Matter Quantum Corrections to the Graviton Self-Energy and the Newtonian Potential

DANIEL BURNS

daniel.burns@hep.manchester.ac.uk

School of Physics and Astronomy, University of Manchester, United Kingdom

based on

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Outline

- **Introduction: the graviton mass**
- **Theoretical Framework**
  - Abelian Higgs model
  - Diffeomorphism symmetry and the master Ward identity
- **Minimisation of the effective action**
  - The cosmological constant
  - Graviton tadpoles
- **The graviton low-energy theorem**
- **Quantum corrections to the Newtonian potential**

Introduction: the graviton mass

- General relativity (GR), as a classical theory, contains a **massless** spin-2 particle: the graviton.
- Treating GR as a quantum theory, it is not clear that the graviton **remains** massless.
- Graviton is massless if the graviton self-energy has no **longitudinal** modes i.e.

$$p_\mu \Pi^{\mu\nu,\rho\sigma}(p) = 0$$

- GR with matter interactions is perturbatively non-renormalisable. ['t Hooft, Veltman, 1974; Deser, van Nieuwenhuizen, 1974; ...]
- We shall use effective field theory approach to gravity. [Donoghue, 1994]

Theoretical framework: Abelian Higgs model

- Study the Abelian Higgs (AH) model in curved spacetime:

$$S = S_G + S_M = \int d^4x\sqrt{-g}\left(\Lambda + \frac{1}{\kappa^2}R + \mathcal{L}_M\right), \quad \kappa^2 = 16\pi G$$

- Matter sector:

$$S_M = \int d^4x\sqrt{-g}\left[-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + g^{\mu\nu}(\nabla_\mu\phi)^\dagger\nabla_\nu\phi - \lambda\left(\phi^\dagger\phi - \frac{\mu^2}{2\lambda}\right)^2\right] + S_f \,,$$

$$F_{\mu\nu} \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu, \quad \phi = \frac{1}{\sqrt{2}}(v + \mathcal{H} + i\mathcal{G}), \quad \nabla_\mu\phi = \partial_\mu\phi - ie\mathcal{A}_\mu\phi$$

- Fermion sector:

$$\mathcal{S}_f = \int d^4x\sqrt{-g}\left[\frac{1}{2}(\nabla_\mu\bar{\psi}^Q)ie^\mu_a\gamma^a\psi^Q - \frac{1}{2}\bar{\psi}^Q ie^\mu_a\gamma^a(\nabla_\mu\psi^Q) - m_\psi\bar{\psi}^Q\psi^Q\right]$$

Theoretical framework: Abelian Higgs model

- Within the background field method (BFM), **split** fields into background and quantum fields:

$$\mathcal{H} = \bar{H} + H^Q, \quad \mathcal{G} = \bar{G} + G^Q, \quad \mathcal{A}_\mu = \bar{A}_\mu + A^Q_\mu$$
$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa(\bar{h}_{\mu\nu} + h^Q_{\mu\nu}) = \bar{g}_{\mu\nu} + \kappa h^Q_{\mu\nu}$$

- Add gauge fixing terms and ghost actions for diffeomorphism symmetry and $U(1)$ gauge invariance.
- Gauge fixing parameters: ξ_D, σ for diffs, ξ_G for $U(1)$.
- Masses:

$$m_H^2 = 2\lambda v^2 \quad m_A^2 = e^2 v^2, \quad m_G^2 = \xi_G m_A^2, \quad m_c^2 = \xi_G m_A^2$$

Diffeomorphism symmetry in the BFM

- Invariant under $x^\mu \to x'^\mu = x^\mu + \kappa\epsilon^\mu(x)$, $\epsilon^\mu << 1$.
- Field transformations are Lie derivatives with respect to $\kappa\epsilon^\mu(x)$
- In BFM, can distribute the gauge transformations **arbitrarily**, as long as full field transforms correctly.
- BFM transformations:

$$\bar{H}' = \bar{H} + \kappa\epsilon^\alpha\partial_\alpha\bar{H}, \qquad H'^Q = H^Q + \kappa\epsilon^\alpha\partial_\alpha H^Q,$$

$$\bar{G}' = \bar{G} + \kappa\epsilon^\alpha\partial_\alpha\bar{G}, \qquad G'^Q = G^Q + \kappa\epsilon^\alpha\partial_\alpha G^Q$$

$$\bar{A}'_\mu = \bar{A}_\mu + \kappa\epsilon^\alpha\partial_\alpha\bar{A}_\mu + \kappa(\partial_\mu\epsilon^\alpha)\bar{A}_\alpha, \qquad A'^Q_\mu = A^Q_\mu + \kappa\epsilon^\alpha\partial_\alpha A^Q_\mu + \kappa(\partial_\mu\epsilon^\alpha)A^Q_\alpha.$$

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu + \kappa(\bar{h}^\alpha_\nu\partial_\mu\epsilon_\alpha + \bar{h}^\alpha_\mu\partial_\nu\epsilon_\alpha + \epsilon_\alpha\partial^\alpha\bar{h}_{\mu\nu}) ,$$

$$h'^Q_{\mu\nu} = h^Q_{\mu\nu} + \kappa(h^Q_{\alpha\nu}\partial_\mu\epsilon^\alpha + h^Q_{\alpha\mu}\partial_\nu\epsilon^\alpha + \epsilon_\alpha\partial^\alpha h^Q_{\mu\nu}) .$$

Diffeomorphism symmetry: Master Ward identity

- To quantise the theory, we introduce the generating functional:

$$Z = N \int \mathcal{D}\Phi \exp\left[iS + \int d^4x \sqrt{-\bar{g}} \left(J_h^{\mu\nu} h_{\mu\nu}^Q + \bar{J}_\psi \psi^Q + \bar{\psi}^Q J_\psi + J_H H^Q + J_G G^Q + J_A^\mu A_\mu^Q \right) \right],$$

$$\mathcal{D}\Phi \equiv \mathcal{D}h_{\mu\nu}^Q \, \mathcal{D}A_\mu^Q \, \mathcal{D}H^Q \, \mathcal{D}G^Q \, \mathcal{D}\bar{\psi}^Q \, \mathcal{D}\psi^Q$$

- Action remains invariant under diffeomorphisms.
- **Path integral measure** remains invariant under diffeomorphisms, as $\int d^4x \, \partial_\mu \epsilon^\mu = 0$
- **Source transformations**:

$$J_h'^{\mu\nu} = J_h^{\mu\nu} + \kappa(\epsilon^\alpha \partial_\alpha J_h^{\mu\nu} - J_h^{\nu\alpha} \partial^\mu \epsilon_\alpha - J_h^{\mu\alpha} \partial^\nu \epsilon_\alpha) \, ,$$

$$J_H' = J_H + \kappa \epsilon^\alpha \partial_\alpha J_H \, ,$$

$$J_G' = J_G + \kappa \epsilon^\alpha \partial_\alpha J_G \, ,$$

$$J_\psi' = J_\psi + \kappa \epsilon^\alpha \partial_\alpha J_\psi \, ,$$

$$\bar{J}_\psi' = \bar{J}_\psi + \kappa \epsilon^\alpha \partial_\alpha \bar{J}_\psi \, ,$$

$$J_A'^\mu = J_A^\mu + \kappa(\epsilon^\alpha \partial_\alpha J_A^\mu - \epsilon_\alpha \partial^\mu J_A^\alpha) \, .$$

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$$J_H' = J_H + \kappa\epsilon^\alpha \partial_\alpha J_H,$$

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J_A'^\mu &= J_A^\mu + \kappa(\epsilon^\alpha \partial_\alpha J_A^\mu - \epsilon_\alpha \partial^\mu J_A^\alpha)\ .
\end{aligned}$$

Graviton sector of Ward identity

- Master Ward identity:

$$\left[\delta^\alpha_\mu \partial_\nu + \kappa\left(\bar{h}^\alpha_\nu \partial_\mu + \partial_\mu \bar{h}^\alpha_\nu + \frac{1}{2}\partial^\alpha \bar{h}_{\mu\nu}\right)\right]\frac{\delta\bar{\Gamma}}{\delta\bar{h}_{\mu\nu}(x)} + \kappa\left(\partial^\alpha \bar{A}_\mu - \partial_\mu \bar{A}^\alpha - \bar{A}^\alpha\partial_\mu\right)\frac{\delta\bar{\Gamma}}{\delta\bar{A}_\mu}$$
$$+ \kappa\partial^\alpha\bar{H}\frac{\delta\bar{\Gamma}}{\delta\bar{H}} + \kappa\partial^\alpha\bar{G}\frac{\delta\bar{\Gamma}}{\delta\bar{G}} + \frac{\delta\bar{\Gamma}}{\delta\psi}\kappa\partial^\alpha\psi + \kappa\partial^\alpha\bar{\psi}\frac{\delta\bar{\Gamma}}{\delta\bar{\psi}} \;=\; 0\,.$$

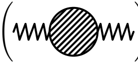
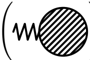
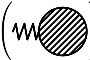
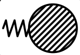
where we have defined the object:

$$\bar{\Gamma}[\bar{h}_{\mu\nu},\psi,\bar{\psi},\bar{H},\bar{G},\bar{A}_\mu] \;\equiv\; \Gamma[\bar{h}_{\mu\nu},\psi,\bar{\psi},\bar{H},\bar{G},\bar{A}_\mu,0,0,0,0]\,.$$

- Setting matter field arguments to zero:

$$\left[\delta^\alpha_\mu \partial_\nu + \kappa\left(\bar{h}^\alpha_\nu \partial_\mu + \partial_\mu \bar{h}^\alpha_\nu + \frac{1}{2}\partial^\alpha \bar{h}_{\mu\nu}\right)\right]\frac{\delta\bar{\Gamma}}{\delta\bar{h}_{\mu\nu}(x)} = 0$$

- Ward identity:

$p_\mu \Big($  $\Big)^{\mu\nu,\rho\sigma} + \frac{\kappa}{2}\Big[\eta^{\nu\rho}p_\mu\Big($  $\Big)^{\mu\sigma} + \eta^{\nu\sigma}p_\mu\Big($  $\Big)^{\mu\rho} - p^\nu\Big($  $\Big)^{\rho\sigma}\Big] = 0\,.$

Minimisation of the effective action

- **Minimisation** may remove the longitudinal modes from the self-energy.
- In the BFM, we have the generic condition:

$$\left.\frac{\delta\Gamma}{\delta X}\right|_{X=0} = 0, \qquad X \in \{h_{\mu\nu}, \psi, \bar{\psi}, H, G, A_\mu\}$$

- Writing $\Gamma = \Gamma^{(0)} + \Gamma^{(n\geq 1)}$, the graviton equation becomes:

$$\frac{\delta\Gamma}{\delta h_{\mu\nu}} = \frac{1}{2}\bar{g}^{\mu\nu}\left(\frac{1}{\kappa}\bar{R} + \kappa(\Lambda_0 + \Lambda_0^H)\right) - \frac{1}{\kappa}\bar{R}^{\mu\nu} - \frac{\kappa}{2}\bar{T}^{\mu\nu} + \frac{\delta\Gamma^{(n\geq 1)}}{\delta h_{\mu\nu}} = 0$$

$$\Lambda_0^H = -\frac{\lambda_0}{4}\left(v_0^2 - \frac{\mu_0^2}{\lambda_0}\right)^2$$

- Minimisation depends on the handling of the **bare cosmological constant**.
- Writing $\Lambda_0 = \Lambda + \delta\Lambda$, we impose:

$$\Gamma[0] = \Lambda_0 + \Lambda_0^H + \Gamma^{(n\geq 1)}[0] = \Lambda$$

- $\delta\Lambda_H = 0$ at the one-loop level. Setting $\Lambda = 0$, we obtain

$$\delta\Lambda + \Gamma^{(1)}[0] = 0.$$

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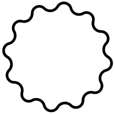
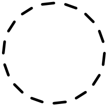
Minimisation: The cosmological constant

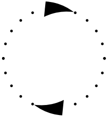
- In dimensional regularisation ['t Hooft, Veltman, 1972], we can calculate $\Gamma^{(1)}[0]$.
- The sum



A_μ +



G +



$$c \;=\; \frac{3i}{2(4\pi)^2}\left(\frac{m_A^2}{2}A_0(m_A^2) - \frac{m_A^4}{12}\right)$$

is independent of the $U(1)$ gauge fixing parameter ξ_G.

- Here, $A_0(m^2)$ is the scalar integral

$$A_0(m^2) \equiv (2\pi\mu)^{4-d}\int \frac{d^dk}{i\pi^2}\,\frac{1}{k^2-m^2} = m^2\left[\frac{1}{\bar\epsilon} + 1 - \ln\left(\frac{m^2}{\mu^2}\right)\right], \qquad \frac{1}{\bar\epsilon} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi$$

- Total cosmological constant counterterm $\delta\Lambda$ is independent of the gauge fixing parameters ξ_G, ξ_D, σ.

Minimisation: Graviton tadpoles

- Define:

$$\frac{\delta\Gamma^{(1)}}{\delta h_{\mu\nu}} = iT_h^{\mu\nu}$$

- Renormalisation of the tadpole graphs proceeds through additive renormalisation of quantum field i.e. $h_{\mu\nu} \to h_{\mu\nu} + \delta h_{\mu\nu}$.
- One-loop minimisation becomes:

$$\int d^4y \left[\left.\frac{\delta^2 S}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(y)}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} \delta h_{\rho\sigma}(y)\right] + \frac{\kappa}{2}\eta^{\mu\nu}\delta\Lambda + T_h^{\mu\nu} = 0$$

- Solving for counterterm $\delta h_{\mu\nu}$:

$$\delta h_{\rho\sigma}(x) = -\int d^4y\, \Delta_{\mu\nu\rho\sigma}(x-y)\left(T_h^{\rho\sigma}(y) + \frac{\kappa}{2}\eta^{\rho\sigma}\delta\Lambda\right).$$

Minimisation: Graviton tadpoles

- Consider gauge sector once more:

$$= \frac{3i\kappa}{4(4\pi)^2}\eta^{\mu\nu}\left(\frac{m_A^2}{2}A_0(m_A^2) - \frac{m_A^4}{12}\right)$$

- Again, this is independent of ξ_G. This is in contrast to Higgs tadpole renormalisation, which is gauge *dependent*.
- Try to compute $\delta h_{\mu\nu}$:

$$T_h^{\mu\nu} + \frac{\kappa}{2}\eta^{\mu\nu}\delta\Lambda = \frac{3i\kappa}{4(4\pi)^2}\left(\frac{m_A^2}{2}A_0(m_A^2) - \frac{m_A^4}{12}\right) - \frac{\kappa}{2}\frac{3i}{2(4\pi)^2}\left(\frac{m_A^2}{2}A_0(m_A^2) - \frac{m_A^4}{12}\right) = 0\ !$$

- This occurs for all sectors *independently*.

The graviton low-energy theorem

- Consider the Higgs low-energy theorem in its canonical form [Pilaftsis, 1998]:

$$\frac{\partial}{\partial v}\Gamma = \frac{\delta\Gamma}{\delta\bar{H}(0)}$$

- Derived from the shift symmetry:

$$v \to v + s, \qquad \bar{H} \to \bar{H} - s,$$

- Relates Higgs tadpole contributions to the v dependence of the cosmological constant.
- Similar symmetry in the gravitational sector:

$$\eta'_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu}, \qquad \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{\kappa}s_{\mu\nu}$$

- Symmetry gives the shift Ward identity:

$$\boxed{\kappa\frac{\partial}{\partial\eta_{\mu\nu}}\Gamma = \frac{\delta\Gamma}{\delta\bar{h}_{\mu\nu}(0)}}$$

- We call this the **Graviton low-energy theorem** (GLET).

GLET in action: The graviton self-energy

- Consider the counterterm in the effective action to cancel $\Gamma^{(n\geq 1)}[0]$:

$$\Delta S = \int d^4x \sqrt{-g}\,\delta\Lambda\ ,$$

- The GLET produces the relation

$$T_h^{\mu\nu} + \frac{\kappa}{2}\eta^{\mu\nu}\delta\Lambda = 0\ !$$

- The renormalised self-energy:

$$\Pi_{\rm R}^{\mu\nu,\rho\sigma}(p) = \Pi^{\mu\nu,\rho\sigma}(p) - \frac{\kappa^2}{4}P^{\mu\nu\rho\sigma}\delta\Lambda + \Delta\Pi^{\mu\nu,\rho\sigma}(p)$$

$$P^{\mu\nu\rho\sigma} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}$$

- Using the GLET, one can show

$$\frac{\kappa}{2}P^{\mu\nu\rho\sigma}\,\delta\Lambda = -\eta^{\nu\rho}\,T_h^{\sigma\mu} - \eta^{\nu\sigma}\,T_h^{\rho\mu} + \eta^{\mu\nu}\,T_h^{\rho\sigma}$$

- Therefore

$$\Pi_{\rm R}^{\mu\nu,\rho\sigma}(p) = \Pi^{\mu\nu,\rho\sigma}(p) + \frac{\kappa}{2}\left(\eta^{\nu\rho}\,T_h^{\sigma\mu} + \eta^{\nu\sigma}\,T_h^{\rho\mu} - \eta^{\mu\nu}\,T_h^{\rho\sigma}\right) + \Delta\Pi^{\mu\nu,\rho\sigma}(p)$$

- and so:

$$p_\mu \Pi_R^{\mu\nu,\rho\sigma}(p) = 0$$

- Graviton is massless after minimisation and renormalisation of the cosmological constant.

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- and so:

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GLET in action: The graviton self-energy

- To check these results, we computed the graviton self-energy at the one-loop level.
- Renormalisation proceeds in the $\overline{MS}$ scheme. [Bardeen et al., 1974]
- As model is Lorentz covariant:

$$\begin{aligned}\Pi_{\mathrm{R}}^{\mu\nu,\rho\sigma}(p) = {} & p^\mu p^\nu p^\rho p^\sigma F_1(p^2) \,+\, \eta^{\mu\nu}\eta^{\rho\sigma}F_2(p^2) \,+\, \Big(\eta^{\mu\rho}\eta^{\nu\sigma} \,+\, \eta^{\nu\rho}\eta^{\mu\sigma}\Big)F_3(p^2) \\ & + \Big(\eta^{\mu\nu}p^\rho p^\sigma + \eta^{\rho\sigma}p^\mu p^\nu\Big)F_4(p^2) \\ & + \Big(\eta^{\mu\rho}p^\nu p^\sigma + \eta^{\nu\rho}p^\mu p^\sigma \,+\, \eta^{\mu\sigma}p^\nu p^\rho + \eta^{\nu\sigma}p^\mu p^\rho\Big)F_5(p^2)\end{aligned}$$

- Ward identity relates these form factors:

$$\begin{aligned} p^2F_1 + F_4 + 2F_5 &= 0, \\ F_2 + p^2F_4 &= 0, \\ F_3 + p^2F_5 &= 0. \end{aligned}$$

Higgs contribution to the self-energy

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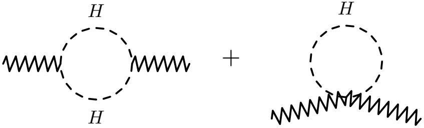


- We find the form factors:

$$F_1(p^2) = \frac{\kappa^2}{3600(4\pi)^2(p^2)^2}\left[\Big(\alpha_1+\alpha_4\Big)B_0(p^2,m_H^2,m_H^2) + \Big(\alpha_2+\alpha_5\Big)A_0(m_H^2) + \Big(\alpha_3+\alpha_6\Big)\right] ,$$

$$F_2(p^2) = \frac{\kappa^2}{3600(4\pi)^2}\Big(\alpha_1 B_0(p^2,m_H^2,m_H^2) + \alpha_2 A_0(m_H^2) + \alpha_3\Big) ,$$

$$F_3(p^2) = \frac{\kappa^2}{7200(4\pi)^2}\Big(\alpha_4 B_0(p^2,m_H^2,m_H^2) + \alpha_5 A_0(m_H^2) + \alpha_6\Big) ,$$

$$F_4(p^2) = -\frac{\kappa^2}{3600(4\pi)^2p^2}\Big(\alpha_1 B_0(p^2,m_H^2,m_H^2) + \alpha_2 A_0(m_H^2) + \alpha_3\Big) ,$$

$$F_5(p^2) = -\frac{\kappa^2}{7200(4\pi)^2p^2}\Big(\alpha_4 B_0(p^2,m_H^2,m_H^2) + \alpha_5 A_0(m_H^2) + \alpha_6\Big) ,$$

$$\begin{aligned}
\alpha_1 &= 15\left[8m_H^4 + 16m_H^2p^2 + 3(p^2)^2\right] ,\\
\alpha_2 &= -30\Big(4m_H^2 + 3p^2\Big) ,\\
\alpha_3 &= 120m_H^4 + 220m_H^2p^2 - 42(p^2)^2 ,\\
\alpha_4 &= 15\Big(p^2 - 4m_H^2\Big)^2 ,\\
\alpha_5 &= -30\Big(8m_H^2 + p^2\Big) ,\\
\alpha_6 &= 16\left[15m_H^4 - 10m_H^2p^2 + (p^2)^2\right] .
\end{aligned}$$

The self-energy for the Abelian Higgs model

- $B_0(p^2, m_1^2, m_2^2)$ is the 2-point scalar integral:

$$B_0(p^2, m_1^2, m_2^2) \equiv (2\pi\mu)^{4-d} \int \frac{d^d k}{i\pi^2} \frac{1}{k^2 - m_1^2} \frac{1}{(k+p)^2 - m_2^2}$$
$$= \frac{1}{\bar{\epsilon}} + 2 - \ln\left(\frac{m_1 m_2}{\mu^2}\right)$$
$$+ \frac{1}{p^2}\left[(m_2^2 - m_1^2)\ln\left(\frac{m_1}{m_2}\right) + \lambda^{1/2}(p^2, m_1^2, m_2^2)\cosh^{-1}\left(\frac{m_1^2 + m_2^2 - p^2}{2m_1 m_2}\right)\right]$$

- Total self-energy:

$$\Pi_{\mathrm{R}}^{\mu\nu,\rho\sigma}(p) = \sum_{i=1}^{N_0} \Pi_0^{\mu\nu,\rho\sigma}(p, m_{0,i}) + \sum_{i=1}^{N_{1/2}} \Pi_{1/2}^{\mu\nu,\rho\sigma}(p, m_{\frac{1}{2},i}) + \sum_{i=1}^{N_1} \Pi_1^{\mu\nu,\rho\sigma}(p, m_{1,i}) ,$$

- Each sector independently satisfies the Ward identity and the GLET.

Applying quantum corrections: The Newtonian potential

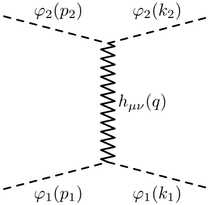
- Newtonian potential is derivable from **single graviton exchange** between two scalar fields.
- It can be **quantum corrected** through radiative corrections to this process.
- **Massless** matter corrections already well established [Hamber, Liu, 1995; Donoghue et al., 2003; ...]:

$$V(r) = -\frac{Gm_1m_2}{r}\left[1 + 3\frac{G(m_1+m_2)}{rc^2} + \frac{41\hbar G}{10\pi c^3 r^2} + \left(\frac{9}{4}N_0 + 3N_{\frac{1}{2}} + 12N_1\right)\frac{\hbar G}{45\pi c^3 r^2} + O(\hbar^2)\right]$$

- How do **massive** matter fields contribute?

Obtaining the Newtonian potential

- Single graviton exchange between two scalar fields φ_1, φ_2 of masses m_1 and m_2:



$$= i\mathcal{M}_{\text{tree}} = iV^{\mu\nu}_{\varphi_1\varphi_1 h}(p_1, -k_1)\Delta_{0,\mu\nu,\rho\sigma}(q)V^{\rho\sigma}_{\varphi_2\varphi_2 h}(p_2, -k_2)$$

with $q_\mu = (p_1 - k_1)_\mu = (p_2 - k_2)_\mu$.

- Ward identity for scalar vertices:

$$(p_1 - k_1)_\mu V^{\mu\nu}_{\varphi_1\varphi_1 h}(p_1, -k_1) \ = \ 0 \ , \qquad (p_2 - k_2)_\mu V^{\mu\nu}_{\varphi_2\varphi_2 h}(p_2, -k_2) \ = \ 0 \ ,$$

- Graviton propagator:

$$\begin{aligned}\Delta_{0,\mu\nu,\rho\sigma}(q) = \frac{1}{q^2 + i\epsilon}\Bigg[& P^{\mu\nu\rho\sigma} - \left(4(1+\xi_D) + \frac{8}{\sigma - 1} + \frac{3 - \xi_D}{(\sigma - 1)^2}\right)\frac{q^\mu q^\nu q^\rho q^\sigma}{(q^2)^2} \\ & + \left(2 + \frac{1}{\sigma - 1}\right)\left(\frac{q^\mu q^\nu}{q^2}\eta^{\rho\sigma} + \frac{q^\rho q^\sigma}{q^2}\eta^{\mu\nu}\right) \\ & + (\xi_D - 1)\left(\frac{q^\mu q^\rho}{q^2}\eta^{\nu\sigma} + \frac{q^\mu q^\sigma}{q^2}\eta^{\nu\rho} + \frac{q^\nu q^\rho}{q^2}\eta^{\mu\sigma} + \frac{q^\nu q^\sigma}{q^2}\eta^{\mu\rho}\right)\Bigg] \ ,\end{aligned}$$

Obtaining the Newtonian potential

- Non-relativistic limit:

$$\mathcal{M}_{\text{tree}}(\vec{q}) = -\frac{\kappa^2 m_1^2 m_2^2}{|\vec{q}|^2}$$

- Convert this to a potential by using the Born approximation [Donoghue et al., 2003]:

$$V(\vec{r}) = \frac{1}{2m_1}\frac{1}{2m_2}\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \mathcal{M}_{\text{tree}}(\vec{q}) \ .$$

- Using the well-known result

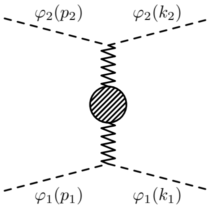
$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{|\vec{q}|^2} = \frac{1}{4\pi r} \ ,$$

- we obtain:

$$\boxed{V(r) = -\frac{Gm_1 m_2}{r}}$$

Newtonian potential: Matter quantum corrections

- Matter quantum corrections:



$$= i\mathcal{M}_{1-\mathrm{loop}} = iV^{\mu\nu}_{\varphi_1\varphi_1 h}(p_1, -k_1)\Delta_{\mu\nu,\rho\sigma}(q)V^{\rho\sigma}_{\varphi_2\varphi_2 h}(p_2, -k_2)$$

- The resummed propagator:

$$\Delta_{\mu\nu,\rho\sigma} = \Delta_{0\,\mu\nu,\rho\sigma} - \Delta_{0\,\mu\nu,\alpha\beta}\Pi_{\mathrm{R}}^{\alpha\beta,\gamma\delta}\Delta_{0\,\gamma\delta,\rho\sigma} + \Delta_{0\,\mu\nu,\alpha\beta}\Pi_{\mathrm{R}}^{\alpha\beta,\gamma\delta}\Delta_{0\,\gamma\delta,\lambda\kappa}\Pi_{\mathrm{R}}^{\lambda\kappa,\epsilon\zeta}\Delta_{0\,\epsilon\zeta,\rho\sigma} + \cdots$$

- Ward identity:

$$q_\mu \Pi_R^{\mu\nu,\rho\sigma}(q) = 0$$

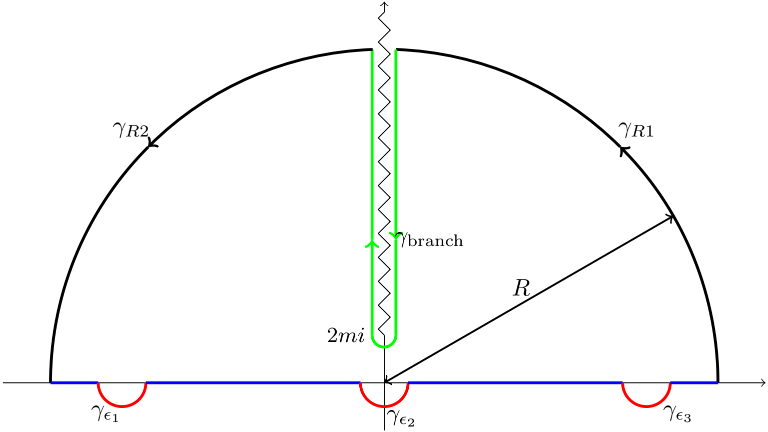
- Non-relativistic limit:

$$\mathcal{M}_{1-\mathrm{loop}}(\vec{q}) = -\kappa^2 m_1^2 m_2^2 \left[\frac{4}{3}\left(\frac{1}{|\vec{q}|^2 + 4F_3(-|\vec{q}|^2)}\right) + \frac{1}{3}\left(\frac{1}{3F_2(-|\vec{q}|^2) + 2F_3(-|\vec{q}|^2) - |\vec{q}|^2}\right)\right]$$

Newtonian potential: Matter quantum corrections

- Convert q to complex variable and evaluate as a contour integral.

$$\int_{-R}^{R} dq \left(\frac{q}{r} e^{iqr} \mathcal{M}_{1-\text{loop}}(q) \right)$$



Newtonian potential: Matter quantum corrections

- Quantum corrected Newtonian potential:

$$V(r) = -\frac{Gm_1 m_2}{r}\Big(\alpha + \Delta V(r)\Big)$$

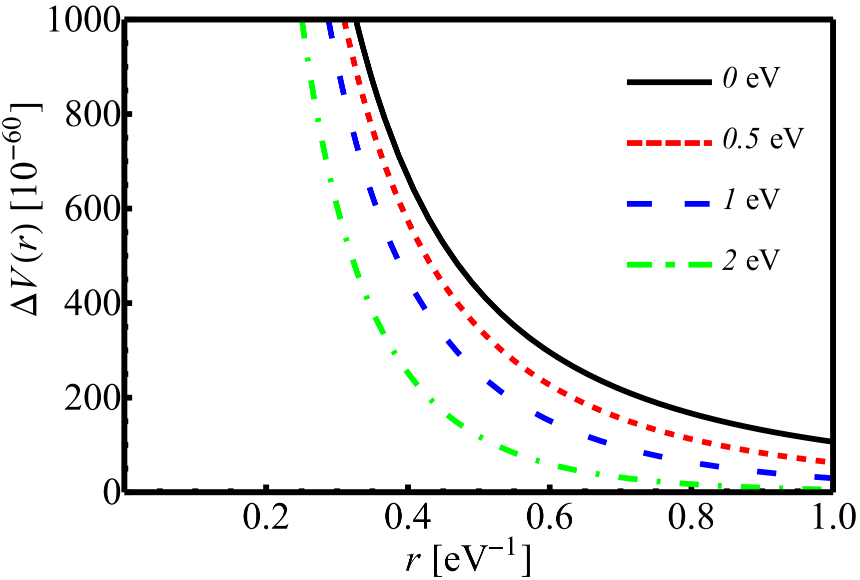
- Newtonian potential is scaled by

$$\alpha = \frac{1}{4}\left[\frac{4}{3}\left(1 - 4\sum_{i=1}^{n} a_i\right)^{-1} - \frac{1}{3}\left(1 + 2\sum_{i=1}^{n} a_i + 3\sum_{i=1}^{n} b_i\right)^{-1}\right],$$

$$a_i = \left.\frac{\partial F_{2,i}(q^2)}{\partial q^2}\right|_{q^2=0}, \qquad b_i = \left.\frac{\partial F_{3,i}(q^2)}{\partial q^2}\right|_{q^2=0}.$$

- The modification for a massive scalar field ($\hat{r}_H = 2m_H r$):

$$\Delta V_H(r) = -\frac{Gm_H^2}{360\pi}\left[\frac{1}{2}\pi\left(7\hat{r}_H^2 - 45\right)\hat{r}_H^2\Big(\boldsymbol{L}_{-1}(\hat{r}_H)K_0(\hat{r}_H) + \boldsymbol{L}_0(\hat{r}_H)K_1(\hat{r}_H)\Big) - \frac{7\pi\hat{r}_H^3}{2} + \frac{45\pi\hat{r}_H}{2}\right.$$
$$\left. + 7\hat{r}_H^3 K_1(\hat{r}_H) - 7\hat{r}_H^2 K_0(\hat{r}_H) - 38\hat{r}_H K_1(\hat{r}_H) + 60K_0(\hat{r}_H) - 36K_2(\hat{r}_H)\right].$$


1000
800
600
400
200
0
0.2
0.4
0.6
0.8
1.0
$\Delta V(r)\ [10^{-60}]$
$r\ [\mathrm{eV}^{-1}]$
0 eV
0.5 eV
1 eV
2 eV

Summary

- Diffeomorphism invariance alone doesn't guarantee that the graviton is massless.
- Graviton mass is only zero after minimisation, renormalisation of the cosmological constant and through the graviton low-energy theorem.
- Shown this explicitly by calculating the graviton self-energy at the one-loop level and showing that it is transverse.
- Used the same work to produce corrections to the Newtonian potential.
- Newtonian potential is rescaled by a factor dependent on the masses of the matter fields of the model.
- Correction to Newtonian potential exhibits an exponential fall-off dependence on the distance r.