All-loop non-Abelian Thirring model

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based on works with
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Introduction and motivation

Integrable models

- Consider a classical integrable seed theory, like a Hamiltonian system or a CFT.
- A classically integrable Hamiltonian 1d system, its eom can be recast to:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, M]\,, \qquad I_n = \mathrm{Tr}\, L^n\,, \qquad \{I_n, I_m\} = 0\,, \qquad n = 1, \ldots, N,$$

  where (L, M) form the Lax pair.
- Deforming the theory and keeping integrability is far from trivial.
- The larger number of deformation parameters, the difficulty rises exponentially.

Introduction and motivation

Exact β-functions

- In a renormalizable field theory, its quantum behaviour is depicted by:
  1. The n-point correlation functions.
  2. The dependence of the coupling with the energy scale.
- Their dependence is encoded within the RG flow equations (1st order non-linear):

$$\beta_\lambda := \mu \frac{d\lambda}{d\mu} ,$$

  which is usually determined perturbatively; finite number counterterms might be needed.

- Can we obtain an all-loop β-function and effective action, resuming all the counterterms?
- If this is feasible then we can discover new fixed points towards the IR.

We study some of these aspects for the non-Abelian bosonized Thirring model.

SYNOPSIS

- Elements of the non-Abelian Thirring model: fermionic and bosonized
- The resumed action:
  - Symmetries
  - Constraints on the (desired) β-function
- Derivation of the all-loop isotropic β-function and its properties
- The anisotropic $SU(2)$ case and the Lagrange and Darboux–Halphen systems
- Discussion & Outlook

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Fermionic model

Exactly solvable QFT describing self-interacting massless Dirac fields in 1+1 dimensions.

- An 1+1 dimensional action with fermions in the fundamental representation of $SU(N)$

  $$\text{Dashen–Frishman 73\&75}: \quad \mathcal{L}_{int} = -\frac{g_B}{2} J_\mu J^\mu - \frac{g_V}{2} J^a_\mu J^{a\mu}, \quad \mu = 0, 1,$$

  where $J^a_\mu = \bar{\Psi} t^a \gamma_\mu \Psi$, with $a = 1, \ldots, N^2 - 1$, are the $SU(N)$ currents and J_μ the $U(1)$.

- For $N = 1$ we recover the Abelian case (prototype) Thirring 58.

- It is invariant under $SU(N) \times U(1)$ (vector) and $U(1)_{\text{Axial}}$.

- The non-Abelian term breaks $SU(N)_{\text{Axial}}$, i.e. $\partial^\mu J^{5a}_\mu = g_V f_{abc} J^{b\mu} J^{5c}_\mu$.

- The theory is scale-invariant only for $g_V = 0$ and $g_V = \frac{4\pi}{n+1}$.

- There is a current algebra at level one $J^a_\pm(z) J^b_\pm(0) = \frac{\delta_{ab}}{z^2} + \frac{f_{abc} J^c_\pm(0)}{z}$.

Bosonized version

The bosonized non-Abelian Thirring model is described by: $S = S_0 + k\,\frac{\lambda_{ab}}{\pi}\int J^a_+\,J^b_-$.

Where S_0 is a CFT, with left-right level $k \in \mathbb{N}^*$ where:

$$J^a_\pm(z)J^b_\pm(0) = \frac{1}{k}\,\frac{\delta_{ab}}{z^2} + \frac{f_{abc}\,J^c_\pm(0)}{z}\,,$$
$$J^a_+ = -i\,\mathrm{Tr}(t^a\,\partial_+ g\,g^{-1})\,,\quad J^a_- = -i\,\mathrm{Tr}(t^a\,g^{-1}\,\partial_- g)\,,\quad D_{ab} = \mathrm{Tr}(t^a g t^b g^{-1})\,.$$

Examples of S_0: WZW or free fermion theory with currents realised in the quark representation.

Consider S_0 been the WZW action Witten 83:

$$S_{\mathrm{WZW},k}(g) = -\frac{k}{2\pi}\int \mathrm{Tr}\left(g^{-1}\partial_+ g\,g^{-1}\partial_- g\right) + \frac{ik}{6\pi}\int_B \mathrm{Tr}\left(g^1\mathrm{d}g\right)^3\,,$$

invariant under the left-right current algebra symmetry: $g \mapsto \Omega(\sigma_+)\,g\,\Omega(\sigma_-)$.

Bosonized version

Symmetries of the model

- The left-right current algebra symmetry breaks down completely for a generic matrix λ.
- It is invariant under the generalized parity symmetry:

$$\lambda \mapsto \lambda^T \,, \qquad g \mapsto g^{-1} \,, \qquad \sigma^{\pm} \mapsto \sigma^{\mp} \,.$$

Quantum aspects of the model

- The model is not conformal; the perturbation is not exactly marginal. The all-loop RG

  Kutasov 89 : $\lambda_{ab} = \lambda\delta_{ab} \,, \quad \mu \frac{d\lambda}{d\mu} = -\frac{1}{k} \frac{c_G \lambda^2}{2(1+\lambda)^2} \,, \quad f_{acd} f_{bcd} = c_G \, \delta_{ab} \,.$

  For general symmetric couplings λ_{ab}, see: Gerganov–LeClair–Moriconi 01
- The corresponding effective action is invariant under the inversion of the coupling:

  Kutasov 89 : $\qquad \lambda \mapsto \lambda^{-1} \,, \qquad k \mapsto -k \,, \qquad k \gg 1 \,.$

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THE RESUMMED ACTION

By a gauging procedure we can construct the following action Sfetsos 13

$$S_{k,\lambda}(g) = S_{WZW,k} + \frac{k}{\pi}\int J_+^a \left(\lambda^{-1} - D^T\right)^{-1}_{ab} J_-^b \,.$$

These models interpolate between a CFT and a σ-model whose target space is a group manifold.

Symmetries

- For $\lambda_{ab} \ll 1$ we get the non-Abelian Thirring model $S = S_0 + k\,\frac{\lambda_{ab}}{\pi}\int J_+^a J_-^b$.
- It is also invariant under the generalized parity symmetry:

$$\lambda \mapsto \lambda^T\,, \qquad g \mapsto g^{-1}\,, \qquad \sigma^{\pm} \mapsto \sigma^{\mp}\,.$$

- Weak-strong duality:

$$S_{-k,\lambda^{-1}}(g^{-1}) = S_{k,\lambda}(g)\,.$$

THE RESUMMED ACTION

Constraints on the β-function

Assuming that the β function at one-loop in $1/k$ takes the form: $\beta_\lambda = \mu \frac{d\lambda}{d\mu} = -\frac{1}{k} f(\lambda)$.

- Let's consider the isotropic case $\lambda_{ab} = \lambda\, \delta_{ab}$.
- From CFT perturbations we expect that:

$$f(\lambda) \simeq \frac{1}{2} c_G \lambda^2 + \mathcal{O}(\lambda^3).$$

- Due to the weak–strong duality we have the constraint:

$$\lambda^2 f(\lambda^{-1}) = f(\lambda)\,.$$

Let us now compute $f(\lambda)$.

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DERIVING THE β-FUNCTIONS

Consider a 1+1-dimensional non-linear σ-model with action

$$S = \frac{1}{2\pi\alpha'} \int (G_{\mu\nu} + B_{\mu\nu}) \partial_+ X^\mu \partial_- X^\nu \, .$$

The one-loop β-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ read:
Ecker–Honerkamp 71, Friedan 80, Braaten–Curtright–Zachos 85

$$\mu \frac{\mathrm{d}G_{\mu\nu}}{\mathrm{d}\mu} + \mu \frac{\mathrm{d}B_{\mu\nu}}{\mathrm{d}\mu} = R^-_{\mu\nu} + \nabla^-_\nu \xi_\mu \, ,$$

where the last term corresponds to field redefinitions (diffeomorphisms).

Generalities:

- The Ricci tensor and the covariant derivative includes torsion, i.e. $H = \mathrm{d}B$.
- The σ-model is renormalizable within the zoo of metrics and 2-forms.
- It is not given that the RG flows will retain the form at hand of $G_{\mu\nu}$ and $B_{\mu\nu}$.

ISOTROPIC CASE

It turns out that the RG flow retains the form of the σ-model, the coupling λ is flowing.

The β-function reads: Itsios–Sfetsos–KS (2014)

$$\beta_\lambda = \mu \frac{d\lambda}{d\mu} = -\frac{c_G \lambda^2}{2k(1+\lambda)^2}, \qquad 0 \leqslant \lambda \leqslant 1, \qquad \text{and } k \text{ does not flow}.$$

Properties of the flow

- It behaves accordingly around $\lambda \ll 1 \Longrightarrow \beta_\lambda \simeq -\frac{c_G \lambda^2}{2k} + \mathcal{O}(\lambda^3)$.
- It is invariant under the weak–strong duality, i.e. $\lambda \mapsto \lambda^{-1}, \quad k \mapsto -k$.
- The β-function can be solved explicitly:

$$\lambda - \lambda^{-1} + 2\ln\lambda = -\frac{c_G}{2k}(t - t_0),$$

where UV at $\lambda \to 0$ and IR at $\lambda \to 1^-$.

$$S = S_{WZW,k} + k\,\frac{\lambda}{\pi}\int J_+^a J_-^a \Leftrightarrow S_{k,\lambda}(g) = S_{WZW,k} + \frac{k}{\pi}\int J_+^a \left(\lambda^{-1} - D^T\right)^{-1}_{ab} J_-^b$$

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The anisotropic $SU(2)$ case

Beyond the isotropic case

Consider the $SU(2)$ case and $\lambda_{ab} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, then the RG flows read

$$\mu \frac{\mathrm{d}\lambda_1}{\mathrm{d}\mu} = -\frac{2}{k} \frac{(\lambda_2 - \lambda_1\lambda_3)(\lambda_3 - \lambda_1\lambda_2)}{(1-\lambda_2^2)(1-\lambda_3^2)}, \qquad \text{and cyclic in 1,2,3}.$$

Properties

- In agreement with the literature LeClair–Sierra 04.
- For small coupling $\lambda_i \ll 1$, we get the Lagrange system:

$$\mu \frac{\mathrm{d}\lambda_1}{\mathrm{d}\mu} = -\frac{2}{k} \lambda_2\lambda_3 + \mathcal{O}(\lambda^3).$$

- For couplings around one we get the Darboux–Halphen system:

$$\mu \frac{\mathrm{d}x_1}{\mathrm{d}\mu} = \frac{x_1^2 - (x_2 - x_3)^2}{2\, x_2\, x_3} + \mathcal{O}\left({}^1\!/_k\right), \qquad \lambda_i = 1 - \frac{x_i}{k}, \qquad k \gg 1.$$

- These were studied by Lagrange 1788, Halphen 1881 and they admit a Lax pair formulation Takhtajan 92. What about the interpolating system?

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Based on symmetries and RG flows we conjectured our resummed action

$$S_{k,\lambda}(g) = S_{WZW,k} + \frac{k}{\pi}\int J_+^a \left(\lambda^{-1} - D^T\right)^{-1}_{ab} J_-^b \,.$$

enraptures the all-loop anisotropic Thirring model $S = S_{WZW,k} + k\frac{\lambda_{ab}}{\pi}\int J_+^a J_-^b$.

Extra properties

- The model turns to be classically integrable for a number special cases:
  1. Semi-simple group with isotropic coupling
     Sfetsos 13, Itsios–Sfetsos–KS–Torrieli 14
  2. Symmetric cosets with isotropic coupling
     Hollowood–Miramontes–Schmidtt 13
  3. $SU(2)$ case and diagonalizable λ_{ab} Sfetsos–KS 14.
- Type-II supergravity embedding with non-trivial RR fluxes Sfetsos–Thompson 14

New fixed points??? CFT with different left-right levels could do the trick.