Symmetric solitonic excitations of the (1+1)-dimensional Abelian-Higgs “classical vacuum” and the fermionic sector

C.E. Tsagkarakis

In collaboration with

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Contents

Introduction

- Classical solutions may describe sufficiently the effective dynamics of nonlinear quantum theories.
- **Solitons** give rise to particle like structures in nonlinear field theories, so they are relevant for the phenomenological description of a wide class of physical systems ranging from elementary particles to superconductors and Bose-Einstein condensates.
- **Final goal**: The study of a $SU(2) - Higgs - fermions$ model.
- In order to achieve our goal we fist study a simpler model which is the $(1+1)$-dimensional Abelian-Higgs-fermions.
- The Abelian-Higgs model in $(1+1)$ dimensions shares distinguished ground as it may reveal important features of superconductivity such as the Meissner effect.
  - The appearance of the condensate spontaneously breaks the U(1) symmetry, giving rise to a finite mass to the gauge field .

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Formulation and Equations of Motion

- The Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\Phi^*\Phi) + \bar{\tilde{\psi}}(i\not{D} - m_F)\tilde{\psi},$$

where $F_{\mu\nu}$ is the $U(1)$ field strength tensor and $D_\mu = \tilde{\partial}_\mu + ie\tilde{A}_\mu$, $V(\Phi^*\Phi) = \mu^2\Phi^*\Phi + \lambda(\Phi^*\Phi)^2$.

- In the case that $\lambda > 0$ while $\mu^2 < 0$, and in the standard gauge selection (Unitary gauge), we expand Φ around its vev $\tilde{v}/\sqrt{2}$ $\Phi = \frac{1}{\sqrt{2}}(\tilde{v} + \tilde{H})$, where $\tilde{v}^2 = -\mu^2/\lambda$

- We obtain the following equations of motion for $\tilde{A}_\mu$, $\tilde{H}$ and $\tilde{\psi}$

$$\begin{aligned}
(\tilde{\Box} &+ m_A^2)\tilde{A}_\mu - \tilde{\partial}_\mu(\tilde{\partial}_\nu\tilde{A}^\nu) + 2e^2\tilde{v}\tilde{H}\tilde{A}_\mu + e^2\tilde{H}^2\tilde{A}_\mu - e\bar{\tilde{\psi}}\gamma_\nu\tilde{\psi} = 0\\
(\tilde{\Box} &+ m_H^2)\tilde{H} + 3\lambda\tilde{v}\tilde{H}^2 + \lambda\tilde{H}^3 - e^2\tilde{A}_\mu\tilde{A}^\mu(\tilde{v} + \tilde{H}) = 0\\
(i\tilde{\not{\partial}} &- m_F)\tilde{\psi} - e\gamma^\mu\tilde{A}_\mu\tilde{\Psi} = 0
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where $m_A^2 = e^2\tilde{v}^2$, $m_H^2 = 2\lambda\tilde{v}^2$.

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Equations of Motion

- We choose the field representation to be $\tilde{A}_0 = \tilde{A}_1 = \tilde{A}_3 = 0$, and $\tilde{A}_2 = \tilde{A} \neq 0$.
- The non vanishing $\tilde{A}_2$ field is a function solely of $(\tilde{x}, \tilde{t})$ and as a result the Lorentz condition $\partial_\nu \tilde{A}^\nu = 0$ is fulfilled automatically.
- Equations in a dimensionless form by rescaling the fields: $\tilde{A} = (m_A/e)A, \tilde{H} = (m_A/e)H, \tilde{\psi} = (m_A^{3/2}/e)\psi, \tilde{\upsilon} = (m_A/e)\upsilon$ and space-time coordinates as: $\tilde{x} = x/m_A, \tilde{t} = t/m_A$.
- The equations of motion are reduced to

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\begin{aligned}
(\Box &+ 1)A + 2AH + H^2A + \bar{\psi}\gamma^2\psi = 0 \\
(\Box &+ q^2)H + \frac{1}{2}q^2H^3 + \frac{3}{2}q^2H^2 + HA^2 + A^2 = 0 \\
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Multiscale expansion

- We employ the **Multiple Scale Perturbation Theory (MSPT)**, which uses a formal **small parameter** $0 < \epsilon \ll 1$.
- We expand the space-time coordinates to a set of **independent variables** and their derivatives:
  $x_0 = x,\ x_1 = \epsilon x,\ x_2 = \epsilon^2 x, \ldots, \quad t_0 = t,\ t_1 = \epsilon t,\ t_2 = \epsilon^2 t, \ldots,$
  $\partial_x = \partial_{x_0} + \epsilon \partial_{x_1} + \ldots, \quad \partial_t = \partial_{t_0} + \epsilon \partial_{t_1} + \ldots.$
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  - The gauge and the scalar field amplitudes are of the same order.
  - This scenario corresponds to a strong breaking of the underlying gauge symmetry, far beyond the related critical point.
  - The minimum of the potential occurs at the bottom of a deep well, while the potential shape is almost symmetric.

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Ordering...

- The equations of motion for the fields reduce to the following system of equations up to $\mathcal{O}(\epsilon^3)$
- In the **first order** $\mathcal{O}(\epsilon)$:

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\begin{aligned}
(\Box_0 &+ 1)A^{(1)} = 0, \quad A^{(1)} = fe^{it_0} + f^* e^{-it_0} \\
(\Box_0 &+ q^2)H^{(1)} = 0, \quad H^{(1)} = le^{iqt_0} + l^* e^{-iqt_0} \\
(i\partial\!\!\!/_0 &- q_F)\psi^{(1)} = 0, \quad \psi^{(1)} = aw_1 e^{-iq_F t_0} + bw_2 e^{-iq_F t_0}
\end{aligned}
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where $f = f(x_i, t_i)$,$l = l(x_i, t_i)$,$a = a(x_i, t_i)$ and $b = b(x_i, t_i)$ are functions of the slow variables that have to be determined (the index $i = 1, 2, \ldots$ refers to the slow scales), while $w_1 = (1, 0, 0, 0)^T$ and while $w_2 = (0, 1, 0, 0)^T$.

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- In the **second order** $\mathcal{O}(\epsilon^2)$:

$$(\Box_0 + 1)\, A^{(2)} = -2\partial_{\mu_0}\partial^{\mu_1} A^{(1)} - 2H^{(1)}A^{(1)} - \bar{\psi}^{(1)}\gamma^2\psi^{(1)}$$

$$(\Box_0 + q^2)H^{(2)} = -2\partial_{\mu_0}\partial^{\mu_1} H^{(1)} - \left(\frac{3q^2}{2}H^{(1)2} + A^{(1)2}\right)$$

$$(i\partial\!\!\!/_{\mu_0} - q_F)\psi^{(2)} = -i\partial\!\!\!/_{\mu_1}\psi^{(1)} + \gamma^2 A^{(1)}\psi^{(1)}$$

  - **Solvability Conditions**: $\partial_{\mu_0}\partial^{\mu_1} A^{(1)} = 0$, $\partial_{\mu_0}\partial^{\mu_1} H^{(1)} = 0$, $\gamma^0\partial_{t_1}\psi^{(1)} = 0$ as these terms are **Secular terms**, i.e.in resonance with the operators on the left side implying a **linear growing** of $A^{(2)}$, $H^{(2)}$,$\psi^{(2)}$ with time which makes **the perturbation scheme invalid.**
  - The condition is satisfied by choosing $f = f(x_1, t_2)$ and $l = l(x_1, t_2)$ and $a = a(x_1, t_2), b = (x_1, t_2)$
  - Notice that $\bar{\psi}^{(1)}\gamma^2\psi^{(1)} = 0$ i.e. there is no fermionic current in this order.

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Second order solutions...

- **For the gauge field**: $A^{(2)} = \frac{f \cdot l}{d} e^{i\Theta_+} + \frac{f \cdot l^*}{s} e^{i\Theta_-} + c.c.$
  where $d = q + q^2/2$, $s = -q + q^2/2$ and $\Theta_\pm \equiv 1 \pm q$
- **For the Higgs field**:
  $H^{(2)} = \frac{l^2}{2} e^{-2iqt_0} + \frac{f^2}{4-q^2} e^{-2it_0} - \frac{3q^2|l|^2+2|f|^2}{q^2} + c.c.$
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  $$\psi^{(2)} = i\big(F_1 w_4 e^{-iq_F t_0} + F_2 w_4 e^{-i(1+q_F)t_0} + F_3 w_4 e^{+i(1-q_F)t_0} + \\ + F_4 w_3 e^{-iq_F t_0} + F_5 w_3 e^{-i(1+q_F)t_0} + F_6 w_3 e^{+i(1-q_F)t_0}\big)$$
  where $F_1 = -\partial_{x_1} a/2q_F, F_2 = f \cdot a/1 + 2q_F, F_3 = f^* \cdot a/2q_F - 1$
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- In the **third order** $\mathcal{O}(\epsilon^3)$:
  For the gauge field:

$$(\Box_0 + 1) A^{(3)} = -2\partial_{\mu_0}\partial^{\mu_1} A^{(2)} - (\Box_1 + 2\partial_{\mu_0}\partial^{\mu_2}) A^{(1)}$$

$$-2\left(H^{(2)}A^{(1)} + A^{(2)}H^{(1)}\right) - H^{(1)2}A^{(1)} - \left(\bar{\psi}^{(1)}\gamma^2\psi^{(2)} + \bar{\psi}^{(2)}\gamma^2\psi^{(1)}\right).$$

For the Higgs field:

$$\left(\Box_0 + q^2\right) H^{(3)} = -2\partial_{\mu_0}\partial^{\mu_1} H^{(2)} - (\Box_1 + 2\partial_{\mu_0}\partial^{\mu_2}) H^{(1)}$$

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The system of CNLS equations

- Collecting all **secular terms** **the solvability condition** reduces to system of CNLS equations for the functions $f(x_1, t_2)$, $l(x_1, t_2)$, $a(x_1, t_2)$, $b(x_1, t_2)$:

$$
\begin{aligned}
i\partial_{t_2} f &= -\frac{1}{2}\partial_{x_1}^2 f + g_{11}|f|^2 f + g_{12}|l|^2 f + \frac{2q_F}{4q_F^2 - 1}(|a|^2 + |b|^2) f, \\
iq\partial_{t_2} l &= -\frac{1}{2}\partial_{x_1}^2 l + g_{21}|f|^2 l + g_{22}|l|^2 l, \\
i\partial_{t_2} a &= -\frac{1}{2q_F}\partial_{x_1}^2 a + \frac{4q_F}{4q_F^2 - 1}|f|^2 a, \\
i\partial_{t_2} b &= -\frac{1}{2q_F}\partial_{x_1}^2 b + \frac{4q_F}{4q_F^2 - 1}|f|^2 b.
\end{aligned}
$$

where

$$
\begin{aligned}
g_{11} &= -\left(\frac{2}{q^2} + \frac{1}{q^2 - 4}\right), \quad g_{12} = -\left(2 - \frac{4}{q^2 - 4}\right) \\
g_{21} &= g_{12}, \qquad g_{22} = -3q^2.
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The modulation instability

- We explore the impact of **modulation instability (MI)** mechanism to the solution space of our model by examining the stability of plane wave solutions of the CNLS.
- MI reveals the localized structures that the system supports i.e. instability of plane waves leads to unstable background for $\tanh -$*shaped* solutions.
- We consider the following ansatz:

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\begin{aligned}
f(x_1, t_2) &= (f_0 + \delta f)\, e^{-i\Omega_f t_2}, \\
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- The frequencies $\Omega_{f,l,a,b}$ satisfy the dispersion relations:

$$\begin{aligned}
\Omega_f &= g_{11}|f_0|^2 + g_{12}|l_0|^2 + \frac{2q_F}{4q_F^2-1}(|a_0|^2+|b_0|^2) \\
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- The small amplitude perturbations are complex functions of the form $\delta f = u_f + iv_f$, $\delta l = u_l + iv_l$, $\delta a = u_a + iv_a$ and $\delta a = u_a + iv_a$.
- The **real** functions u_j, v_j are considered to be of the general form:

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- Substituting to the CNLS leads to an homegenous algebraic system of eight equations, the determinant of which has to be zero.
- This compatibility condition leads to the following equation:

$$B\Omega^8 + C\Omega^6 + D\Omega^4 + E\Omega^4 + G\Omega^4 + L = 0,$$

where the coefficients B, C, D, G, L are products of g_{ij},q_F, K and the amplitudes of the plane waves.
- Requiring real roots for the above equation we are led to the following stability condition $g_{22} > 0$, which cannot be satisfied for any real value of the parameter q
- So plane wave solutions are **unstable** which implies that $\tanh -$*shaped* **solutions prove to be unstable.**
- Thus we argue that localized solutions in the form of **kinks**, are not supported in the setting where A and H are of the same order,in contrast to **Oscillons**.

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Oscillons

- The CNLS system admits analytical soliton solutions.
- Due to the fact that tanh $-$*shaped* solutions prove to be unstable, we look for **bright soliton solutions** for all the fields using the following ansatz:

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\begin{aligned}
f &= f_0 sech\left(x_1\right) e^{-i\nu_f t_2} \\
l &= l_0 sech\left(x_1\right) e^{-i\nu_l t_2} \\
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- The requirement for the squared amplitudes to be positive defines regions for the parameters q and q_F:

$$f_0^2 = (1 - 4q_F^2)/4q_F^2 > 0 => q_F > 1/2$$

$$a_0^2 + b_0^2 = (1 + f_0^2 g_{11} + g_{12} lo^2)(-1 + 2q_F)(1 + 2q_F)/2q_F > 0$$

$$l_0^2 = (-1 - f_0^2 g_{21})/g_{22} > 0$$

- These inequalities along with the fact that the mass of condesate coresponds to the mass of the Higgs field and it is given as $m_H = 2m_F$ which leads to the relation $q = 2q_F$ set the restriction:

$$0.76 < q < 1$$

Phenomelogy of superconductors

- In the case of **Oscillons**, a **magnetic** and an **electric** field originate from the gauge field A which was chosen to be in the $\hat{y}$ direction:

$$\vec{B}(x,t) = \partial_x A(x,t)\hat{z}, \quad \vec{\mathcal{E}}(x,t) = -\partial_t A(x,t)\hat{y}$$

.

- The above equations, describe the electric field in the y direction which produces a magnetic field in the z direction, while both fields are localized around the origin of the x axis resembling the **Meissner** effect.
- Finally it is important to notice the fermionic sector is also localized around the origin of the x axis implying that in the superconductor the are no free fermions as they are condensed.

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- The above equations, describe the electric field in the y direction which produces a magnetic field in the z direction, while both fields are localized around the origin of the x axis resembling the **Meissner** effect.
- Finally it is important to notice the fermionic sector is also localized around the origin of the x axis implying that in the superconductor the are no free fermions as they are condensed.

SU(2)-Higgs Model

- The Lagrangian $SU(2)$-Higgs field dynamics is described by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} \quad + \quad (D_\mu \phi)^\dagger (D^\mu \phi) - V(\Phi^\dagger \Phi),$$

where $F^a_{\mu\nu}$ is the $SU(2)$ field strength tensor and

$$D_\mu = \partial_\mu + igA^a_\mu \frac{\sigma^a}{2}, \qquad V(\Phi^\dagger \Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$$

- In the the case $\lambda > 0$ while $\mu^2 < 0$, so in the standard gauge selection (Unitary gauge), we expand Φ

$$\Phi = \left(0, \frac{1}{\sqrt{2}}(v + H)\right)^T \quad , v^2 = -\frac{\mu^2}{\lambda}$$

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- Emploing the **MSPT** the gauge fields are introduced as:

$$A_1^1 = A_2^2 = A_3^3 = A = \mathcal{O}(\epsilon) \;\; ; \;\; A_0^1, A_0^2, A_0^3 = \mathcal{O}(\epsilon^2)$$
$$A_2^1, A_3^1, A_1^2, A_3^2, A_1^3, A_2^3 = \mathcal{O}(\epsilon^\nu), \qquad \nu \geq 3,$$

- Following the same procedure the equations of motions are reduced to:

$$(\Box + 1)A + 2AH + H^2 A + \frac{8}{3}A^3 = 0$$
$$(\Box + q^2)H + \frac{1}{2}q^2 H^3 + \frac{3}{2}q^2 H^2 + HA^2 + A^2 = 0$$

where $q = m_H/m_A$.

- $\underline{1^{st} \text{ case}}$: **Strong Higgs scenario.**

$$H = \mathcal{O}(\epsilon) \qquad , \qquad H(1) \neq 0$$

- We end up to a system of two CNLS equations:

$$\begin{aligned} i\partial_{t_2} f &= -\frac{1}{2}\partial_{x_1}^2 f + g_{11}|f|^2 f + g_{12}|l|^2 f, \\ iq\partial_{t_2} l &= -\frac{1}{2}\partial_{x_1}^2 l + g_{21}|f|^2 l + g_{22}|l|^2 l, \end{aligned}$$

where $g_{ij} \equiv g_{ij}(q)$ are the following functions of q:

$$\begin{aligned} g_{11} &= 4 - \frac{2}{q^2} - \frac{1}{q^2-4}, \; g_{12} = -2 + \frac{2}{q(q-2)} + \frac{2}{q(q+2)} \\ g_{21} &= g_{12}, \qquad\qquad\qquad\qquad g_{22} = -3q^2. \end{aligned}$$

- Due to modulation instability only bright-bright soliton solutions for the gauge and the Higgs field may be stable.
- For $0.53 < q < 1.63$ and $1.88 < q < 2$

$$f_{bb} = a_1 \mathrm{sech}\left(\beta_{bb} x_1\right) e^{-i\nu_1 t_2},$$
$$l_{bb} = a_2 \mathrm{sech}\left(\beta_{bb} x_1\right) e^{-i\nu_2 t_2},$$

- These solutions will be set as initial conditions to the initial system of equations of motion in order to investigate their stability and longevity.

- 2^{nd} case: **Weak Higgs scenario.**

$$H = \mathcal{O}(\epsilon^2) \qquad , \qquad H(1) = 0$$

- We derive the following **NLS equation** for gauge field:

$$i\partial_{t_2} f + \frac{1}{2}\partial_{x_1}^2 f + s|f|^2 f = 0$$

where $s = -4 + \frac{2}{q^2} + \frac{1}{q^2-4}$

The soliton solutions

- 1^{st} case: $s > 0$ i.e. $0 < q < 0.68$ or $2 < q < 2.07$.
  Oscillons:
$$f = f_0 \mathrm{sech}\left(\sqrt{|s|}u_0 x_1\right) e^{-i\omega t_2}, \tag{3.1}$$
  where f_0 is a free parameter characterizing the amplitude of the soliton, while the soliton frequency is $\omega = -1/2 f_0^2 s$.
- The approximate solutions for A and H can be expressed in terms of the original variables x and t as follows:

$$A \approx 2\epsilon f_0 \mathrm{sech}\left(\epsilon f_0 \sqrt{|s|} x\right) \cos\left[\left(1 - \frac{1}{2}(\epsilon f_0)^2 s\right) t\right], \tag{3.2}$$

$$H \approx (\epsilon f_0)^2 \mathrm{sech}^2\left(\epsilon f_0 \sqrt{|s|} x\right) \times \left\{\frac{-2}{q^2} - \frac{1}{q^2 - 4} 2\cos\left[2\left(1 - \frac{1}{2}(\epsilon f_0)^2 s\right) t\right]\right\},$$

We note that oscillons acquire **vanishing** values at $|x| \to \infty$.

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- 2^{nd} case: $s < 0$ i.e. $0.68 < q < 2$ and $q > 2$.
  Oscillating kinks:

$$f = f_0 \tanh\left(\sqrt{|s|} f_0 x_1\right) e^{-i\omega t_2},$$

  while the soliton frequency now is $\omega = f_0^2 |s|$.
- The approximate solutions for A and H in this case are :

$$\begin{aligned} A &\approx 2\epsilon f_0 \tanh\left(\sqrt{|s|}\epsilon f_0 x\right) \cos\left[\left(1 + (\epsilon f_0)^2 |s|\right) t\right], \qquad (3.3) \\ H &\approx (\epsilon f_0)^2 B \tanh^2\left(\sqrt{|s|}\epsilon f_0 x\right) \\ &\times \left\{\frac{-2}{q^2} - \frac{1}{q^2 - 4} \cos\left[2\left(1 + \epsilon^2 f_0^2 |s|\right) t\right]\right\}. \qquad (3.4) \end{aligned}$$

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Discussion and conclusions

- **Gauge theories,both Abelian and Non-Abelian with spontaneous broken symmetry have a common type of solutions, which are the solitonic NLS solutions.**