phi}_A{}^{\bar{\delta}}{}_{\bar{\beta}} = 0\,.$$

- It follows then that the connections are all anti-symmetric,

$$\Gamma_{ABC} = -\Gamma_{ACB}\,,$$

$$\Phi_{Apq} = -\Phi_{Aqp}\,, \qquad \Phi_{A\alpha\beta} = -\Phi_{A\beta\alpha}\,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = -\bar{\Phi}_{A\bar{q}\bar{p}}\,, \qquad \bar{\Phi}_{A\bar{\alpha}\bar{\beta}} = -\bar{\Phi}_{A\bar{\beta}\bar{\alpha}}\,,$$

and as usual,

$$\Phi_{A}{}^{\alpha}{}_{\beta} = \tfrac{1}{4}\Phi_{Apq}(\gamma^{pq})^{\alpha}{}_{\beta}\,, \qquad \bar{\Phi}_{A}{}^{\bar{\alpha}}{}_{\bar{\beta}} = \tfrac{1}{4}\bar{\Phi}_{A\bar{p}\bar{q}}(\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}\,.$$

- **Further, the master derivative is compatible with the whole NS-NS sector,**

$$\mathcal{D}_A d = \nabla_A d := -\tfrac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \tfrac{1}{2}\Gamma^B{}_{BA} = 0\,,$$

$$\mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = 0\,,$$

$$\mathcal{D}_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma_{AB}{}^C \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0\,.$$

- It follows that

$$\mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0\,, \qquad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0\,,$$

and the connections are related to each other,

$$\Gamma_{ABC} = V_B{}^p D_A V_{Cp} + \bar{V}_B{}^{\bar{p}} D_A \bar{V}_{C\bar{p}}\,,$$

$$\Phi_{Apq} = V^B{}_p \nabla_A V_{Bq}\,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla_A \bar{V}_{B\bar{q}}\,.$$

- **Further, the master derivative is compatible with the whole NS-NS sector,**

$$\mathcal{D}_A d = \nabla_A d := -\tfrac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \tfrac{1}{2}\Gamma^B{}_{BA} = 0\,,$$

$$\mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = 0\,,$$

$$\mathcal{D}_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma_{AB}{}^C \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0\,.$$

- It follows that

$$\mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0\,, \qquad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0\,,$$

and the connections are related to each other,

$$\Gamma_{ABC} = V_B{}^p D_A V_{Cp} + \bar{V}_B{}^{\bar{p}} D_A \bar{V}_{C\bar{p}}\,,$$

$$\Phi_{Apq} = V^B{}_p \nabla_A V_{Bq}\,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla_A \bar{V}_{B\bar{q}}\,.$$

- **The connections assume the following most general forms:**

$$\Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} ,$$

$$\Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq} ,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} .$$

Here

$$\begin{aligned}\Gamma^0_{CAB} = \quad & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & -\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) ,\end{aligned}$$

Jeon-Lee-JHP 2011

and, with the corresponding derivative, $\nabla^0_A = \partial_A + \Gamma^0_A$,

$$\Phi^0_{Apq} = V^B{}_p \nabla^0_A V_{Bq} = V^B{}_p \partial_A V_{Bq} + \Gamma^0_{ABC} V^B{}_p V^C{}_q ,$$

$$\bar{\Phi}^0_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla^0_A \bar{V}_{B\bar{q}} = \bar{V}^B{}_{\bar{p}} \partial_A \bar{V}_{B\bar{q}} + \Gamma^0_{ABC} \bar{V}^B{}_{\bar{p}} \bar{V}^C{}_{\bar{q}} .$$

- **The connections assume the following most general forms:**

$$\Gamma_{CAB} = \Gamma^{0}_{CAB} + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \,,$$

$$\Phi_{Apq} = \Phi^{0}_{Apq} + \Delta_{Apq} \,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^{0}_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} \,.$$

- The extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the **torsion** of SDFT, which must be covariant and, in order to maintain $\mathcal{D}_A d = 0$, must satisfy

$$\Delta_{Apq} V^{Ap} = 0 \,, \qquad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0 \,.$$

  Otherwise they are arbitrary.

- As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{pq}\psi_A \,, \qquad \bar{\psi}_{\bar{p}}\gamma_A\psi_{\bar{q}} \,, \qquad \bar{\rho}\gamma_{Apq}\rho \,, \qquad \bar{\psi}_{\bar{p}}\gamma_{Apq}\psi^{\bar{p}} \,,$$

  where we set $\psi_A = \bar{V}_A{}^{\bar{p}}\psi_{\bar{p}}$, $\gamma_A = V_A{}^p\gamma_p$.

- **The connections assume the following most general forms:**

$$\Gamma_{CAB} = \Gamma^{0}_{CAB} + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \,,$$

$$\Phi_{Apq} = \Phi^{0}_{Apq} + \Delta_{Apq} \,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^{0}_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} \,.$$

- The extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the **torsion** of SDFT, which must be covariant and, in order to maintain $\mathcal{D}_A d = 0$, must satisfy

$$\Delta_{Apq} V^{Ap} = 0 \,, \qquad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0 \,.$$

  Otherwise they are arbitrary.

- As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{pq}\psi_A \,, \quad \bar{\psi}_{\bar{p}}\gamma_A\psi_{\bar{q}} \,, \quad \bar{\rho}\gamma_{Apq}\rho \,, \quad \bar{\psi}_{\bar{p}}\gamma_{Apq}\psi^{\bar{p}} \,,$$

  where we set $\psi_A = \bar{V}_A{}^{\bar{p}}\psi_{\bar{p}}$, $\gamma_A = V_A{}^p\gamma_p$.

- **The connections assume the following most general forms:**

$$\Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} ,$$

$$\Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq} ,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} .$$

- The extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the **torsion** of SDFT, which must be covariant and, in order to maintain $\mathcal{D}_A d = 0$, must satisfy

$$\Delta_{Apq} V^{Ap} = 0 , \qquad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0 .$$

Otherwise they are arbitrary.

- As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{pq}\psi_A , \qquad \bar{\psi}_{\bar{p}}\gamma_A\psi_{\bar{q}} , \qquad \bar{\rho}\gamma_{Apq}\rho , \qquad \bar{\psi}_{\bar{p}}\gamma_{Apq}\psi^{\bar{p}} ,$$

where we set $\psi_A = \bar{V}_A{}^{\bar{p}}\psi_{\bar{p}}$, $\gamma_A = V_A{}^p\gamma_p$.

- The 'torsionless' connection,

$$\Gamma^0_{CAB} = 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_D P_{EC}$$
$$-\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) ,$$

further obeys

$$\Gamma^0_{ABC} + \Gamma^0_{BCA} + \Gamma^0_{CAB} = 0\,,$$

and

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma^0_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma^0_{DEF} = 0\,.$$

- **In fact, the torsionless connection,**

$$\Gamma^{0}_{CAB} = 2\left(P\partial_{C}P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_{D}P_{EC} - \tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_{D}d + (P\partial^{E}P\bar{P})_{[ED]}\right),$$

is the unique solution to the following constraints:

$$\Gamma_{CAB} + \Gamma_{CBA} = 0 \quad \Longrightarrow \quad \nabla_{A}\mathcal{J}_{BC} = 0\,,$$

$$\nabla_{A}P_{BC} = \nabla_{A}\bar{P}_{BC} = 0\,,$$

$$\nabla_{A}d = 0\,,$$

$$\Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0 \quad \Longrightarrow \quad \hat{\mathcal{L}}(\partial) = \hat{\mathcal{L}}(\nabla)\,,$$

$$(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{DEF}\Gamma_{DEF} = 0\,.$$

- In this way, Γ^{0}_{ABC} is the **DFT analogy of the Christoffel connection.**
However, unlike Christoffel symbol, the DFT-diffeomorphism cannot transform it to vanish point-wise. This can be viewed as the failure of the Equivalence Principle applied to an extended object, *i.e.* string.

Precisely the same expression was re-derived by Hohm-Zwiebach.

- **In fact, the torsionless connection,**

$$\Gamma^{0}_{CAB} = 2\left(P\partial_{C}P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_{D}P_{EC}$$
$$-\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_{D}d + (P\partial^{E}P\bar{P})_{[ED]}\right),$$

is the unique solution to the following constraints:

$$\Gamma_{CAB} + \Gamma_{CBA} = 0 \quad \Longrightarrow \quad \nabla_{A}\mathcal{J}_{BC} = 0\,,$$

$$\nabla_{A}P_{BC} = \nabla_{A}\bar{P}_{BC} = 0\,,$$

$$\nabla_{A}d = 0\,,$$

$$\Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0 \quad \Longrightarrow \quad \hat{\mathcal{L}}(\partial) = \hat{\mathcal{L}}(\nabla)\,,$$

$$(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{DEF}\Gamma_{DEF} = 0\,.$$

- In this way, Γ^{0}_{ABC} is the **DFT analogy of the Christoffel connection.**
  However, unlike Christoffel symbol, the DFT-diffeomorphism cannot transform it to vanish point-wise. This can be viewed as the failure of the Equivalence Principle applied to an extended object, *i.e.* string.

Precisely the same expression was re-derived by Hohm-Zwiebach.

Semi-covariant Riemann curvature

- The usual curvatures for the three connections,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E}\Gamma_{BED} - \Gamma_{BC}{}^{E}\Gamma_{AED}\,,$$

$$F_{ABpq} = \partial_A \Phi_{Bpq} - \partial_B \Phi_{Apq} + \Phi_{Apr}\Phi_{B}{}^{r}{}_{q} - \Phi_{Bpr}\Phi_{A}{}^{r}{}_{q}\,,$$

$$\bar{F}_{AB\bar{p}\bar{q}} = \partial_A \bar{\Phi}_{B\bar{p}\bar{q}} - \partial_B \bar{\Phi}_{A\bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}\bar{r}}\bar{\Phi}_{B}{}^{\bar{r}}{}_{\bar{q}} - \bar{\Phi}_{B\bar{p}\bar{r}}\bar{\Phi}_{A}{}^{\bar{r}}{}_{\bar{q}}\,,$$

are, from $[\mathcal{D}_A, \mathcal{D}_B]V_{Cp} = 0$ and $[\mathcal{D}_A, \mathcal{D}_B]\bar{V}_{C\bar{p}} = 0$, related to each other,

$$R_{ABCD} = F_{CDpq}V_A{}^{p}V_B{}^{q} + \bar{F}_{CD\bar{p}\bar{q}}\bar{V}_A{}^{\bar{p}}\bar{V}_B{}^{\bar{q}}\,.$$

- However, the crucial object in DFT turns out to be

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right),$$

which we name semi-covariant Riemann curvature.

Semi-covariant Riemann curvature

- The usual curvatures for the three connections,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E}\Gamma_{BED} - \Gamma_{BC}{}^{E}\Gamma_{AED}\,,$$

$$F_{ABpq} = \partial_A \Phi_{Bpq} - \partial_B \Phi_{Apq} + \Phi_{Apr}\Phi_{B}{}^{r}{}_{q} - \Phi_{Bpr}\Phi_{A}{}^{r}{}_{q}\,,$$

$$\bar{F}_{AB\bar{p}\bar{q}} = \partial_A \bar{\Phi}_{B\bar{p}\bar{q}} - \partial_B \bar{\Phi}_{A\bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}\bar{r}}\bar{\Phi}_{B}{}^{\bar{r}}{}_{\bar{q}} - \bar{\Phi}_{B\bar{p}\bar{r}}\bar{\Phi}_{A}{}^{\bar{r}}{}_{\bar{q}}\,,$$

  are, from $[\mathcal{D}_A, \mathcal{D}_B]V_{Cp} = 0$ and $[\mathcal{D}_A, \mathcal{D}_B]\bar{V}_{C\bar{p}} = 0$, related to each other,

$$R_{ABCD} = F_{CDpq}V_{A}{}^{p}V_{B}{}^{q} + \bar{F}_{CD\bar{p}\bar{q}}\bar{V}_{A}{}^{\bar{p}}\bar{V}_{B}{}^{\bar{q}}\,.$$

- **However, the crucial object in DFT turns out to be**

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right)\,,$$

  which we name semi-covariant Riemann curvature.

Properties of the semi-covariant curvature

- Under arbitrary variation of the connection, $\delta\Gamma_{ABC}$, it transforms as

$$\delta S^{0}_{ABCD} = \mathcal{D}_{[A}\delta\Gamma^{0}_{B]CD} + \mathcal{D}_{[C}\delta\Gamma^{0}_{D]AB}\,,$$

$$\delta S_{ABCD} = \mathcal{D}_{[A}\delta\Gamma_{B]CD} + \mathcal{D}_{[C}\delta\Gamma_{D]AB} - \tfrac{3}{2}\Gamma_{[ABE]}\delta\Gamma^{E}{}_{CD} - \tfrac{3}{2}\Gamma_{[CDE]}\delta\Gamma^{E}{}_{AB}\,.$$

- It also satisfies precisely the same symmetric property as the ordinary Riemann curvature,

$$S_{ABCD} = \tfrac{1}{2}\left(S_{[AB][CD]} + S_{[CD][AB]}\right)\,,$$

$$S^{0}_{[ABC]D} = 0\,,$$

as well as projection property,

$$S_{p\bar{p}q\bar{q}} = S_{ABCD}V^{A}{}_{p}\bar{V}^{B}{}_{\bar{p}}V^{C}{}_{q}\bar{V}^{D}{}_{\bar{q}} = 0\,.$$

Properties of the semi-covariant curvature

- Under arbitrary variation of the connection, $\delta\Gamma_{ABC}$, it transforms as

$$\delta S^{0}_{ABCD}=\mathcal{D}_{[A}\delta\Gamma^{0}_{B]CD}+\mathcal{D}_{[C}\delta\Gamma^{0}_{D]AB}\,,$$

$$\delta S_{ABCD}=\mathcal{D}_{[A}\delta\Gamma_{B]CD}+\mathcal{D}_{[C}\delta\Gamma_{D]AB}-\tfrac{3}{2}\Gamma_{[ABE]}\delta\Gamma^{E}{}_{CD}-\tfrac{3}{2}\Gamma_{[CDE]}\delta\Gamma^{E}{}_{AB}\,.$$

- It also satisfies precisely the same symmetric property as the ordinary Riemann curvature,

$$S_{ABCD}=\tfrac{1}{2}\left(S_{[AB][CD]}+S_{[CD][AB]}\right)\,,$$

$$S^{0}_{[ABC]D}=0\,,$$

as well as projection property,

$$S_{p\bar{p}q\bar{q}}=S_{ABCD}V^{A}{}_{p}\bar{V}^{B}{}_{\bar{p}}V^{C}{}_{q}\bar{V}^{D}{}_{\bar{q}}=0\,.$$

‘Semi-covariance’

- Generically, under DFT-diffeomorphisms, the variation of the semi-covariant derivative carries anomalous terms which are dictated by the six-index projectors,

$$\delta_X \left(\nabla_C T_{A_1 \cdots A_n}\right) \equiv \hat{\mathcal{L}}_X \left(\nabla_C T_{A_1 \cdots A_n}\right) + \sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE}\partial_F\partial_{[D}X_{E]}\,T_{\cdots B \cdots}\,.$$

- Hence, it is not DFT-diffeomorphism covariant,

$$\delta_X \neq \hat{\mathcal{L}}_X\,.$$

- However, the characteristic property of our ‘semi-covariant’ derivative/curvature is that, the anomaly can be easily projected out, and can thus produce completely covariant derivatives/curvatures.

‘Semi-covariance’

- Generically, under DFT-diffeomorphisms, the variation of the semi-covariant derivative carries anomalous terms which are dictated by the six-index projectors,

$$\delta_X\left(\nabla_C T_{A_1\cdots A_n}\right) \equiv \hat{\mathcal{L}}_X\left(\nabla_C T_{A_1\cdots A_n}\right) + \sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE}\partial_F\partial_{[D}X_{E]}T_{\cdots B\cdots}\,.$$

- Hence, it is not DFT-diffeomorphism covariant,

$$\delta_X \neq \hat{\mathcal{L}}_X\,.$$

- However, the characteristic property of our ‘semi-covariant’ derivative/curvature is that, the anomaly can be easily projected out, and can thus produce completely covariant derivatives/curvatures.

Completely covariant derivatives

- **For O(D, D) tensors:**

$$P_{C}{}^{D}\bar{P}_{A_1}{}^{B_1}\bar{P}_{A_2}{}^{B_2}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_{D}T_{B_1B_2\cdots B_n}\,,\qquad \bar{P}_{C}{}^{D}P_{A_1}{}^{B_1}P_{A_2}{}^{B_2}\cdots P_{A_n}{}^{B_n}\nabla_{D}T_{B_1B_2\cdots B_n}\,,$$

$$\left.\begin{array}{l} P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}T_{BD_1D_2\cdots D_n}\,,\\ \bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_{A}T_{BD_1D_2\cdots D_n}\end{array}\right\}\quad\text{Divergences}\,,$$

$$\left.\begin{array}{l} P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}\nabla_{B}T_{D_1D_2\cdots D_n}\,,\\ \bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_{A}\nabla_{B}T_{D_1D_2\cdots D_n}\end{array}\right\}\quad\text{Laplacians}\,,$$

and

$$\mathfrak{D}_{A}{}^{C}\bar{P}_{B_1}{}^{D_1}\cdots\bar{P}_{B_n}{}^{D_n}T_{CD_1\cdots D_n}\,,\qquad \bar{\mathfrak{D}}_{A}{}^{C}P_{B_1}{}^{D_1}\cdots P_{B_n}{}^{D_n}T_{CD_1\cdots D_n}\,,$$

where we set a pair of semi-covariant second order differential operators,

$$\mathfrak{D}_{A}{}^{B}:=(P_{A}{}^{B}P^{CD}-2P_{A}{}^{D}P^{BC})(\nabla_{C}\nabla_{D}-S_{CD})\,,\qquad \bar{\mathfrak{D}}_{A}{}^{B}:=(\bar{P}_{A}{}^{B}\bar{P}^{CD}-2\bar{P}_{A}{}^{D}\bar{P}^{BC})(\nabla_{C}\nabla_{D}-S_{CD})\,,$$

which are relevant to the DFT fluctuation analysis Ko-Meyer-Melby-Thompson-JHP

Completely covariant derivatives

- **For local Lorentz tensors, Spin$(1,D{-}1)_L \times$ Spin$(D{-}1,1)_R$:**

$$\mathcal{D}_p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}_{\bar{p}} T_{q_1q_2\cdots q_n}\,,$$

$$\mathcal{D}^p T_{p\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p}q_1q_2\cdots q_n}\,,$$

$$\mathcal{D}_p\mathcal{D}^p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}_{\bar{p}}\mathcal{D}^{\bar{p}} T_{q_1q_2\cdots q_n}\,,$$

$$\mathfrak{D}_p{}^q T_{q\bar{p}_1\bar{p}_2\cdots\bar{p}_n}\,, \qquad \bar{\mathfrak{D}}_{\bar{p}}{}^{\bar{q}} T_{\bar{q}p_1p_2\cdots p_n}\,.$$

These are the 'pull-back' of the previous page using the DFT-vielbeins, such as

$$\mathcal{D}_p := V^A{}_p\mathcal{D}_A\,, \qquad \mathcal{D}_{\bar{p}} := \bar{V}^A{}_{\bar{p}}\mathcal{D}_A\,.$$

Completely covariant derivatives

- Following the aforementioned general prescription, **completely covariant Yang-Mills field strength** is given by two opposite projections, or

$$\mathcal{F}_{p\bar{q}} = V^{M}{}_{p}\bar{V}^{N}{}_{\bar{q}}\mathcal{F}_{MN}\,,$$

where $\mathcal{F}_{MN}$ is the semi-covariant field strength of a YM potential, $\mathcal{V}_M$,

$$\mathcal{F}_{MN} := \nabla_M\mathcal{V}_N - \nabla_N\mathcal{V}_M - i\,[\mathcal{V}_M,\mathcal{V}_N]\,.$$

Unlike the Riemannian case, the Γ-connections are not canceled out.

- Further, we may freely impose "gauged" section condition to halve the off-shell degrees:

$$(\partial_M - i\,\mathcal{V}_M)(\partial^M - i\,\mathcal{V}^M) = 0\,,$$

which implies $\mathcal{V}^M\partial_M = 0, \partial_M\mathcal{V}^M = 0, \mathcal{V}_M\mathcal{V}^M = 0$, like the coordinate gauge potential.

For consistency, the above condition is preserved under all the symmetry transformations: $\mathbf{O}(D,D)$ rotations, diffeomorphisms, and the Yang-Mills gauge symmetry,

$$[\mathbf{g}\mathcal{V}^M\mathbf{g}^{-1} - i(\partial^M\mathbf{g})\mathbf{g}^{-1}]\partial_M = 0\,,\quad (\hat{\mathcal{L}}_X\mathcal{V}^M)\partial_M = [X^N\partial_N\mathcal{V}^M + (\partial^M X_N - \partial_N X^M)\mathcal{V}^N]\partial_M = 0\,.$$

Completely covariant derivatives

- Following the aforementioned general prescription, **completely covariant Yang-Mills field strength** is given by two opposite projections, or

$$\mathcal{F}_{p\bar{q}} = V^{M}{}_{p}\bar{V}^{N}{}_{\bar{q}}\mathcal{F}_{MN}\,,$$

where $\mathcal{F}_{MN}$ is the semi-covariant field strength of a YM potential, $\mathcal{V}_M$,

$$\mathcal{F}_{MN} := \nabla_M\mathcal{V}_N - \nabla_N\mathcal{V}_M - i\,[\mathcal{V}_M,\mathcal{V}_N]\,.$$

Unlike the Riemannian case, the Γ-connections are not canceled out.

- Further, we may freely impose "gauged" section condition to halve the off-shell degrees:

$$(\partial_M - i\,\mathcal{V}_M)(\partial^M - i\,\mathcal{V}^M) = 0\,,$$

which implies $\mathcal{V}^M\partial_M = 0, \partial_M\mathcal{V}^M = 0, \mathcal{V}_M\mathcal{V}^M = 0$, like the coordinate gauge potential.

For consistency, the above condition is preserved under all the symmetry transformations: $\mathbf{O}(D,D)$ rotations, diffeomorphisms, and the Yang-Mills gauge symmetry,

$$[\mathbf{g}\mathcal{V}^M\mathbf{g}^{-1} - i(\partial^M\mathbf{g})\mathbf{g}^{-1}]\partial_M = 0\,,\quad (\hat{\mathcal{L}}_X\mathcal{V}^M)\partial_M = [X^N\partial_N\mathcal{V}^M + (\partial^M X_N - \partial_N X^M)\mathcal{V}^N]\partial_M = 0\,.$$

Completely covariant derivatives

- $\mathbf{O}(D,D)$ covariant **Killing equations** in DFT:

$$\hat{\mathcal{L}}_X \mathcal{H}_{MN} = 0 \qquad \Longleftrightarrow \qquad (P\nabla_M)(\bar{P}X)_N - (\bar{P}\nabla_N)(PX)_M = 0\,,$$

$$\hat{\mathcal{L}}_X d = 0 \qquad \Longleftrightarrow \qquad \nabla_M X^M = 0\,.$$

JHP-Rey-Rim-Sakatani

c.f. Chris Blair

Completely covariant derivatives

- **Dirac operators for fermions,** $\rho^{\alpha},\ \psi^{\alpha}_{\bar{p}},\ \rho'^{\bar{\alpha}},\ \psi'^{\bar{\alpha}}_{p}$:

$$\gamma^{p}\mathcal{D}_{p}\rho=\gamma^{A}\mathcal{D}_{A}\rho\,,\qquad\qquad \gamma^{p}\mathcal{D}_{p}\psi_{\bar{p}}=\gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}}\,,$$

$$\mathcal{D}_{\bar{p}}\rho\,,\qquad\qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}}=\mathcal{D}_{A}\psi^{A}\,,$$

$$\bar{\psi}^{A}\gamma_{p}(\mathcal{D}_{A}\psi_{\bar{q}}-\tfrac{1}{2}\mathcal{D}_{\bar{q}}\psi_{A})\,,$$

$$\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'=\bar{\gamma}^{A}\mathcal{D}_{A}\rho'\,,\qquad\qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'_{p}=\bar{\gamma}^{A}\mathcal{D}_{A}\psi'_{p}\,,$$

$$\mathcal{D}_{p}\rho'\,,\qquad\qquad \mathcal{D}_{p}\psi'^{p}=\mathcal{D}_{A}\psi'^{A}\,,$$

$$\bar{\psi}'^{A}\bar{\gamma}_{\bar{p}}(\mathcal{D}_{A}\psi'_{q}-\tfrac{1}{2}\mathcal{D}_{q}\psi'_{A})\,.$$

Incorporation of fermions into DFT   1109.2035

Completely covariant derivatives

- **For R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\beta}}$:**

$$\mathcal{D}_{+}\mathcal{C} := \gamma^{A}\mathcal{D}_{A}\mathcal{C} + \gamma^{(D+1)}\mathcal{D}_{A}\mathcal{C}\bar{\gamma}^{A}\,,$$

$$\mathcal{D}_{-}\mathcal{C} := \gamma^{A}\mathcal{D}_{A}\mathcal{C} - \gamma^{(D+1)}\mathcal{D}_{A}\mathcal{C}\bar{\gamma}^{A}\,.$$

- Especially for the torsionless case, the corresponding operators are **nilpotent,**

$$(\mathcal{D}^{0}_{+})^{2}\mathcal{C} = 0\,, \qquad\qquad (\mathcal{D}^{0}_{-})^{2}\mathcal{C} = 0\,,$$

and hence, they define **O(D,D) covariant cohomology.**

- **The field strength of the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$, is then defined by**

$$\mathcal{F} := \mathcal{D}^{0}_{+}\mathcal{C}\,.$$

- Thanks to the nilpotency, the **R-R gauge symmetry** is simply realized

$$\delta\mathcal{C} = \mathcal{D}^{0}_{+}\Delta \qquad \Longrightarrow \qquad \delta\mathcal{F} = \mathcal{D}^{0}_{+}(\delta\mathcal{C}) = (\mathcal{D}^{0}_{+})^{2}\Delta = 0\,.$$

Completely covariant derivatives

- **For R-R potential,** $\mathcal{C}^{\alpha}{}_{\bar{\beta}}$**:**

$$\mathcal{D}_{+}\mathcal{C} := \gamma^{A}\mathcal{D}_{A}\mathcal{C} + \gamma^{(D+1)}\mathcal{D}_{A}\mathcal{C}\bar{\gamma}^{A}\,,$$

$$\mathcal{D}_{-}\mathcal{C} := \gamma^{A}\mathcal{D}_{A}\mathcal{C} - \gamma^{(D+1)}\mathcal{D}_{A}\mathcal{C}\bar{\gamma}^{A}\,.$$

- Especially for the torsionless case, the corresponding operators are **nilpotent,**

$$(\mathcal{D}^{0}_{+})^{2}\mathcal{C} = 0\,, \qquad\qquad (\mathcal{D}^{0}_{-})^{2}\mathcal{C} = 0\,,$$

and hence, they define **O**(D, D) **covariant cohomology.**

- **The field strength of the R-R potential,** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$**, is then defined by**

$$\mathcal{F} := \mathcal{D}^{0}_{+}\mathcal{C}\,.$$

- Thanks to the nilpotency, the **R-R gauge symmetry** is simply realized

$$\delta\mathcal{C} = \mathcal{D}^{0}_{+}\Delta \qquad \Longrightarrow \qquad \delta\mathcal{F} = \mathcal{D}^{0}_{+}(\delta\mathcal{C}) = (\mathcal{D}^{0}_{+})^{2}\Delta = 0\,.$$

Completely covariant curvatures

- **Scalar curvature:**

$$\mathcal{S} := (P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}$$

c.f. $S_{AB}{}^{AB} = 0\,.$

- **"Ricci" curvature:**

$$S^{0}_{p\bar{q}} = V^{A}{}_{p}\bar{V}^{B}{}_{\bar{q}}S^{0}_{AB}$$

where we set $S_{AB} = S_{ACB}{}^{C}\,.$

Jeon-Lee-JHP 2011

- **Further, we have conserved "Einstein" curvature,**

$$G^{AB} = 2(P^{AC}\bar{P}^{BD} - \bar{P}^{AC}P^{BD})S_{CD} - \tfrac{1}{2}\mathcal{J}^{AB}\mathcal{S}\,, \qquad \nabla_{A}G^{AB} = 0\,.$$

JHP-Rey-Rim-Sakatani 2015

Combining all the results above, we are now ready to spell

- $D=10$ **Maximally Supersymmetric Double Field Theory**

- **Lagrangian**:

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho' + i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\,.$$

where $\bar{\mathcal{F}}^{\bar{\alpha}}{}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_{+}^{-1}\mathcal{F}^{T}C_{+}$.

- As they are contracted with the DFT-vielbeins properly,
  every term in the Lagrangian is completey covariant.

c.f. Democratic SUGRA *à la* Bergshoeff-Kallosh-Ortin-Roest-Van Proeyen
& Generalized Geometry *à la* Coimbra-Strickland-Constable-Waldram

- **Lagrangian**:

$$\mathcal{L}_{\text{Type II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho' + i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\,.$$

where $\bar{\mathcal{F}}^{\bar{\alpha}}{}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_{+}^{-1}\mathcal{F}^{T}C_{+}$.

- As they are contracted with the DFT-vielbeins properly,
every term in the Lagrangian is completey covariant.

c.f. Democratic SUGRA *à la* Bergshoeff-Kallosh-Ortin-Roest-Van Proeyen
& Generalized Geometry *à la* Coimbra-Strickland-Constable-Waldram

- **Lagrangian**:

$$\mathcal{L}_{\mathrm{Type\,II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_q\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho' + i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big].$$

- **Torsions:** The semi-covariant curvature, S_{ABCD}, is given by the connection,

$$\Gamma_{ABC} = \Gamma^{0}_{ABC} + i\tfrac{1}{3}\bar{\rho}\gamma_{ABC}\rho - 2i\bar{\rho}\gamma_{BC}\psi_{A} - i\tfrac{1}{3}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} + 4i\bar{\psi}_{B}\gamma_{A}\psi_{C}$$
$$+ i\tfrac{1}{3}\bar{\rho}'\bar{\gamma}_{ABC}\rho' - 2i\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A} - i\tfrac{1}{3}\bar{\psi}'^{p}\bar{\gamma}_{ABC}\psi'_{p} + 4i\bar{\psi}'_{B}\bar{\gamma}_{A}\psi'_{C},$$

which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold:
$\mathcal{D}^{\star}_{A}$ for the unprimed fermions and $\mathcal{D}'^{\star}_{A}$ for the primed fermions, set by

$$\Gamma^{\star}_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{4}\bar{\rho}\gamma_{BC}\psi_{A} + i\tfrac{5}{24}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 2i\bar{\psi}_{B}\gamma_{A}\psi_{C} + i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A},$$

$$\Gamma'^{\star}_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}'\bar{\gamma}_{ABC}\rho' + i\tfrac{5}{4}\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A} + i\tfrac{5}{24}\bar{\psi}'^{p}\bar{\gamma}_{ABC}\psi'_{p} - 2i\bar{\psi}'_{B}\bar{\gamma}_{A}\psi'_{C} + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_{A}.$$

- **Lagrangian**:

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho' + i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big] .$$

- **Torsions:** The semi-covariant curvature, S_{ABCD}, is given by the connection,

$$\Gamma_{ABC} = \Gamma^{0}_{ABC} + i\tfrac{1}{3}\bar{\rho}\gamma_{ABC}\rho - 2i\bar{\rho}\gamma_{BC}\psi_{A} - i\tfrac{1}{3}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} + 4i\bar{\psi}_{B}\gamma_{A}\psi_{C}$$
$$+ i\tfrac{1}{3}\bar{\rho}'\bar{\gamma}_{ABC}\rho' - 2i\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A} - i\tfrac{1}{3}\bar{\psi}'^{p}\bar{\gamma}_{ABC}\psi'_{p} + 4i\bar{\psi}'_{B}\bar{\gamma}_{A}\psi'_{C} ,$$

which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold:
$\mathcal{D}^{\star}_{A}$ for the unprimed fermions and $\mathcal{D}'^{\star}_{A}$ for the primed fermions, set by

$$\Gamma^{\star}_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{4}\bar{\rho}\gamma_{BC}\psi_{A} + i\tfrac{5}{24}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 2i\bar{\psi}_{B}\gamma_{A}\psi_{C} + i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A} ,$$
$$\Gamma'^{\star}_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}'\bar{\gamma}_{ABC}\rho' + i\tfrac{5}{4}\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A} + i\tfrac{5}{24}\bar{\psi}'^{p}\bar{\gamma}_{ABC}\psi'_{p} - 2i\bar{\psi}'_{B}\bar{\gamma}_{A}\psi'_{C} + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_{A} .$$

- **Maximal supersymmetry transformation rules are also completely covariant,**

$$\delta_\varepsilon d = -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,,$$

$$\delta_\varepsilon V_{Ap} = i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,,$$

$$\delta_\varepsilon \bar{V}_{A\bar{p}} = iV_A{}^{q}(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,$$

$$\delta_\varepsilon \mathcal{C} = i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,$$

$$\delta_\varepsilon \rho = -\gamma^p\hat{\mathcal{D}}_p\varepsilon + i\tfrac{1}{2}\gamma^p\varepsilon\,\bar{\psi}'_p\rho' - i\gamma^p\psi^{\bar{q}}\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p\,,$$

$$\delta_\varepsilon \rho' = -\bar{\gamma}^{\bar{p}}\hat{\mathcal{D}}'_{\bar{p}}\varepsilon' + i\tfrac{1}{2}\bar{\gamma}^{\bar{p}}\varepsilon'\,\bar{\psi}_{\bar{p}}\rho - i\bar{\gamma}^{\bar{q}}\psi'_p\bar{\varepsilon}\gamma^p\psi_{\bar{q}}\,,$$

$$\delta_\varepsilon \psi_{\bar{p}} = \hat{\mathcal{D}}_{\bar{p}}\varepsilon + (\mathcal{F} - i\tfrac{1}{2}\gamma^q\rho\,\bar{\psi}'_q + i\tfrac{1}{2}\psi^{\bar{q}}\,\bar{\rho}'\bar{\gamma}_{\bar{q}})\bar{\gamma}_{\bar{p}}\varepsilon' + i\tfrac{1}{4}\varepsilon\bar{\psi}_{\bar{p}}\rho + i\tfrac{1}{2}\psi_{\bar{p}}\bar{\varepsilon}\rho\,,$$

$$\delta_\varepsilon \psi'_p = \hat{\mathcal{D}}'_p\varepsilon' + (\bar{\mathcal{F}} - i\tfrac{1}{2}\bar{\gamma}^{\bar{q}}\rho'\bar{\psi}_{\bar{q}} + i\tfrac{1}{2}\psi'^q\bar{\rho}\gamma_q)\gamma_p\varepsilon + i\tfrac{1}{4}\varepsilon'\bar{\psi}'_p\rho' + i\tfrac{1}{2}\psi'_p\bar{\varepsilon}'\rho'\,,$$

where

$$\hat{\Gamma}_{ABC} = \Gamma_{ABC} - i\tfrac{17}{48}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_A + i\tfrac{1}{4}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 3i\bar{\psi}'_B\bar{\gamma}_A\psi'_C\,,$$

$$\hat{\Gamma}'_{ABC} = \Gamma_{ABC} - i\tfrac{17}{48}\bar{\rho}'\bar{\gamma}_{ABC}\rho' + i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + i\tfrac{1}{4}\bar{\psi}'^p\bar{\gamma}_{ABC}\psi'_p - 3i\bar{\psi}_B\gamma_A\psi_C\,.$$

$D = 10$ Maximal SDFT [1210.5078]

- **Lagrangian**:

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho' + i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\,.$$

- The Lagrangian is **pseudo:** It is necessary to impose a **self-duality** of the R-R field strength by hand,

$$\tilde{\mathcal{F}}_{-} := \left(1 - \gamma^{(D+1)}\right)\left(\mathcal{F} - i\tfrac{1}{2}\rho\bar{\rho}' + i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}'_{p}\bar{\gamma}^{\bar{q}}\right) \equiv 0\,.$$

- **Lagrangian**:

$$\mathcal{L}_{\mathrm{Type\,II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho' + i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\,.$$

- The Lagrangian is **pseudo:** It is necessary to impose a **self-duality** of the R-R field strength by hand,

$$\tilde{\mathcal{F}}_{-} := \left(1 - \gamma^{(D+1)}\right)\left(\mathcal{F} - i\tfrac{1}{2}\rho\bar{\rho}' + i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}'_{p}\bar{\gamma}^{\bar{q}}\right) \equiv 0\,.$$

$D = 10$ Maximal SDFT [1210.5078]

- Under the $\mathcal{N} = 2$ SUSY transformation rule, the Lagrangian transforms as

$$\delta_\varepsilon \mathcal{L}_{\text{Type II}} = -\tfrac{1}{8} e^{-2d} \bar{V}^A{}_{\bar{q}} \delta_\varepsilon V_{Ap} \mathrm{Tr}\left(\gamma^p \tilde{\mathcal{F}}_- \bar{\gamma}^{\bar{q}} \overline{\tilde{\mathcal{F}}_-}\right) \ + \ \text{total derivatives}\,,$$

where the precise self-duality relation appears quadratically,

$$\tilde{\mathcal{F}}_- := \left(1 - \gamma^{(D+1)}\right)\left(\mathcal{F} - i\tfrac{1}{2}\rho\bar{\rho}' + i\tfrac{1}{2}\gamma^p \psi_{\bar{q}} \bar{\psi}'_p \bar{\gamma}^{\bar{q}}\right)\,.$$

This verifies, to the full order in fermions, **the supersymmetric invariance of the action, modulo the self-duality.**

- For a nontrivial consistency check, the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

$$\delta_\varepsilon \tilde{\mathcal{F}}_- = -i\left(\tilde{\mathcal{D}}_{\bar{p}}\rho + \gamma^p \tilde{\mathcal{D}}_p \psi_{\bar{p}} - \gamma^p \mathcal{F} \bar{\gamma}_{\bar{p}} \psi'{}_p\right) \bar{\varepsilon}' \bar{\gamma}^{\bar{p}} - i\gamma^p \varepsilon \left(\tilde{\mathcal{D}}'_p \bar{\rho}' + \tilde{\mathcal{D}}'_{\bar{p}} \bar{\psi}'_p \bar{\gamma}^{\bar{p}} - \bar{\psi}_{\bar{p}} \gamma_p \mathcal{F} \bar{\gamma}^{\bar{p}}\right)\,.$$

$D = 10$ Maximal SDFT [1210.5078]

- Under the $\mathcal{N} = 2$ SUSY transformation rule, the Lagrangian transforms as

$$\delta_\varepsilon \mathcal{L}_{\rm Type\,II} = -\tfrac{1}{8} e^{-2d}\, \bar{V}^A{}_{\bar{q}} \delta_\varepsilon V_{Ap} \mathrm{Tr}\left(\gamma^p \tilde{\mathcal{F}}_- \bar{\gamma}^{\bar{q}} \overline{\tilde{\mathcal{F}}_-}\right) \;+\; \text{total derivatives}\,,$$

where the precise self-duality relation appears quadratically,

$$\tilde{\mathcal{F}}_- := \left(1 - \gamma^{(D+1)}\right)\left(\mathcal{F} - i\tfrac{1}{2}\rho\bar{\rho}' + i\tfrac{1}{2}\gamma^p \psi_{\bar{q}} \bar{\psi}'_p \bar{\gamma}^{\bar{q}}\right) .$$

This verifies, to the full order in fermions, **the supersymmetric invariance of the action, modulo the self-duality.**

- For a nontrivial consistency check, the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

$$\delta_\varepsilon \tilde{\mathcal{F}}_- = -i\left(\tilde{\mathcal{D}}_{\bar{p}}\rho + \gamma^p \tilde{\mathcal{D}}_p \psi_{\bar{p}} - \gamma^p \mathcal{F} \bar{\gamma}_{\bar{p}} \psi'{}_p\right) \bar{\varepsilon}' \bar{\gamma}^{\bar{p}} - i\gamma^p \varepsilon \left(\tilde{\mathcal{D}}'_p \bar{\rho}' + \tilde{\mathcal{D}}'_{\bar{p}} \bar{\psi}'_p \bar{\gamma}^{\bar{p}} - \bar{\psi}_{\bar{p}} \gamma_p \mathcal{F} \bar{\gamma}^{\bar{p}}\right) .$$

Equations of Motion

- DFT-vielbein:

$$S_{p\bar{q}}+\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})+i\bar{\rho}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\rho+2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho-i\bar{\psi}^{\bar{p}}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}}+i\bar{\rho}'\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\rho'+2i\bar{\psi}'_p\tilde{\mathcal{D}}_{\bar{q}}\rho'-i\bar{\psi}'^q\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\psi'_q=0.$$

  This is **DFT-generalization of Einstein equation.**

- DFT-dilaton:

$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

  Namely, **the on-shell Lagrangian vanishes!**

- R-R potential:

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=0\,,$$

  which is automatically met by the self-duality, together with the nilpotency of $\mathcal{D}^0_+$,

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=\mathcal{D}^0_-\left(\gamma^{(D+1)}\mathcal{F}\right)=-\gamma^{(D+1)}\mathcal{D}^0_+\mathcal{F}=-\gamma^{(D+1)}(\mathcal{D}^0_+)^2\mathcal{C}=0\,.$$

- The 1.5 formalism works: The variation of the Lagrangian induced by that of the connection is trivial, $\delta\mathcal{L}_{\mathrm{Type\,II}}=\delta\Gamma_{ABC}\times 0\,.$

Equations of Motion

- DFT-vielbein:

$$S_{p\bar{q}}+\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})+i\bar{\rho}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\rho+2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho-i\bar{\psi}^{\bar{p}}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}}+i\bar{\rho}'\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\rho'+2i\bar{\psi}'{}_p\tilde{\mathcal{D}}_{\bar{q}}\rho'-i\bar{\psi}'^{q}\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\psi'_q=0.$$

  This is **DFT-generalization of Einstein equation.**

- DFT-dilaton:

$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

  Namely, **the on-shell Lagrangian vanishes!**

- R-R potential:

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=0\,,$$

  which is automatically met by the self-duality, together with the nilpotency of $\mathcal{D}^0_+$,

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=\mathcal{D}^0_-\left(\gamma^{(D+1)}\mathcal{F}\right)=-\gamma^{(D+1)}\mathcal{D}^0_+\mathcal{F}=-\gamma^{(D+1)}(\mathcal{D}^0_+)^2\mathcal{C}=0\,.$$

- The 1.5 formalism works: The variation of the Lagrangian induced by that of the connection is trivial, $\delta\mathcal{L}_{\mathrm{Type\,II}}=\delta\Gamma_{ABC}\times 0\,.$

Equations of Motion

- DFT-vielbein:

$$S_{p\bar{q}}+\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})+i\bar{\rho}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\rho+2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho-i\bar{\psi}^{\bar{p}}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}}+i\bar{\rho}'\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\rho'+2i\bar{\psi}'{}_p\tilde{\mathcal{D}}_{\bar{q}}\rho'-i\bar{\psi}'^{q}\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\psi'_q=0.$$

This is **DFT-generalization of Einstein equation.**

- DFT-dilaton:

$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

Namely, **the on-shell Lagrangian vanishes!**

- R-R potential:

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=0\,,$$

which is automatically met by the self-duality, together with the nilpotency of $\mathcal{D}^0_+$,

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=\mathcal{D}^0_-\left(\gamma^{(D+1)}\mathcal{F}\right)=-\gamma^{(D+1)}\mathcal{D}^0_+\mathcal{F}=-\gamma^{(D+1)}(\mathcal{D}^0_+)^2\mathcal{C}=0\,.$$

- The 1.5 formalism works: The variation of the Lagrangian induced by that of the connection is trivial, $\delta\mathcal{L}_{\mathrm{Type\,II}}=\delta\Gamma_{ABC}\times 0\,.$

Equations of Motion

- DFT-vielbein:

$$S_{p\bar{q}}+\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})+i\bar{\rho}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\rho+2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho-i\bar{\psi}^{\bar{p}}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}}+i\bar{\rho}'\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\rho'+2i\bar{\psi}'{}_p\tilde{\mathcal{D}}_{\bar{q}}\rho'-i\bar{\psi}'^q\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\psi'_q=0.$$

  This is **DFT-generalization of Einstein equation.**

- DFT-dilaton:

$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

  Namely, **the on-shell Lagrangian vanishes!**

- R-R potential:

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=0\,,$$

  which is automatically met by the self-duality, together with the nilpotency of $\mathcal{D}^0_+$,

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=\mathcal{D}^0_-\left(\gamma^{(D+1)}\mathcal{F}\right)=-\gamma^{(D+1)}\mathcal{D}^0_+\mathcal{F}=-\gamma^{(D+1)}(\mathcal{D}^0_+)^2\mathcal{C}=0\,.$$

- The 1.5 formalism works: The variation of the Lagrangian induced by that of the connection is trivial, $\delta\mathcal{L}_{\mathrm{Type\,II}}=\delta\Gamma_{ABC}\times 0\,.$

Equations of Motion

- DFT-vielbein:

$$S_{p\bar{q}}+\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})+i\bar{\rho}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\rho+2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho-i\bar{\psi}^{\bar{p}}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}}+i\bar{\rho}'\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\rho'+2i\bar{\psi}'{}_p\tilde{\mathcal{D}}_{\bar{q}}\rho'-i\bar{\psi}'^{q}\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\psi'_q=0.$$

  This is **DFT-generalization of Einstein equation.**

- DFT-dilaton:

$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

  Namely, **the on-shell Lagrangian vanishes!**

- R-R potential:

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=0\,,$$

  which is automatically met by the self-duality, together with the nilpotency of $\mathcal{D}^0_+$,

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=\mathcal{D}^0_-\left(\gamma^{(D+1)}\mathcal{F}\right)=-\gamma^{(D+1)}\mathcal{D}^0_+\mathcal{F}=-\gamma^{(D+1)}(\mathcal{D}^0_+)^2\mathcal{C}=0\,.$$

- The 1.5 formalism works: The variation of the Lagrangian induced by that of the connection is trivial, $\delta\mathcal{L}_{\mathrm{Type\,II}}=\delta\Gamma_{ABC}\times 0\,.$

Twofold spin and Standard Model

1506.05277

- In principle, fermions live on a locally inertial frame.
- Local Lorentz symmetry means the arbitrariness of the locally inertial frame at each spacetime point.
- SDFT manifests twofold local Lorentz symmetries: $\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R$, and as a consequence it unifies type IIA and IIB supergravities.
- Left and right string modes perceive or live on two different locally inertial frames. Duff
- **SDFT predicts the fermions in Standard Model are twofold: $\mathbf{Spin}(1,3)_L \times \mathbf{Spin}(3,1)_R$.**
  (Even after Scherk-Schwarz compactification, the spin group remains still twofold.)
- Employing the completely covariant DFT-geometry described above, we can couple Standard Model to stringy backgrounds in a covariant way: **It is possible to Double Field Theorize the Standard Model, without introducing any extra physical degrees.**
- Doing so, one has to decide the spin group for each fermion (Yukawa coupling). No experimental evidence of proton decay seems to indicate that **the quarks and the leptons may belong to different spin groups.**
- If so, this constrains the possible higher order corrections to SM.
- Further, DFT forbids the theta term and hence solves the strong CP problem.

- In principle, fermions live on a locally inertial frame.
- Local Lorentz symmetry means the arbitrariness of the locally inertial frame at each spacetime point.
- SDFT manifests twofold local Lorentz symmetries: $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$, and as a consequence it unifies type IIA and IIB supergravities.
- Left and right string modes perceive or live on two different locally inertial frames. Duff
- **SDFT predicts the fermions in Standard Model are twofold: $\mathbf{Spin}(1,3)_L \times \mathbf{Spin}(3,1)_R$.** (Even after Scherk-Schwarz compactification, the spin group remains still twofold.)
- Employing the completely covariant DFT-geometry described above, we can couple Standard Model to stringy backgrounds in a covariant way: **It is possible to Double Field Theorize the Standard Model, without introducing any extra physical degrees.**
- Doing so, one has to decide the spin group for each fermion (Yukawa coupling). No experimental evidence of proton decay seems to indicate that **the quarks and the leptons may belong to different spin groups.**
- If so, this constrains the possible higher order corrections to SM.
- Further, DFT forbids the theta term and hence solves the strong CP problem.

- In principle, fermions live on a locally inertial frame.
- Local Lorentz symmetry means the arbitrariness of the locally inertial frame at each spacetime point.
- SDFT manifests twofold local Lorentz symmetries: $\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R$, and as a consequence it unifies type IIA and IIB supergravities.
- Left and right string modes perceive or live on two different locally inertial frames. Duff
- **SDFT predicts the fermions in Standard Model are twofold: $\mathbf{Spin}(1,3)_L \times \mathbf{Spin}(3,1)_R$.**
  (Even after Scherk-Schwarz compactification, the spin group remains still twofold.)
- Employing the completely covariant DFT-geometry described above, we can couple Standard Model to stringy backgrounds in a covariant way: **It is possible to Double Field Theorize the Standard Model, without introducing any extra physical degrees.**
- Doing so, one has to decide the spin group for each fermion (Yukawa coupling). No experimental evidence of proton decay seems to indicate that **the quarks and the leptons may belong to different spin groups.**
- If so, this constrains the possible higher order corrections to SM.
- Further, DFT forbids the theta term and hence solves the strong CP problem.

- In principle, fermions live on a locally inertial frame.
- Local Lorentz symmetry means the arbitrariness of the locally inertial frame at each spacetime point.
- SDFT manifests twofold local Lorentz symmetries: $\mathbf{Spin}(1, D{-}1)_L \times \mathbf{Spin}(D{-}1, 1)_R$, and as a consequence it unifies type IIA and IIB supergravities.
- Left and right string modes perceive or live on two different locally inertial frames. Duff
- **SDFT predicts the fermions in Standard Model are twofold: $\mathbf{Spin}(1,3)_L \times \mathbf{Spin}(3,1)_R$.**
  (Even after Scherk-Schwarz compactification, the spin group remains still twofold.)
- Employing the completely covariant DFT-geometry described above, we can couple Standard Model to stringy backgrounds in a covariant way: **It is possible to Double Field Theorize the Standard Model, without introducing any extra physical degrees.**
- Doing so, one has to decide the spin group for each fermion (Yukawa coupling).
  No experimental evidence of proton decay seems to indicate that
  the quarks and the leptons may belong to different spin groups.
- If so, this constrains the possible higher order corrections to SM.
- Further, DFT forbids the theta term and hence solves the strong CP problem.

Outlook : things to do

Outlook : things to do

Revist and Double Field Theorize 20th centrury physics covariantly,

including string theory itself.