Weakly Constrained Double Field Theory

(Tomography in String Theory)

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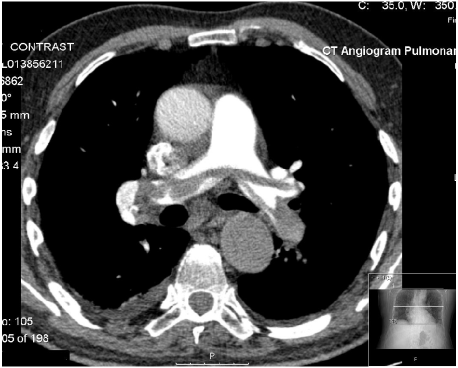
Duality Symmetries in String and M-Theories

- Tomography refers to imaging by sections or sectioning, through the use of any kind of penetrating wave. (Wikipedia)



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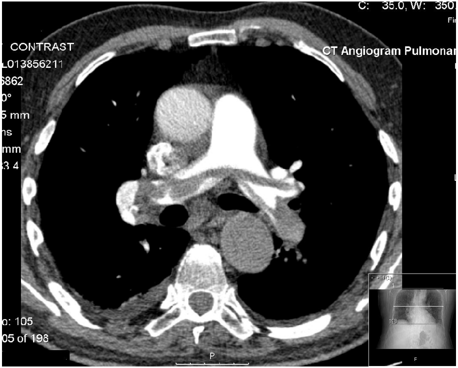

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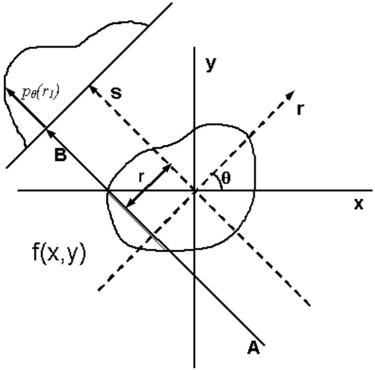
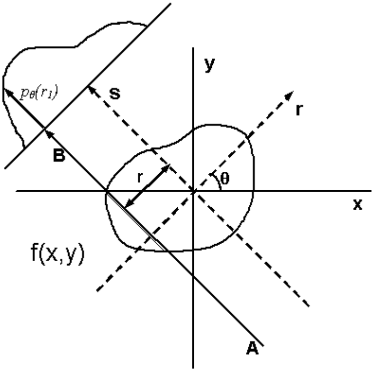


- If we denote a measured intensity as $I(r,\theta)$ and the input intensity as I_0, then they are related by

$$I = I_0 \exp(\int \mu(x,y)\mathrm{d}s)$$

- Then the logarithm of the intensity ratio $P_\theta(r)$ yields a *projction* or *profile*

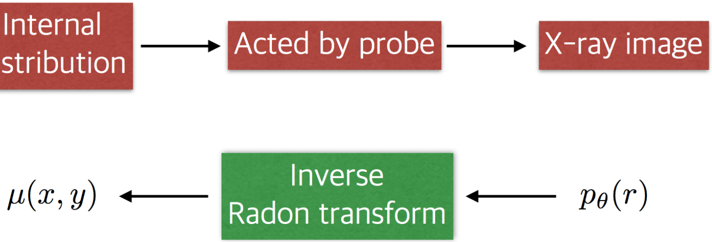
$$P_\theta(r) = -\log(I/I_0) = \int \mu(x,y)\mathrm{d}s\,.$$



- How we can represent the attenuation coefficient μ in terms of $P_\theta(r)$?
- Inverse Radon (X-ray) transform

$$\mu(x, y) = \int_0^{2\pi} P_\theta(r) \mathrm{d}\theta$$

- Relations



- Radon (X-ray) transform can be generalized to $\mathbb{R}^n$ as well as any dimensional tori T^n.
- Inverse Radon transform is closely related to the Penrose transform.

A massless field = $\sum$ fields on all possible light-cones

Outline

- Radon (X-ray) transform on a torus
- Binary operation for weakly constrained fields
- Weakly constrained DFT
- Conclusion

Radon (X-ray) transform on a torus

- Consider a doubled torus T^{2d} with periodic coordinates X^I

$$X^I \sim X^I + 1\,, \qquad I = 1, 2, \cdots, 2d$$

$$X^I = \begin{pmatrix} x^i \\ \tilde{x}_i \end{pmatrix}\,, \qquad i = 1, 2 \cdots, d$$

- $I, J, \cdots$ are $\mathbf{O}(d, d)$ vector indices with a $\mathbf{O}(d, d)$ metric

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- A closed d-dimensional plane $\mathcal{D}(X^I, \Pi)$ on a T^{2d} passing through a point $X^I \in T^{2d}$ is parametrized as

$$\mathcal{D}(X^I, \Pi) = \{X^I + t_i \Pi^{iI} | 0 \leq t_i < 1 \text{ and } \Pi \in \mathcal{P}_d\}$$

$\mathcal{P}_d$ is a set of $d \times 2d$ integer matrices of rank d, whose Smith normal form is

$$\Pi = L D_0 V$$

where $L \in PSL(d, \mathbb{Z})$, $V \in PSL(2d, \mathbb{Z})$ and $D_0 = (\mathbb{1}_d \; 0_d)$

- Since the closed d-dimensional plane is defined as a section or cutting plane of T^{2d}, and the Π determines how to slice, we will call Π as slicing matrix .

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- A closed d-dimensional *null*-plane is parametrized

$$\mathcal{D}^0(X^I,\Pi) = \{X^I + t_i \Pi^{iI} | 0 \leq t_i < 1 \text{ and } \Pi \in \mathcal{P}_d^0\}$$

$\mathcal{P}_d^0$ is a subset of the $\mathcal{P}_d$ such that for an arbitrary element $\Pi \in \mathcal{P}_d^0$, the row vectors Π^i are mutually orthogonal and null

$$\Pi^i{}_I \mathcal{J}^{IJ} (\Pi^t)_J{}^j = 0$$

Since the tangent vectors for $\mathcal{D}^0(X^I,\Pi)$ are Π^i, it is a null-plane.

- For $\Pi \in \mathcal{P}_d^0$, the Smith normal form of Π is given by

$$\Pi = L D_0 V$$

where $L \in PSL(d,\mathbb{Z})$ and $V \in \mathbf{O}(d,d;\mathbb{Z})$.

- Note that the parametrization of d-plane is not unique, but there is a $PSL(d, \mathbb{Z})$ equivalence relation

$$\Pi^i{}_I \sim a^i{}_j \Pi^j{}_I \,, \qquad a^i{}_j \in PSL(d, \mathbb{Z})$$

- If two slicing matrices Π' and Π are related by $PSL(d, \mathbb{Z})$ rotation, then they parametrize the same d-plane because the $a \in PSL(d, \mathbb{Z})$ can be absorbed into the parameter t^i by redefining $t'_i = t_j a^j{}_i$.

- Radon (X-ray) transform on a torus is an integral transform mapping a continuous function $f(X^I)$ on a T^{2d} to the integrals of this function over the d-dimensional closed planes $\mathcal{D}(X^I, \Pi)$

$$\mathcal{R}f(X^I;\Pi) = \int_0^1 \cdots \int_0^1 \mathrm{d}t_1 \cdots \mathrm{d}t_d f\big(X^I + t_i \Pi^{iI}\big)$$

  where X^I is a point on the T^{2d} and $\Pi^{iI} \in \mathcal{P}_d$.
- X-ray transform for T^{2d} is an injective mapping, and it is possible to define the inverse transformation [Abouelaz, Rouviere, 2011]
- In general, the X-ray transform can be applied to any continuous functions, but we will focus only on weakly constrained fields.

- Let us consider a null plane wave $e_K = e^{2\pi i K_I X^I}$ with an integer momentum K_I satisfying

$$K_I K^I = 0$$

- Then the t integrals in X-ray transform can be done for the e_K trivially

$$\mathcal{R}\, e_K(X^I; \Pi) = \int \mathrm{d}^d t\, e^{2\pi i K_I (X^I + t_i \Pi^{iI})} = e^{2\pi i K_I X^I} \int \mathrm{d}^d t\, e^{2\pi i K_I t_i \Pi^{iI}}$$

$$= e_K\, \delta_{\Pi^{iI} K_I, 0}$$

- Then we have two constraints on K^I for a given Π:

$$(1) \quad \Pi^{iI} K_I = 0\,, \qquad i = 1 \cdots d$$
$$(2) \quad K_I K^I = 0$$

- The first constraint eliminates d degrees of freedom of K^I. Thus K^I is expanded by d-momentum ℓ_i

$$K_I = \ell_i \Psi^i{}_I$$

where $\Psi^i{}_I$ is a $d \times 2d$ integer valued matrix of rank d.

- From the second condition, the row vectors of Ψ^i should be mutually null and orthogonal vectors

$$\Psi^i{}_I \mathcal{J}^{IJ} \Psi^j{}_J = 0$$

and the Ψ^i become a basis of a maximal null subspace N

- Also Ψ and Π are orthogonal by the (1)

- Recall that the orthogonal complement of a maximal null subspace N is identical with itself, $N = N_\perp$.
- Since Π generates $N_\perp$, we can identify Π and Ψ without loss of generality. Then the doubled momentum K_I is represented by

$$K_I = \ell_i \Pi^i{}_I\,, \qquad \text{and} \qquad \Pi^i{}_I \mathcal{J}^{IJ} \Pi^i{}_J = 0$$

  Thus Π defines a null d-dimensional plane $\mathcal{D}^0(X^I, \Pi \in \mathcal{P}^0_d)$.
- The X-ray transform of the e_K can be rewritten by d-dimensional momenta ℓ_i

$$\mathcal{R}e_K(X^I; \Pi^i) = e^{2\pi i \ell_i \Pi^i{}_I X^I} = e^{2\pi i \ell_i z^i}\,, \qquad z^i = \Pi^i{}_I X^I$$

- After X-ray transform, the Fourier basis e_K on T^{2d} reduces to a Fourier basis of d-dimensional null plane defined by $\Pi^i{}_I$.

- To get a X-ray transform for an arbitrary function $f(X^I)$, we carry out Fourier expansion and use the previous result $\mathcal{R}e_K(X^I;\Pi)$

$$\mathcal{R}f(z^i;\Pi^i) = \sum_{K\in\mathbb{Z}^{2d}} \tilde{f}_K e^{2\pi i K_I X^I} \delta_{\Pi^{iI}K_I,0}$$

$$= \sum_{l_i} \tilde{f}'_{l_i}\, e^{2\pi i l_i z^i} \; ,$$

  where $\tilde{f}'_{l_i} = \tilde{f}_{l_i \Pi^i{}_I}$, and it is reduced to the usual d-dimensional Fourier expansion. This is known as Fourier slice theorem.

- The X-ray transform maps a $2d$-dimensional weakly constrained field to a d-dimensional strongly constrained field on a d-dimensional null plane.

- Inverse X-ray transform : Reconstruction of the original $2d$-dimensional weakly constrained field $f(X^I)$ in terms of d-dimensional strongly constrained fields $\mathcal{R}f(z^i;\Pi)$ [Abouelaz, 2011]

$$f(X^I) = \sum_{\Pi\in\mathcal{P}_d^0} \varphi(\Pi)\hat{f}_\Pi(z^i)$$

where $\varphi(\Pi)$ is a weight factor for convergence of this series

$$\varphi(\Pi^i) = \exp(-\|\Pi\|^2) = \exp(-\sum_{i,I}(\Pi^i{}_I)^2)$$

- The $\hat{f}_\Pi(z^i)$ is defined in terms of $\mathcal{R}f(z^i;\Pi)$

$$\hat{f}_\Pi(z^i;\Pi^i) = \int_{T^{2d}} \mathrm{d}^{2d}Y \sum_K \tfrac{1}{\psi(K)}\mathcal{R}f(\Pi^i{}_I Y^I)e^{2\pi i K_I(X^I - Y^I)}$$

$$= \tfrac{1}{\psi(0)}\mathcal{R}f(z^i;\Pi)$$

- Each $\hat{f}_{\Pi}(z^i)$ is strongly constrained field on a null plane $\mathcal{D}^0(X^I, \Pi)$. Hence,

Weakly constrained fields can be represented as a collection of strongly constrained fields through inverse X-ray transform.

- From inverse X-ray transform, a null plane wave e_K is reconstructed by X-ray image fields $\hat{e}_{K,\Pi}$ as

$$e_K = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi)\, \hat{e}_{K,\Pi}$$

- Using the previous result $\mathcal{R} e_K = e_K \delta_{\Pi^{iI} K_I}$

$$\begin{aligned} \hat{e}_{K,\Pi}(z^i) &= \sum_{K'} \frac{1}{\psi(K')} e^{2\pi i K'_I X^I} \int_{T^{2d}} \mathrm{d}^{2d} Y\, e^{-2\pi i (K'_I - K_I) Y^I} \delta_{\Pi^i{}_I K^I, 0} \\ &= \sum_{K'} \frac{1}{\psi(K')} e^{2\pi i K'_I X^I} \delta_{K_I, K'_I} \delta_{\Pi^i{}_I K^I, 0} \\ &= \frac{1}{\psi(K)} e^{2\pi i K_I X^I} \delta_{\Pi^i{}_I K^I, 0} \\ &= \frac{1}{\psi(K)} e_K \delta_{\Pi^i{}_I K^I, 0} = \frac{1}{\psi(K)} \mathcal{R} e_K \end{aligned}$$

- By definition of $\psi(K) = \sum_\Pi \varphi(\Pi)\delta_{\Pi^i{}_I K^I,0}$

$$\begin{aligned} e_K &= \sum_{\Pi\in\mathcal{P}_d^0} \varphi(\Pi)\tfrac{1}{\psi(K)} e^{2\pi i K_I X^I} \delta_{\Pi^i{}_I K^I,0} \\ &= e^{2\pi i K_I X^I} \sum_{\Pi\in\mathcal{P}_d^0} \varphi(\Pi)\tfrac{1}{\psi(K)} \delta_{\Pi^i{}_I K^I,0} \\ &= e^{2\pi i K_I X^I} \end{aligned}$$

- It is easily generalized to an arbitrary function $f(X^I)$ through Fourier expansion.

Binary operations for weakly constrained fields

Binary operations for weakly constrained fields

- Weakly constrained fields form the kernel K of the level matching constraint

$$L_0 - \bar{L}_0 = \partial_I \partial^I$$

- The K is not closed by ordinary product. For arbitrary $f, g \in K$,

$$f \cdot g \notin K$$

- Q: How we can define a binary operation which is compatible with level matching constraint?

$$f \circ g \in K$$

- Using the inverse X-ray transform, the $f \cdot g$ is represented as

$$f \cdot g = \sum_{\Pi, \Pi' \in \mathcal{P}_d^0} \varphi(\Pi)\varphi(\Pi')\hat{f}_\Pi(z^i)\hat{g}_{\Pi'}(z'^i)$$

- To find an additional condition which makes the ordinary product become compatible with level matching constraint, we act the level matching operator $\partial_I \partial^I$ to the product

$$\partial_I \partial^I \left(f \cdot g\right) = 2\partial_I f \partial^I g = 2 \sum_{\Pi, \Pi' \in \mathcal{P}_d^0} \varphi(\Pi)\varphi(\Pi')\Pi^i{}_I \Pi'^{jI} \frac{\partial \hat{f}_\Pi}{\partial z^i} \frac{\partial \hat{g}_{\Pi'}}{\partial z'^j} ,$$

- A simple and natural way to vanish the right-hand side is to impose an orthogonality condition on the slicing matrices

$$\Pi^i{}_I \mathcal{J}^{IJ} \Pi'^j{}_J = 0 .$$

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$$\Pi^i{}_I \mathcal{J}^{IJ} \Pi'^j{}_J = 0 .$$

- Now we assume that Π and Π' are orthogonal.
- Since the row vectors Π^i define a maximal null subspace, their orthogonal complement is identical with the original maximal null subspace. Thus the Π'^i is represented by a linear combination of Π^i

$$\Pi'^i{}_I = a^i{}_j \Pi^j{}_I \,, \qquad a^i{}_j \in PSL(d;\mathbb{Z})$$

- By the equivalence relation, $\mathcal{D}^0(X^I;\Pi)$ and $\mathcal{D}^0(X^I;a\Pi)$ are identical. Then the X-ray image fields $\hat{f}_\Pi$ and $\hat{g}_{\Pi'}$ live on the same plane.

- Moreover, we can absorb the ${a^i}_j$ into the momenta ℓ_i , which is define by the relation $K_I = \ell_i {\Pi^i}_I$ in the Fourier expansion, by redefining ℓ'_i

$$\ell''_i = \ell'_j {a^j}_i$$

- Therefore, without loss of generality, we can always identify Π and Π' if we assume Π and Π' are orthogonal.
- We define a novel binary operation $\circ$ as a product in the space of weakly constrained fields:

$$f(X^I) \circ g(X^I) = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \hat{f}_\Pi(z^i) \cdot \hat{g}_\Pi(z^i) \, .$$

cf. with ordinary product

$$f \cdot g = \sum_{\Pi, \Pi' \in \mathcal{P}_d^0} \varphi(\Pi) \varphi(\Pi') \hat{f}_\Pi(z^i) \hat{g}_{\Pi'}(z'^i)$$

- We can show that the $\circ$-product satisfy the following algebraic properties:
  - Commutativity

$$f \circ g = g \circ f$$

  - Associativity

$$f \circ (g \circ h) = (f \circ g) \circ h$$

  - Distributivity

$$f \circ (g + h) = f \circ g + f \circ h$$

  In addition we can define an identity I satisfying $I \circ f = f \circ I = f$

$$I = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \cdot 1$$

- Leibniz rule

$$\partial_I (f \circ g) = \partial_I f \circ g + f \circ \partial_I g$$

- Hull and Zwiebach defined a projector by inserting an operator $\delta_{L_0-\bar{L}_0,0}$ within the Fourier expansion of a function to satisfy level matching constraint. For massless fields, $N=\bar{N}=1$, the $\delta_{L_0-\bar{L}_0,0}$ is represented as

$$\delta_{L_0-\bar{L}_0,0}=\delta_{\partial_I\partial^I,0}$$

and the projector is defined

$$[\![f]\!]=\sum_{K^I\in\mathbb{Z}^{2d}}\delta_{K_IK^I,0}\tilde{f}_Ke^{2\pi iK_IX^I}$$

- The projector for a product of two weakly constrained fields f and g

$$[\![f \cdot g]\!] = \sum_{K^I, K'^I} \delta_{K_I K'^I, 0} \tilde{f}_K \tilde{g}_{K'} e^{2\pi i (K+K')_I X^I}$$

where K and K' are null vectors.

- One can show that the strong constraint is automatically satisfied

$$[\![\partial_I f \cdot \partial^I g]\!] = 0$$

and it is commutative

$$[\![fg]\!] = [\![gf]\!]$$

but not associative

$$[\![[\![fg]\!]h]\!] \neq [\![[\![gh]\!]f]\!] \neq [\![[\![hf]\!]g]\!] \neq [\![fgh]\!]$$

- This projector is motivated by the string product in string field theory. For effective field theory level, integrating out all the massive modes, it is not clear weather it is a valid physical product. Supergravity is effective field theory of string field theory, but it is the associative theory. How about weakly constrained DFT?
- At least, $\circ$-product is an associative limit of the projector.
- Using $\circ$-product, we can construct a consistent theory.

Weakly Constrained Double Field Theory

- We have to define $\mathbf{O}(d,d;\mathbb{Z})$ group equipped with $\circ$-product. To distinguish with the usual $\mathbf{O}(d,d)$ group, we denote as $\mathbf{O}(d,d;\mathbb{Z})_\circ$.
- Assume that $\mathcal{J}_\circ$ is the $\mathbf{O}(d,d;\mathbb{Z})_\circ$ metric which is defined as

$$\mathcal{J}_\circ = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}$$

where the identity matrix I_d is defined by

$$I_d = \sum_{\Pi} \varphi(\Pi)\, \mathbb{1}_d$$

where $\mathbb{1}_d = \mathrm{diag}(1, \cdots, 1)$. Note that $\mathcal{J}$ is a constant matrix, but it is not the usual $\mathbf{O}(d,d)$ metric

$$\mathcal{J}_{\circ IJ} \neq \mathcal{J}_{IJ} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix} .$$

- $\mathbf{O}(d,d;\mathbb{Z})_\circ$ is defined by a set of $2d \times 2d$ matrices satisfying

$$\mathcal{O}^t \circ \mathcal{J}_\circ \circ \mathcal{O} = \mathcal{J}_\circ$$

  where $\mathcal{O} \in \mathbf{O}(d,d;\mathbb{Z})_\circ$.

- $\mathcal{J}_\circ$ and $\mathcal{O}$ are expanded by inverse X-ray transform

$$\mathcal{J}_\circ = \sum_\Pi \varphi(\Pi)\hat{\mathcal{J}}_\Pi\,, \qquad \mathcal{O} = \sum_\Pi \varphi(\Pi)\,\hat{\mathcal{O}}_\Pi(z_i)$$

- Each X-ray images $\hat{\mathcal{O}}_\Pi$ are usual $\mathbf{O}(d,d;\mathbb{Z})$ elements

$$\hat{\mathcal{O}}_\Pi^t \cdot \hat{\mathcal{J}}_\Pi \cdot \hat{\mathcal{O}}_\Pi = \hat{\mathcal{J}}_\Pi$$

  Thus $\mathbf{O}(d,d;\mathbb{Z})_\circ$ element is represented by a collection of $\mathbf{O}(d,d;\mathbb{Z})$ elements.

- Then we can show that $\mathbf{O}(d,d;\mathbb{Z})_\circ$ forms a group. For arbitrary elements $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \in \mathbf{O}(d,d;\mathbb{Z})$, they satisfy the following the properties:
  - Closure
$$\mathcal{O}_1 \circ \mathcal{O}_2 \in \mathbf{O}(d,d)$$
  - Associativity
$$\mathcal{O}_1 \circ (\mathcal{O}_2 \circ \mathcal{O}_3) = (\mathcal{O}_1 \circ \mathcal{O}_2) \circ \mathcal{O}_3$$
  - Identity
$$A \circ I_{2d} = I_{2d} \circ A = A$$
  - Inverse
$$A \circ A^{-1} = A^{-1} \circ A = I_{2d}$$

- $\mathbf{O}(d,d;\mathbb{Z})_\circ$ tensor transforms as

$$T'_{I_1\cdots I_m}{}^{J_1\cdots J_n}(X') = \mathcal{O}_{I_1}{}^{K_1}\circ\cdots\circ\mathcal{O}_{I_1}{}^{K_m}\circ T_{K_1\cdots K_m}{}^{L_1\cdots L_n}\circ\mathcal{O}^{J_1}{}_{L_1}\circ\cdots\circ\mathcal{O}^{J_n}{}_{L_n}$$

- Since we are assuming torus case only, it should be $\mathbf{O}(d,d;\mathbb{Z})_\circ$ rather than $\mathbf{O}(d,d,\mathbb{R})_\circ$

- Weakly constrained fields are represented by summing the all possible strongly constrained fields. Conversely, we may consider a collection of all possible strongly constrained generalized metric

$$\mathcal{H}_{IJ}(X^I) = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \hat{\mathcal{H}}_{\Pi\, IJ}(z^i)$$

- Weakly constrained generalized metric satisfy following conditions

$$\mathcal{H}_{IJ} = \mathcal{H}_{(IJ)} \qquad \mathcal{H} \circ \mathcal{J}_\circ \circ \mathcal{H}^t = \mathcal{J}_\circ^{-1}$$

- Furthermore, $\mathcal{H}$ is an $\mathbf{O}(d, d; \mathbb{Z})_\circ$ tensor

$$\mathcal{H} \longrightarrow \mathcal{O} \circ \mathcal{H} \circ \mathcal{O}^t$$

- As strongly constrained DFT, we can parametrize $\mathcal{H}$

$$\mathcal{H}_{IJ} = \begin{pmatrix} g^{-1} & g^{-1} \circ B \\ B \circ g^{-1} & g - B \circ g^{-1} \circ B \end{pmatrix}$$

where the g^{-1} is defined by

$$g^{-1} \circ g = g \circ g^{-1} = I_d$$

- Even if we consider weakly constrained DFT, the physical degrees of freedom are same as strongly constrained DFT

$$g(x, \tilde{x}) \,, \qquad B(x, \tilde{x}) \,, \qquad \phi(x, \tilde{x})$$

This is consistent with the result of string field theory.

- Section condition divides the doubled torus into physical torus and its dual torus and ignore the dual torus dependence.
- Polarization Θ provides a consistent way to separate the T^d and $\tilde{T}^d$ within the double torus T^{2d} . [Hull, 2004]
- Parametrization of the $\hat{\mathcal{H}}_\Pi$ and $\hat{d}_\Pi$ using usual physical variables requires explicit polarization for each Π.
- Since the X-ray image fields $\hat{\mathcal{H}}_\Pi$ are function of $z^i = \Pi^i{}_I X^I$ on a null plane $\mathcal{D}^0(X^I, \Pi)$, we can regard the $\mathcal{D}^0(X^I, \Pi)$ as a physical torus T^d . Also, we can always introduce a dual coordinate $\tilde{z}^i$ corresponding to the dual torus $\tilde{T}^d$.

- The polarization vector Θ is explicitly realized using the slicing matrix Π

$$\Theta_{\hat{I}}{}^{I} = \begin{pmatrix} \Pi^{iI} \\ \tilde{\Pi}_{i}{}^{I} \end{pmatrix}$$

and they are $\mathbf{O}(d,d;\mathbb{Z})$ elements

$$\Theta_{\hat{I}}{}^{I} \hat{\mathcal{J}}_{IJ} \left(\Theta^{t}\right)^{J}{}_{\hat{J}} = \hat{\mathcal{J}}_{\hat{I}\hat{J}}$$

- The doubled coordinate X^{I} is also decomposed to physical torus z^{i} and its dual torus $\tilde{z}_{i}$

$$X^{\hat{I}} = \Theta^{\hat{I}}{}_{I} X^{I} = \begin{pmatrix} \tilde{z}_{i} \\ z^{i} \end{pmatrix}$$

- From the $\mathbf{O}(d,d)$ condition, the polarization vector should satisfy

$$\hat{\mathcal{J}}_{\hat{I}\hat{J}} = \begin{pmatrix} \Pi^{iI}\hat{\mathcal{J}}_{IJ}\Pi^{t\,Jj} & \Pi^{iI}\hat{\mathcal{J}}_{IJ}\tilde{\Pi}^{t\,J}{}_{j} \\ \tilde{\Pi}_{i}{}^{I}\hat{\mathcal{J}}_{IJ}\Pi^{t\,Jj} & \tilde{\Pi}_{i}{}^{I}\hat{\mathcal{J}}_{IJ}\tilde{\Pi}^{t\,J}{}_{j} \end{pmatrix} = \begin{pmatrix} 0 & \delta^{i}{}_{j} \\ \delta_{i}{}^{j} & 0 \end{pmatrix} .$$

This implies that the polarization vector $\tilde{\Pi}_i$ is a right-inverse of Π

$$\Pi\,\tilde{\Pi} = \mathbb{1}_d\,, \qquad \text{or} \qquad (\tilde{\Pi}^t)^{I}{}_{i} = (\Pi^{-1})^{I}{}_{i}$$

- The X-ray images for generalized metric $\hat{\mathcal{H}}_{\Pi\,IJ}$ is parametrized by using the polarization Θ, in terms of metric and Kalb-Ramond fields on a null d-plane

$$\Theta_{\hat{I}}{}^{I}\hat{\mathcal{H}}_{\Pi\,IJ}(\Theta^t)^{J}{}_{\hat{J}} = \hat{\mathcal{H}}_{\Pi\,\hat{I}\hat{J}} = \begin{pmatrix} \hat{g}_{\Pi}^{\hat{i}\hat{j}} & -\,\hat{g}_{\Pi}^{\hat{i}\hat{k}}\hat{B}_{\Pi\,\hat{k}\hat{j}} \\ \hat{B}_{\Pi\,\hat{i}\hat{k}}\hat{g}_{\Pi}^{\hat{k}\hat{j}} & \hat{g}_{\Pi\,\hat{i}\hat{j}} - \hat{B}_{\Pi\,\hat{i}\hat{k}}\hat{g}_{\Pi}^{\hat{k}\hat{l}}\hat{B}_{\Pi\,\hat{l}\hat{j}} \end{pmatrix}$$

- Physical degrees of freedom is given by weakly constrained generalized metric.

$$\mathcal{H}_{IJ}(X^I) = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \hat{\mathcal{H}}_{\Pi IJ}(z^i)$$

- Gauge transformation of each $\hat{\mathcal{H}}_{\Pi IJ}(z^i)$ is given by generalized Lie derivative. The gauge transformation of $\mathcal{H}_{IJ}$ should be a collection of generalized Lie derivatives.
- It is natural to speculate that the form of gauge transformation of the weakly constrained fields : replacing all the usual products to $\circ$-product in the generalized Lie derivative

$$\delta_X \mathcal{H}_{IJ} = X^K \circ \partial_K \mathcal{H}_{IJ} + \left(\partial_I X^K - \partial^K X_I\right) \circ \mathcal{H}_{KJ} + \left(\partial_J X^K - \partial^K X_J\right) \circ \mathcal{H}_{IK} \,,$$

$$\delta_X d = X^P \circ \partial_P d - \tfrac{1}{2} \partial_P X^P \,.$$

- It is straight to show that the gauge transform is closed exactly

$$[\delta_X , \delta_Y]\, \mathcal{H}_{MN} = \delta_{[X,Y]_C} \mathcal{H}_{MN} \, ,$$

where the generalized version of C-bracket is defined by

$$[X,Y]_C^M = X^N \circ \partial_N Y^M - \tfrac{1}{2} X^N \circ \partial^M Y_N - (X \leftrightarrow Y)$$

- Under the $\tilde{\partial}^i$-expansion, the gauge transform is expanded by

$$\delta^{(0)}\mathcal{E}_{ij} = \partial_i \Lambda_j - \partial_j \Lambda_i + \xi^k \circ \partial_k \mathcal{E}_{ij} + \partial_i \xi^k \circ \mathcal{E}_{kj} + \partial_j \xi^k \circ \mathcal{E}_{ik}$$

$$\delta^{(1)}\mathcal{E}_{ij} = -\mathcal{E}_{ik} \circ \left(\tilde{\partial}^k \xi^l - \partial^l \xi^k\right) \circ \mathcal{E}_{lj} + \Lambda_k \circ \tilde{\partial}^k \mathcal{E}_{ij} - \tilde{\partial}^k \Lambda_i \circ \mathcal{E}_{kj} - \tilde{\partial}^k \Lambda_j \circ \mathcal{E}_{ik}$$

where $\mathcal{E} = g + B$. This is exactly same as Hohm, Hull and Zwiebach's tilde derivative expansion except the product.

- We propose a weakly constrained DFT action as

$$\mathcal{S}_{\text{WDFT}} = \int \mathrm{d}^{2D}X\, [e^{-2d}]_\circ \circ \mathcal{L}_{\text{WDFT}}$$

where the Lagrangian $\mathcal{L}_{\text{WDFT}}$ is given by

$$\begin{aligned}\mathcal{L}_{\text{WDFT}} = \; & 4\mathcal{H}^{IJ} \circ \partial_I\partial_J d - \partial_I\partial_J\mathcal{H}^{IJ} - 4\mathcal{H}^{IJ} \circ \partial_I d \circ \partial_J d + 4\partial_I\mathcal{H}^{IJ} \circ \partial_J d \\ & + \tfrac{1}{8}\mathcal{H}^{IJ} \circ \partial_I\mathcal{H}^{KL} \circ \partial_J\mathcal{H}_{KL} - \tfrac{1}{2}\mathcal{H}^{IJ} \circ \partial_I\mathcal{H}^{KL} \circ \partial_K\mathcal{H}_{JL}\end{aligned}$$

- The exponentiation of the d, $[e^{-2d}]_\circ$, is defined by

$$\begin{aligned}[e^{-2d}]_\circ &= I - 2d + \tfrac{1}{2}(2d)\circ(2d) - \tfrac{1}{3!}(2d)\circ(2d)\circ(2d) + \cdots \\ &= \sum_{\Pi\in\mathcal{P}_d^0}\sum_{m\geq 0}\varphi(\Pi)\tfrac{1}{m!}\big(-2\hat{d}_\Pi(z^i)\big)^m \\ &= \sum_{\Pi\in\mathcal{P}_d^0}\varphi(\Pi)e^{-2\hat{d}_\Pi}\,.\end{aligned}$$

- Using the definition of $\circ$-product, the action is expanded as

$$\mathcal{S}_{\text{WDFT}} = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \hat{S}_\Pi(z^i)$$

- Using the polarization Θ, we can represent the $\hat{S}_\Pi$ as a strongly constrained DFT action on a null d-dimensional plane with a oblique section conditions

$$\mathcal{H}_{IJ} = \left(\Theta^{-1}\right)_I{}^{\hat{I}} \hat{\mathcal{H}}_{\Pi\,\hat{I}\hat{J}} \left(\Theta^{-1}\right)^{\hat{J}}{}_J$$

$$\hat{\partial}_{\hat{I}} = \Theta_{\hat{I}}{}^I \partial_I$$

- Each $\hat{S}_\Pi$ is a strongly constrained DFT action on a d-dimensional null plane $\mathcal{D}^0(X^I, \Pi)$ with an oblique section condition $\frac{\partial}{\partial \tilde{z}_i} = 0$

$$\hat{S}_\Pi = e^{2\hat{d}_\Pi} \big(4\hat{\mathcal{H}}_\Pi{}^{\hat{I}\hat{J}} \partial_{\hat{I}} \partial_{\hat{J}} \hat{d}_\Pi - \partial_{\hat{I}} \partial_{\hat{J}} \hat{\mathcal{H}}_\Pi{}^{\hat{I}\hat{J}} - 4\hat{\mathcal{H}}_\Pi^{\hat{I}\hat{J}} \partial_{\hat{I}} \hat{d} \partial_{\hat{J}} \hat{d}_\Pi$$
$$+ 4\partial_{\hat{I}} \hat{\mathcal{H}}_\Pi{}^{\hat{I}\hat{J}} \partial_{\hat{J}} \hat{d} + \tfrac{1}{8} \hat{\mathcal{H}}_\Pi{}^{\hat{I}\hat{J}} \partial_{\hat{I}} \hat{\mathcal{H}}_\Pi{}^{\hat{K}\hat{L}} \partial_{\hat{J}} \hat{\mathcal{H}}_{\Pi\,\hat{K}\hat{L}} - \tfrac{1}{2} \hat{\mathcal{H}}_\Pi{}^{\hat{I}\hat{J}} \partial_{\hat{I}} \hat{\mathcal{H}}_\Pi{}^{\hat{K}\hat{L}} \partial_{\hat{K}} \hat{\mathcal{H}}_{\Pi\,\hat{J}\hat{L}} \big)$$

- Hull and Zwiebach constructed weakly constrained DFT action up to cubic order fluctuations of string fields from closed string field theory.
- Up to cubic order perturbation, the projector $[\![fg]\!]$ is unnecessary due to the following identities:

$$\int [\![fg]\!] = \int fg\,, \qquad \int [\![[\![fg]\!]h]\!] = \int [\![[\![gh]\!]f]\!] = \int [\![[\![hf]\!]g]\!] = \int fgh$$

- There is no non-associativity issue, and the action is written in terms of ordinary product.

- For WDFT case, consider fluctuations of weakly constrained generalized metric and dilaton around constant backgrounds $\mathcal{H}^0$ and d^0

$$\mathcal{H}_{IJ} = \mathcal{H}^0_{IJ} + \bar{\mathcal{H}}_{IJ}\,, \qquad d = d^0 + \bar{d}$$

  Here each fluctuations $\bar{\mathcal{H}}_{IJ}$ and $\bar{d}$ are weakly constriained fields
- We have following identities

$$\int \mathrm{d}^{2D}X\ \tfrac{1}{\psi} f \cdot g = \int \mathrm{d}^{2D}X\ f \circ g\,,$$

$$\int \mathrm{d}^{2D}X\ \tfrac{1}{\psi^2} f \cdot g \cdot h = \int \mathrm{d}^{2D}X\ f \circ g \circ h$$

- If we substitute the fluctuation ansatz into the $\mathcal{S}_{\mathrm{WDFT}}$, then we reproduce the Hull, Zwiebach action. Thus up to cubic order perturbation, it is consistent with the string field theory.

- For weakly constrained DFT, one can define a generalised Scherk-Schwarz reduction by defining a rotated generalised frame

$$E'_A = U_A{}^B(X^I) \circ E_B$$

To maintain the torus topology, the twisted matrix $U_A{}^B$ should be restricted (e.g. duality twist).

- $U_A{}^B$ is reconstructed by inverse X-ray transform

$$U = \sum_{\Pi \in \mathcal{P}_d^0} \varphi \hat{U}_\Pi(z^i)$$

- Each $\hat{U}_\Pi$ should satisfy strong constraint. If we relax the section condition for $\hat{U}_\Pi$, then $\hat{U}_\Pi$ does not satisfy even weak constraint. [Marques, Grana 2012]

- Generalized flux

$$\mathcal{F}_{ABC} = 3E_{[A}{}^{I} \circ \partial_I E_B{}^{J} \circ E_{C]J}$$

and it is also reconstructed

$$\mathcal{F}_{ABC} = \sum_{\Pi} \varphi \, \hat{\mathcal{F}}_{\Pi\, ABC}$$

- If we take supergravity limit, the only one of the $\hat{\mathcal{F}}_{\Pi\, ABC}$ survives, and it reduces to usual gauged supergravity.

Doubled sigma model on a weakly constrained background

- Following the main idea of WDFT, we propose that

doubled sigma model on a weakly constrained background	=	Overlapping all possible doubled sigma model on a strongly constrained background

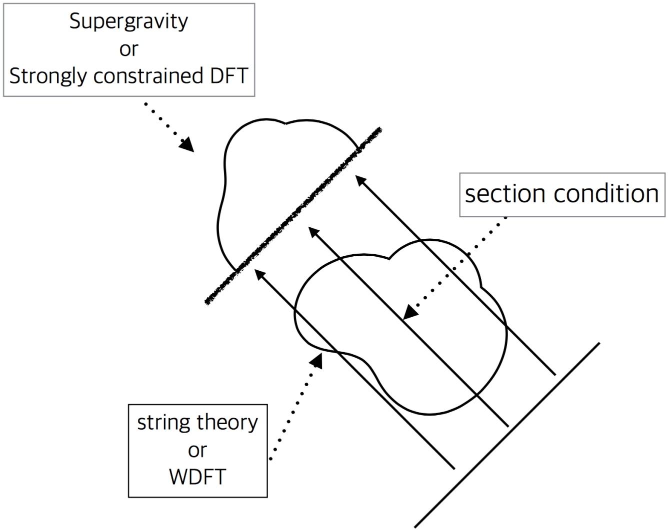
- The action is given by

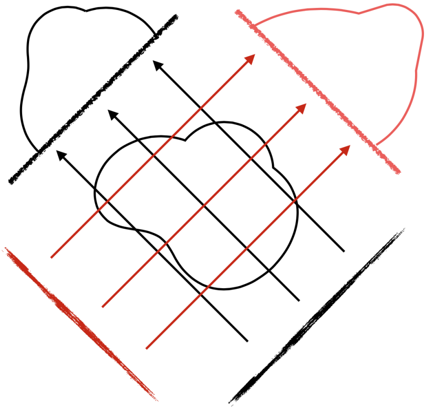
$$\mathcal{S} = \int \mathrm{d}^2\sigma\sqrt{-h}\, h^{ij}\partial_i \mathbb{X}^I \partial_j \mathbb{X}^J\, \mathcal{H}_{IJ}(\mathbb{X}^I)$$

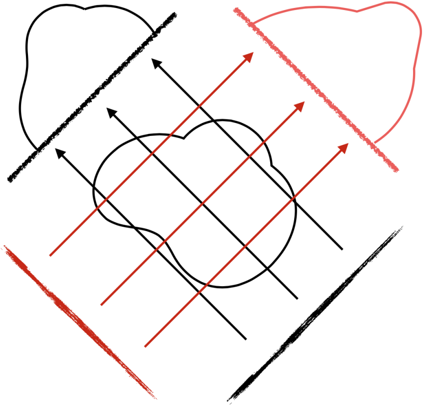
$$= \sum_{\Pi\in\mathcal{P}_d^0} \varphi(\Pi) \int \mathrm{d}^2\sigma\sqrt{-h}\, h^{ij}\partial_i \mathbb{X}^I \partial_j \mathbb{X}^J\, \hat{\mathcal{H}}_{\Pi\, IJ}({\Pi^i}_I \mathbb{X}^I)$$

with the self-dual constraint

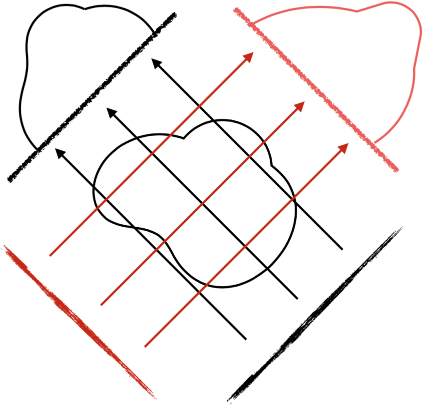
$${\mathcal{H}^I}_J \partial_i \mathbb{X}^J + \frac{1}{\sqrt{-h}} {\epsilon_i}^j \partial_j \mathbb{X}^I = 0$$


Supergravity
or
Strongly constrained DFT
section condition
string theory
or
WDFT





Weakly Constrained DFT = overlapping all the X-ray images



Sugra is just a single image among the infinite number of images.

- This is the string effective theory beyond supergravity limit. WDFT is not rewriting supergravity at all!
- From the X-ray transform and $\circ$-product, WDFT is defined in a very simple and straightforward way.
- If you know strongly constrained DFT (SUGRA), then you can calculate WDFT.
- There are huge number of things to check and calculate.

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Thank you for attention