Black Holes in Higher-Derivative Gravity

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Introduction

- Black holes are the most fundamental objects in any theory of gravity, and as such they can provide powerful probes for investigating some of the subtle global features of the theory.

- In ordinary Einstein gravity, by itself or coupled to "standard" matter, there are powerful no-hair theorems or uniqueness theorems which imply strong restrictions on the parameter space of possible black-hole solutions. For example, in Einstein-Maxwell theory the most general black hole is characterised by its mass, angular momentum and charge.

- More general theories of gravity comprise Einstein gravity with higher-order curvature terms, such as arise in the low-energy limit of string theory. In string theory, there are an infinite number of higher-order terms, involving arbitrarily large powers of the curvature and its covariant derivatives.

- We may first consider a theory with just a finite number of higher-order terms. Of particular interest is the case of Einstein gravity with additional quadratic curvature terms only, since this is actually renormalisable (Stelle 1977), albeit at the price of having ghosts.

- Perhaps there are ways to live with ghosts (Smilga,...), or otherwise find a regime where quadratic curvatures dominate over yet higher order terms (Starobinsky,...). In any case, since it turns out that one can (semi) explicitly probe some of the detailed properties of black holes in Einstein gravity with quadratic curvatures, it is of interest to do so.

- For simplicity, we consider spherically-symmetric black holes. In four dimensions, unlike higher dimensions, any solution of of Einstein gravity remains a solution when quadratic curvature terms are added. Thus the Schwarzschild black hole is a solution in the quadratic theory. The question is whether there exist any other acceptable spherically-symmetric black-hole solutions.

- For example, can there exist non-standard black holes in a regime where cubic and higher terms can be neglected? If so, these solutions would be representative of new solutions even in string theory.

- As we shall see, there in fact exists a second branch of such black holes in Einstein plus quadratic gravity, but only provided the horizon radius is sufficiently large, and the curvature is large.

- The new features of the higher-derivative theory are associated with additional modes in the spectrum of the theory; the massive spin-2 and spin-0 modes. The new black holes involve condensates of the (ghostlike) massive spin-2 modes.

General Expectations in Quadratic Gravity

- Since the Gauss-Bonnet combination of quadratic curvatures is a total derivative in four dimensions, the most general action for the quadratic theory (without cosmological constant) can be taken to be

$$I = \int d^4x \sqrt{-g}\, (R - \alpha\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta\, R^2)\,.$$

  If one linearises the theory around a Minkoswki background, the fluctuations will obey equations which can be separated into a factorised fourth-order spin-2 equation and a second-order spin-0 equation:

$$\Box(\Box - m_2^2) h_{\mu\nu} = 0\,, \qquad (\Box - m_0^2)\phi = 0\,,$$

  where $m_2^2 = 1/(2\alpha)$ and $m_0^2 = 1/(6\beta)$.

- The massive spin-2 and spin-0 modes will lead to terms with Yukawa-type behaviour $1/r\, e^{\pm mr}$ at large distances. Generically, terms with both signs in the exponential will occur, and the terms with the rising exponentials will give rise to fatal pathological behaviour in the asymptotic region.

- The real question, therefore, is whether it is possible to fine tune the parameters in the general solutions so as to be able to remove the rising Yukawa terms.

- Are there any such fine tunings, aside from Schwarzschild?

A Partial No-Hair Theorem

- We can study the static solutions of the theory by considering metrics of the form $ds^2 = -\lambda^2\,dt^2 + h_{ij}dx^i dx^j$, where λ and h_{ij} depend only on the spatial coordinates x^i.

- The equations of motion for the quadratic theory are

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} - 4\alpha\, B_{\mu\nu} + 2\beta\, R(R_{\mu\nu} - \tfrac{1}{4}Rg_{\mu\nu}) + 2\beta(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R = 0\,,$$

where $B_{\mu\nu} = (\nabla^\rho\nabla^\sigma + \frac{1}{2}R^{\rho\sigma})C_{\mu\rho\nu\sigma}$ is the Bach tensor. Taking the trace gives

$$\beta\,(\Box - m_0^2)R = 0\,.$$

- Multiplying by λR and integrating over the spatial domain outside the putative horizon of the black hole gives

$$\int d^3x\sqrt{h}\left[D^i(\lambda R D_i R) - \lambda(D_i R)^2 - m_0^2\,\lambda R^2\right] = 0\,,$$

where D_i are covariant derivatives in the spatial metric h_{ij}. Since by definition λ goes to zero on the horizon, it follows that if $D_i R$ goes to zero sufficently rapidly at infinity then the surface term gives no contribution and hence the non-positivity of the remaining integrand implies

$$R = 0\,.$$

- This partial no-hair theorem (due to W. Nelson) provides a considerable simplification of the problem. It means we immediately have a second-order equation of motion, and in fact if we use the spherically-symmetric ansatz

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,,$$

  we can reduce the equations of motion to two second-order ODEs for $h(r)$ and $f(r)$.

- Nelson tried to go further, by applying a similar technique to the trace-free part of the equations of motion, and claiming this led to a complete no-hair theorem, namely that $R_{\mu\nu} = 0$. This would have meant that the Schwarzschild metric was the only spherically-symmetric static black hole solution in the quadratic theory.

- However, we found that Nelson had some crucial sign errors in his calculation, and what he had thought to be a strictly non-positive integrand actually had a mix of terms with both signs. Thus no definite conclusion could be reached.

- On the basis that "what is not forbidden is allowed," this raises the possibility of non-Schwarzschild spherically-symmetric static black holes in the quadratic theory. With the simplification of knowing that they must satisfy $R = 0$, this means we can wolog drop the R^2 term in the action and just consider Einstein-Weyl gravity.

Numerical Solution of the Equations

- Although simplified by the $R = 0$ condition, the equations for $h(r)$ and $f(r)$ unfortunately appear not to be analytically solvable, and we therefore resort to numerical analysis.

- We begin by making Taylor expansions of $h(r)$ and $f(r)$ near to a putative horizon at $r = r_0$:

$$\begin{aligned} h &= c\Big[(r-r_0) + h_2\,(r-r_0)^2 + h_3\,(r-r_0)^3 + \cdots\Big]\,, \\ f &= f_1\,(r-r_0) + f_2\,(r-r_0)^2 + f_3\,(r-r_0)^3 + \cdots\,. \end{aligned}$$

The constant c is a trivial, absorbable into a rescaling of the time coordinate. Plugging into the equations of motion, one can solve for all the coefficients h_i and f_i with $i \ge 2$ in terms of the two non-trivial parameters r_0 and f_1. The Schwarzschild solution itself corresponds to $f_1 = 1/r_0$, so if we write

$$f_1 = \frac{1+\delta}{r_0}\,,$$

then non-vanishing δ characterises the extent to which the near-horizon solution deviates from Schwarzschild.

- We can now use the shooting method to construct solutions numerically. Namely, we set initial conditions just outside the horizon, by choosing values for r_0 and δ and making use of the near-horizon Taylor expansions. We then integrate the equations out numerically to large r.

- It is convenient to fix the scale size in the problem by making a choice for the coupling constant α for the Weyl-squared term in the action. We take $\alpha = \frac{1}{2}$. The parameters r_0 and δ are then both non-trivial.

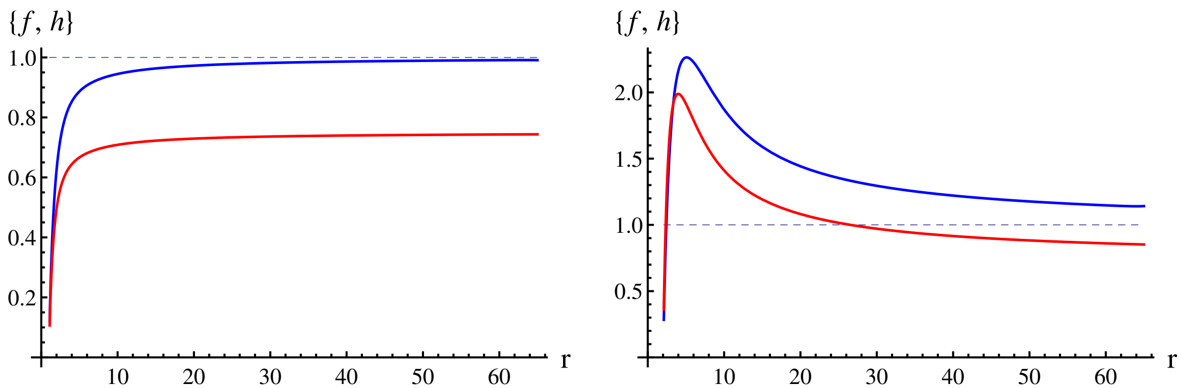
- For generic r_0 and δ, the outward integration runs into a singularity, in which the metric functions $h(r)$ and $f(r)$ rapidly diverge either to $+\infty$ or $-\infty$. By very delicate fine tuning of the parameters, one can systematically extend outwards the limit r_{max} before which the singularity is reached. Increasing the precision allows integrating out further–ad infinitum.

- The functions $h(r)$ and $f(r)$ asymptotically approach constants, with $f(r) \to 1$. The asymptotic constant value for $h(r)$ can be adjusted by choosing the trivial parameter c so that $h(\infty) = 1$. Thus we obtain well behaved asymptotically flat black hole solutions.

- We picked a value for r_0, and then fine tuned δ to get asymptotically Minkowskian behaviour. For any r_0 there is always at least one such solution, with $\delta = 0$ (to within numerical rounding errors), corresponding to the Schwarzschild black hole.

- In addition, if r_0 is greater than a certain minimum value $r_0^{\mathrm{min}} \approx 0.876$, we find that there exists a second choice of $\delta = \delta^*$ that gives a second, non-Schwarzschild, black hole.

- Here are two examples, showing the $f(r)$ (blue) and $h(r)$ (red) metric functions for the non-Schwarzschild black hole, for $r_0 = 1$ (LHS), and $r_0 = 2$ (RHS). In order to avoid an asymptotic overlay of the h and f curves, we have made use of the trivial scaling c so that the function h asymptotically approaches $\frac{3}{4}$ rather than 1.

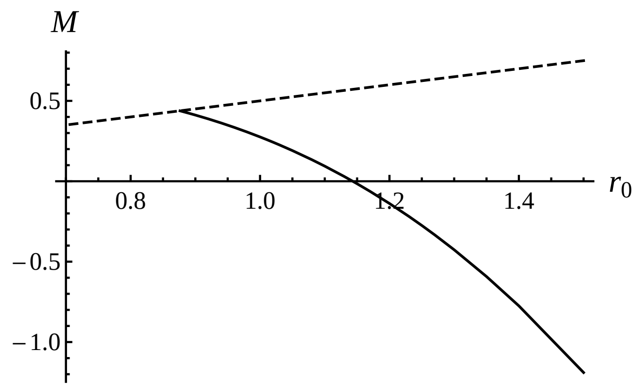


- The metric functions in the $r_0 = 1$ case are looking very like those in Schwarzschild. In the $r_0 = 2$ case they aproach their asymptotic values from above, suggesting negative mass.

Properties of the Non-Schwarzschild Black Holes

- The mass of the non-Schwarzschild black hole is indeed given by the usual ADM formula for asymptotically flat spacetimes, which amounts to $\frac{1}{2}$ the coefficient of the $1/r$ term in g_{tt} (assuming a canonical normalisation for t so that $g_{tt} \to -1$ at infinity).

- The mass of the Schwarzschild (dotted line) and the non-Schwarzschild (solid line) black holes as a function of horizon radius r_0 are shown below:



- The solid curve for the non-Schwarzschild black hole terminates at its left-hand end at $r_0 = r_0^{\text{min}}$. In fact the non-Schwarzschild and Schwarzschild branches coalesce here, at this minimum r_0 for which the non-Schwarzschild branch exists.

Negative Mass?

- The negative mass of the non-Schwarzschild black holes for $r_0 > r_0^{m=0}$ clearly violates the normal behaviour seen for black holes in general relativity, for which the positive energy theorem guarantees the non-negativity of the mass for any system of physically reasonable matter coupled to gravity.
- In fact the behaviour we are seeing here is very like what would happen if we looked at ordinary Schwarzschild black holes in pure Einstein gravity, but with a minus sign in front of the Einstein-Hilbert action. The mass, calculated as a Noether charge for the sign-reversed action, would be negative, and it would become more and more negative as the radius r_0 of the black hole was increased.
- The negativity of the mass for large non-Schwarzschild black holes can be understood as being a consequence of the ghost-like nature of the massive spin-2 modes in quadratic gravity:

$$\frac{m^2}{\Box\,(\Box - m^2)} = \frac{1}{\Box} - \frac{1}{(\Box - m^2)}\,.$$

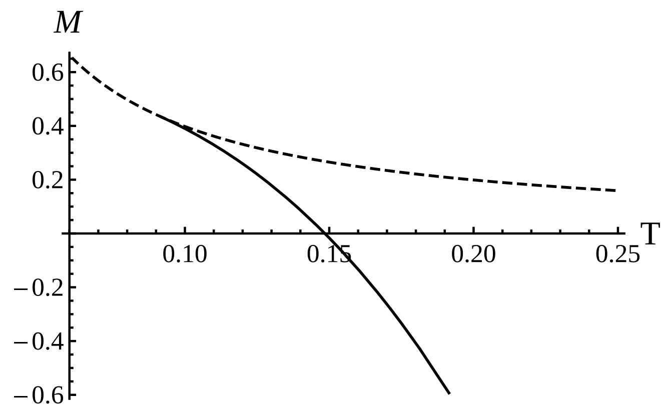
Effectively, we are seeing that whereas a condensation of massless spin-2 gravitons in a normal black hole gives rise to a spacetime with positive energy, a condensation of massive spin-2 modes, which are ghostlike, can give rise to a spacetime with negative energy.

- The non-Schwarzschild black holes form a distinct branch that only meets the Schwarzschild branch at $r_0 = r_0^{\rm min}$. They have positive mass only when r_0 is in the interval

$$0.876 \approx r_0^{\rm min} \le r_0 \le r_0^{m=0} \approx 1.143\,.$$

- For r_0 just a little greater than $r_0^{\rm min}$, the non-Schwarzschild black hole is perturbatively close to the Schwarzschild black hole. Apart from this case, the non-Schwarzschild black holes cannot in general be obtained by a linearised analysis around Schwarzschild.

- By plotting the mass versus the Hawking temperature for the non-Schwarzschild and Schwarzschild black holes, we can see that they both have negative specific heat $C = \partial M/\partial T$, but that of the non-Schwarzschild black hole (solid line) is always more negative, at any given mass:



Conclusions

- Even though Einstein plus quadratic gravity generically has rising as well as falling Yukawa terms in the asymptotic solutions, one can fine tune the parameters in static spherically symmetric solutions and thereby find a second branch of solutions, over and above Schwarzschild. They exist for $r_0 > 0.876\sqrt{2\alpha}$, with positive mass for $r_0 < 1.143\sqrt{2\alpha}$.

- Knowing what the black hole solutions in the theory are allows us to see what one should or should not try to prove analytically.

- For example, it is pointless to try to prove in general that non-Schwarzschild black holes can't exist in Einstein + quadratic gravity.

- It also shows that black holes can exist that are not perturbatively close to Schwarzschild. They have masses $M \le 0.438\sqrt{2\alpha}$. Thus if α is "small," their masses are small and the curvature is large on the horizon, implying cubic and higher curvatures would generically be equally important.

- It seems that one ought to be able to prove the non-existence of non-Schwarzschild black holes whose curvatures are small enough to make the neglect of yet higher-order curvature terms legitimate. If true, this could mean that a no-hair theorem might in fact hold for black holes in string theory.

Final Remarks...

- I came to Texas A&M in 1988, entirely because of Dick Arnowitt's vision for building up a high-energy theory group.
- In the earlier years Dick's guiding hand played a crucial role in the development and the functioning of the group.
- In later years, his wisdom and common sense was invaluable, and will be much missed.
- He had a dry and sardonic sense of humour. I recall a faculty meeting in the early days when a dean or provost was addressing us, and extolling the virtues of branching out into "interdisciplinary research." Knowing that he was speaking to physicists, he reinforced his point by observing: "If you take contributions N and N and multiply them together you get N^2..."
- ...Immediately, speaking *sotto voce*, and audible only to those of us nearby, Dick made the remark: "and if you take ϵ and ϵ and multiply them together, you get ϵ^2."
- Thank you, Dick, for your legacy of immortal contributions to physics, for your insights, and for your humour!