# High Energy Thresholds in SUSY GUTs

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# Outline

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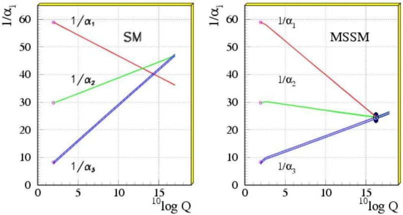
In supersymmetric Grand Unified Theories, **SUSY GUTs**:

- The three fundamental forces of the Standard Model (SM) unify

$$a_1 = a_2 = a_3 = a_{GUT}$$




given $M_{SUSY} \simeq 1\text{TeV}$

- The symmetry $G_{SM} = SU(3) \otimes SU(2) \otimes U(1)$ is incorporated within a simple gauge group $G_{GUT}$ ➜ electric charge quantization / magnetic monopoles.
- All the fermions of one generation belong to one or two representations of the group
- Nucleon instability

# Main characteristics

## Particle Content

The **fermions** of one generation, **along with the right handed neutrino**, are all contained in the same spinorial rep. $\mathbf{16}_L$.

The **generators** of $SO(10)$ belong to the adjoint rep. $\mathbf{45}_V$

The minimal **Higgs** fields content

$$A = \mathbf{45}_H$$
$$C + \overline{C},\ C' + \overline{C'} = \mathbf{16}_H + \overline{\mathbf{16}}_H$$
$$T_1,\ T_2 = \mathbf{10}_H$$
$$+ \text{ singlets}$$

## Doublet-Triplet splitting

Fulfilled through **Dimopoulos - Wilczek (D - W) mechanism,**

assuming **the essential superpotential term** : $\boldsymbol{W_{2/3} = T_1^{\alpha} A^{\alpha\beta} T_2^{\beta}}$,

**and demanding** $\langle A \rangle$ along the $B - L$ direction:

$$\boldsymbol{\langle A \rangle = diag(a, a, a, 0, 0) \otimes i\tau_2}, \text{ with } a \sim M_{GUT} \text{ and } \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

**Breaking** $SO(10)$ The spinorial and the adjoint sector break the symmetry of $SO(10)$ with a different manner, yet simultaneously, thus in low energies the symmetry $G_{SM}$ of the SM is left unbroken.

$$\left.\begin{array}{l} SO(10) \xrightarrow{\mathbf{16}+\overline{\mathbf{16}}} SU(5),\quad \cancel{B-L} \\ SO(10) \xrightarrow{\mathbf{45}} SU(3)_c \times SU(2)_L \times U(1) \times U(1)_{B-L} \end{array}\right\} \ \text{Standard Model}$$

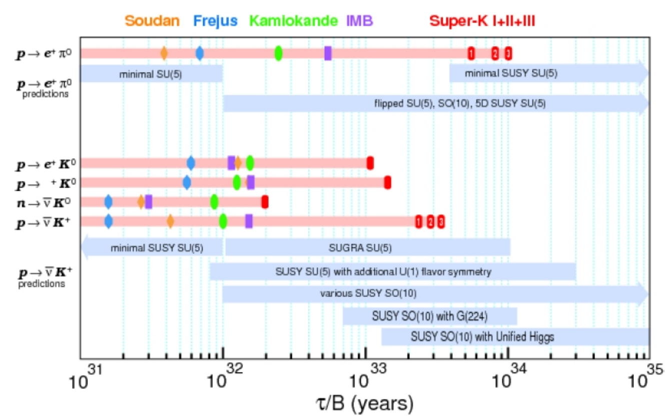
**Superpotential** $W = W_A + W_C + W_{ACC'} + W_{2/3}$

- $W_A$ For desirable $\langle A \rangle$ for D-W mechanism
- $W_C$ For $\langle C \rangle = \langle \overline{C} \rangle = c \sim M_{GUT}$
- $W_{ACC'}$ Gives mass to pseudo - Goldstone bosons
- $W_{2/3}$ For the 2/3 splitting

In total 25 superheavy masses which affect the running of RGEs and are defined by 12 independent parameters.

# Proton Decay Experimental Bounds

Akiri et al. 2011




**Super - Kamiokande :** $\tau(p \to \bar{\nu}K^+) > 4 \times 10^{33}$ yrs.

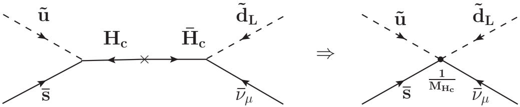
Fundamental Physics at the Intensity Frontier [hep-ex] 1205.2671

Through D = 5 operators. Dominant decay mode for SUSY GUTS.

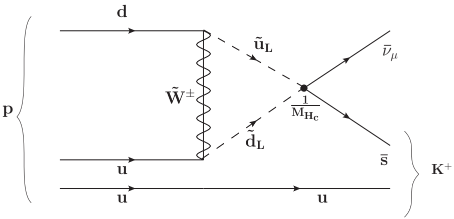
# D = 5 decay operators

D=5 operators are induced **via the exchange** of superheavy color triplet Higgsinos in the $\mathbf{10_H}$. They arise from an effective superpotential which is inverse proportional to a mass parameter (**effective mass** $M_{eff}$):

$$\boxed{M_{eff} = \frac{M_3\, M_3'}{M_2}}$$




After "dressing"



Arnowitt, Nath '93

## Proton decay rate

$$\Gamma(p \to \bar{\nu} K^+) = \sum_i \Gamma(p \to \bar{\nu}_i K^+) = \left(\frac{\beta_p}{M_{eff}}\right)^2 |A|^2 \, |B_i|^2 \, C \qquad \text{with } i = e, \mu, \tau.$$

- $\beta_p$ The hadronic matrix element between the proton and the vacuum state of the 3 quarks operator. Calculation from latice gauge theory Aoki et al. '08
- $A$ Depends on quark masses at 1 GeV in $\overline{\mathrm{MS}}$ and CKM matrix elements
- $B_i$ The functions that describe the dressing of the loop diagrams
- $C$ Contains chiral Lagrangian factors, which convert a Lagrangian involving quark fields to the effective Lagrangian involving mesons and baryons. Chadha et al. '83

# High energy thresholds (HET)

The effect of the low and high energy thresholds in $\overline{DR}$, in the running of the RGEs

$$
\begin{aligned}
a_i^{-1}(\mu) &= a_G^{-1}(M_{GUT}) + \frac{1}{2\,\pi}\,(b_i^{SM} + b_i^{SUSY})\,\ln\frac{M_{GUT}}{\mu} \\
&+ \frac{1}{2\,\pi}\sum_{SM_j} b_i^{SM_j}\,\ln\frac{\mu}{m_{SM_j}} \qquad \Leftarrow \text{SM thres.} \\
&+ \frac{1}{2\,\pi}\sum_{S_k} b_i^{S_k}\,\ln\frac{\mu}{m_{S_k}} \qquad \Leftarrow \text{SUSY thres.} \\
&+ \frac{1}{2\,\pi}\sum_{H_l} b_i^{H_l}\,\ln\frac{M_{GUT}}{m_{H_l}} \qquad \Leftarrow \text{HET} \\
&+ (2-\text{loops effects}).
\end{aligned}
$$

At the GUT breaking scale we impose:

**Gauge coupling unification condition** $a_1(M_{GUT}) = a_2(M_{GUT}) = a_3(M_{GUT}) \equiv a_G$

**Universal boundary conditions for the soft SUSY breaking parameters (mSUGRA/CMSSM )**

$$m_i(M_{GUT}) = m_0 \quad M_i(M_{GUT}) = M_{1/2} \quad A_i(M_{GUT}) = A_0.$$

**[the independent parameters of CMSSM** are: $m_0$, $M_{1/2}$, $A_0$, $\tan\beta$, sign$(\mu)$]
$\tan\beta = \upsilon_2/\upsilon_1$, with $\upsilon_{1,2} = \langle H_{1,2}\rangle$ (the Higgs fields of MSSM).

**The effective mass parameter $M_{eff}$**

- ➜ depends on the values of $a_i(M_Z)$ and
- ➜ parameters stemming from the HET of the model
- ✔ and is restricted from the proton decay

Solving the system of the three 1-loop RGEs, with HET: $\frac{M_{eff}}{M_Z} = e^{h(a'^{-1}_i)} f(x)$.

The **massless parameter $x$** is defined as:

$$x \equiv \frac{a}{2c}, \qquad a \to \langle 45 \rangle, \quad c = \langle 16 \rangle,$$

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# Parametrization of HET

If $\boldsymbol{M_L}$ $\equiv$ is the smallest mass of the superheavy particles and we set:

$$a_G^{-1}(M_L) \equiv a_G^{-1}(M_{GUT}) + \frac{1}{2\,\pi}\, b_i^{GUT} \ln \frac{M_{GUT}}{M_L},$$

and also

$$c_i \equiv \frac{1}{2\,\pi} \sum_{H_l} b_i^{H_l} \ln \frac{M_L}{m_{H_l}}.$$

Then

## New boundary condition

$$\boldsymbol{a_i^{-1}(M_L) = a_G^{-1}(M_L) + c_i},$$

at the scale $M_L$,

which takes into acount the contribution of all the HET through $c_i$.

# The $c_i$ s

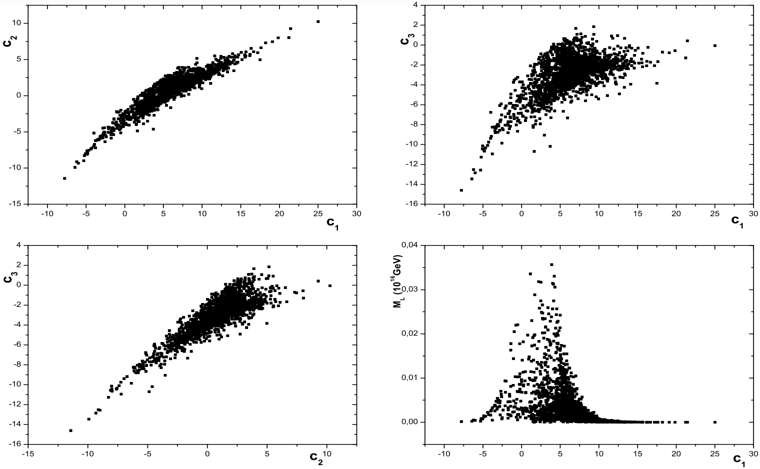
- Include the impact of HETs from $M_{GUT}$ to $M_L$.
- A random sample generator is used, which assigns random numbers to the GUT parameters $p_j$ of the model.
- **The amount of the parameters of the superheavy spectrum is reduced dramatically.**

  Independent mass parameters → New parameters

  $p_j \quad (j = 1, \cdots, 12)$ → $\vec{c} = (c_1, c_2, c_3, M_L, M_{GUT})$

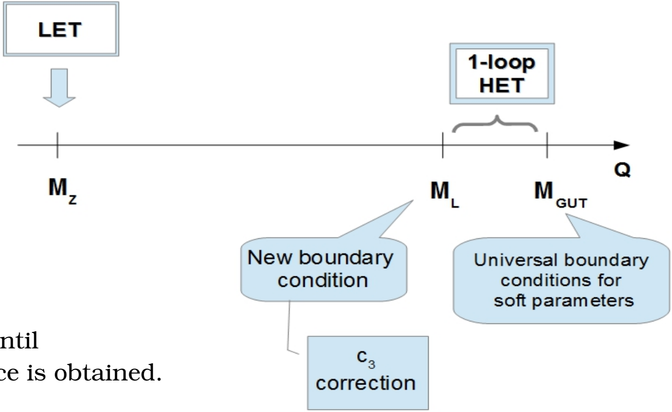
  - **One avoids** time-consuming scans over a multidimensional space.
  - We seek for points in this new parameter region which, when we impose the set criteria, shrinks even more.
  - **The procedure is applicable to almost any GUT model** The method becomes more useful, the more complex and numerous the superheavy particle content of the model appears to be.

# Numerical procedure




The distribution of $c_1$, $c_2$, $c_3$ and $M_L$ by twos, for $x = 5$.

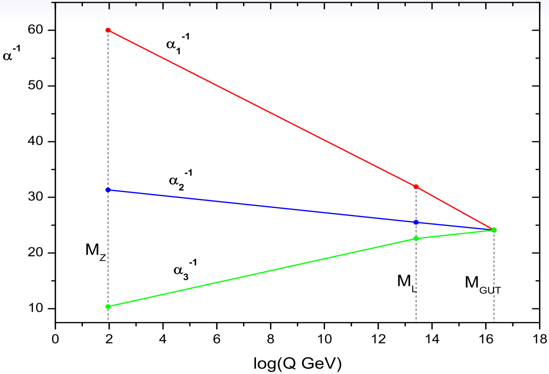
- We produce a large number of random points $\vec{c}_{in} \equiv (c_i^{in}, M_L^{in}, M_{GUT}^{in})$. $M_{GUT}^{in}$ = common and constant.
- We solve numerically the 2-loop RGEs.




Iteration until convergence is obtained.

$$c_3 = a_3^{-1}(M_L) - a_G^{-1}(M_L),$$

The running and unification of the three gauge couplings, according to our method.




Normally $M_L$ and $M_{GUT}$ should not be far apart. For $\dfrac{M_L}{M_{GUT}} > 10^{-3}$. our approach is viable.

From the random number samples, we get in average $\log \frac{M_{GUT}}{M_L} \simeq 2.7$.

# Inputs

- $a_{em}$,
- Fermi coupling constant $G_F$,
- $Z$ boson physical mass $M_?$,
- top and tau physical masses and the running mass $m_b(m_b)$ in $\overline{MS}$.
- Soft supersymmetric breaking parameters $m_0, M_{1/2}, A_0$, the value of $\tan\beta$ and the sign of the Higgsino mixing parameter ($\mu > 0$).
- The ratio $x$. The random sample generator define "slices" of the same $x$ in the space of the vectors $\vec{c}$. Primarily, we choose $x = 5$. Large $x$ moves $M_L$ away from its normal values
- Mainly $M_{GUT} = 2 \cdot 10^{16}\, GeV$ for the most part of our analysis.

# Electroweak precision measurements

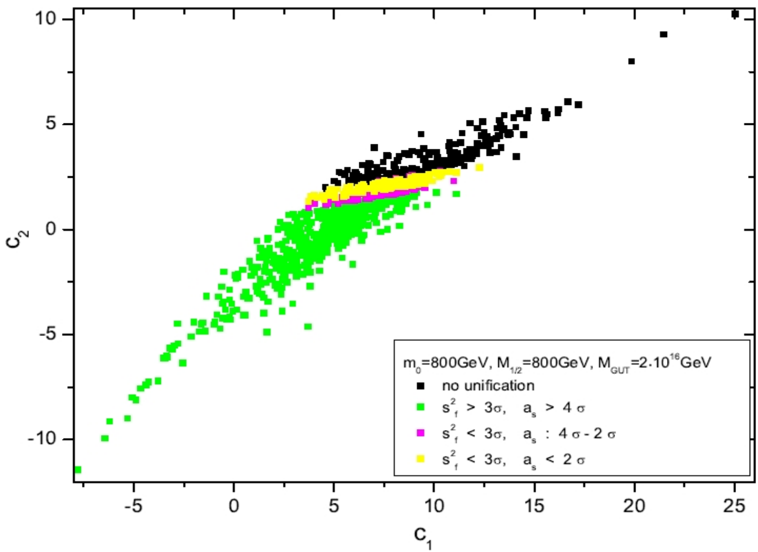
- **Effective mixing angle** ($\overline{sin}^2_f \hat{\partial} \equiv s^2_f$): in general, it occurs with descent values, with error $< 3\sigma$ - ultimately it does not add up to the constraints.

- **Strong coupling constant** at $M_Z$ ($a_{strong} \equiv a_s$)

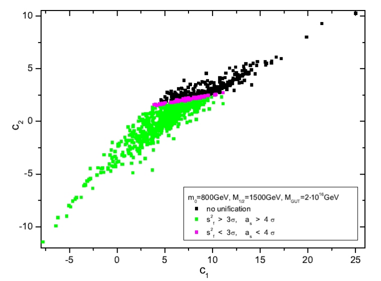
It turns out that $a_{strong}$ prefers region of the parameter space where the values of the soft breaking parameters, $M_{1/2}$ and mostly $m_0$ are **low to central**: **for $m_0$ up to 1400 GeV and for $M_{1/2}$ up to 1300 GeV**

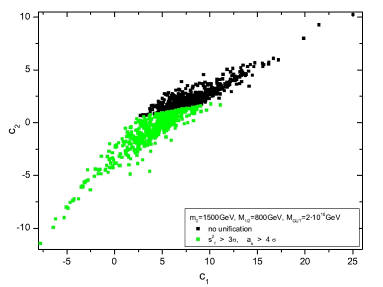
This is expected since

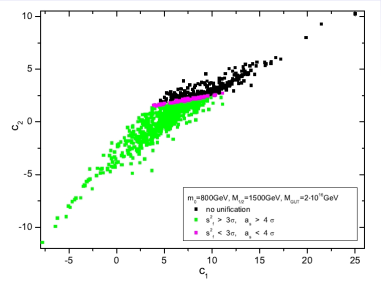
- for large values of $M_{1/2}$ and $m_0$ SUSY is absent, due to decoupling.
- large soft SUSY breaking parameters and in particular $M_{1/2}$, shrink the range of the allowed values of $M_{eff}$,because they affect the wino masses and in turn the $B_i$ in the proton decay rate relation.




For this diagram and those which follow $\tan\beta = 10$ and $A_0 = 100$ GeV.


10
5
0
-5
-10
c2
-5
0
5
10
15
20
25
c1
m0=800GeV, M1/2=1500GeV, MGUT=2·10^16GeV
no unification
s²f > 3σ, as > 4 σ
s²f < 3σ, as < 4 σ


$m_0$=1500GeV, $M_{1/2}$=800GeV, $M_{GUT}$=2·10$^{16}$GeV
no unification
$s^2_t$ > 3σ, $a_s$ > 4 σ
$c_1$
$c_2$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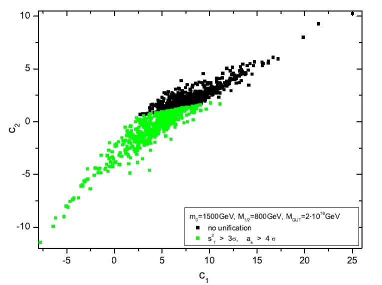
$c_2$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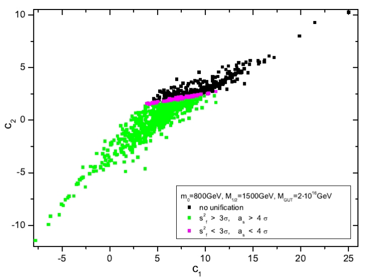
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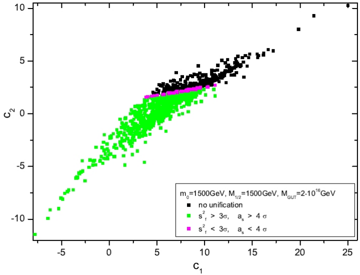
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no unification
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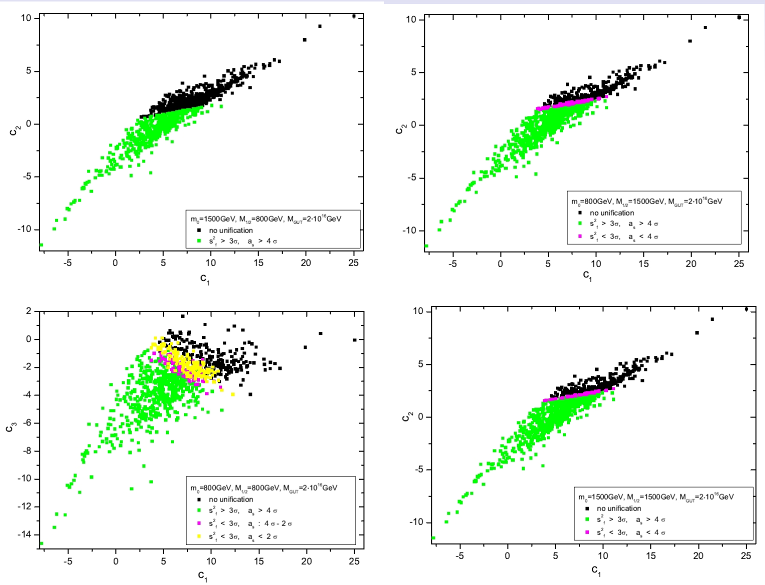
$s^2_f$ < 3σ, $a_s$ < 4 σ



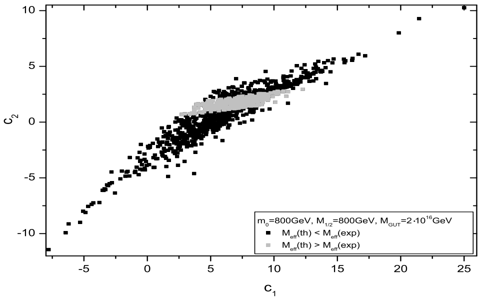
$m_0$=1500GeV, $M_{1/2}$=800GeV, $M_{GUT}$=2·10$^{16}$GeV
no unification
$s^2_t$ > 3σ, $a_s$ > 4 σ
$m_0$=800GeV, $M_{1/2}$=1500GeV, $M_{GUT}$=2·10$^{16}$GeV
no unification
$s^2_t$ > 3σ, $a_s$ > 4 σ
$s^2_t$ < 3σ, $a_s$ < 4 σ
$m_0$=800GeV, $M_{1/2}$=800GeV, $M_{GUT}$=2·10$^{16}$GeV
no unification
$s^2_t$ > 3σ, $a_s$ > 4 σ
$s^2_t$ < 3σ, $a_s$ : 4 σ - 2 σ
$s^2_t$ < 3σ, $a_s$ < 2 σ
$m_0$=1500GeV, $M_{1/2}$=1500GeV, $M_{GUT}$=2·10$^{16}$GeV
no unification
$s^2_t$ > 3σ, $a_s$ > 4 σ
$s^2_t$ < 3σ, $a_s$ < 4 σ
$c_1$
$c_2$
$c_3$

# Proton decay constraint

$$M_{eff}(th) \; > \; M_{eff}(exp) \; \equiv \; \beta_p \, |A| \, \sqrt{\tau_p \, C \sum_i |B_i|^2} \qquad (*)$$

The points which satisfy $(*)$

- give proton lifetime in the range $10^{34} - 10^{37}$ years
- overlap with the majority of the points that yield gauge coupling unification with values of $a_{strong}$ within the $2\sigma$ .
- Raising of $m_0$ and/or $M_{1/2}$ results to an homogeneous reduction of the gray region around a central point.



# Other factors

**The ratio $x$** $x = 0.5$ to $x = 500$,

1. Small values of $x$ favour the gauge coupling unification
2. and fail completely to satisfy the proton decay constraint, since

$$\Gamma_p \sim \frac{1}{M_{eff}^2} \sim \frac{1}{f^2(x)}$$

3. $a_{strong}$ with error$< 2\sigma$ encourages central values of $x$.

**$\tan\beta$** A shift in $\tan\beta$ from 10 to 45

- causes small reduction in the number of points that succeed unification (*Landau poles*) and
- light raise in the number of those who give $a_{strong}$ within the experimental bounds
- It also induces a significant decrease in $\vec{c}$s which fulfil the proton decay constraint (from 22% for $m_0 = M_{1/2} = 800$ GeV, until 100% for $m_0 = 1500$ GeV and $M_{1/2} = 800$ GeV), as expected since $B_i$s depend on $\frac{1}{\sin 2\beta}$.

**Universal trilinear coupling $A_0$** Gentle influence. Small, positive values support our analysis.

**$M_{GUT}$ scale** Pushing $M_{GUT}$ higher offers easier satisfaction of the constraints.

# Higgs mass and SUSY exclusion limits

Important factor, nowadays: the given results and limits by the LHC.

- **Neutral Higgs**

ATLAS and CMS experiments: particle with mass $\sim$ **126 GeV** ($\sqrt{s} = 8$ TeV). This particle has spin equal to zero and mainly positive parity couplings: **Higgs boson**.

- **Gluino and squark exclusion limits**

**Assuming R parity conservation**, strong production of squarks and gluinos, is expected to dominate : ($pp \to \tilde{g}\tilde{g}, \tilde{g}\tilde{q}, \tilde{q}\tilde{q}$)

**ATLAS - CMS** .

CMSSM: $m_{\tilde{g}} \gtrsim 1350$ GeV.

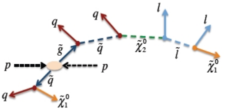
Chatrchyan 2014

Simplified models:

$m_{\tilde{g}} \gtrsim 1200 - 1300$ GeV

$m_{\tilde{q}} \gtrsim 900$ GeV.

Upward tendency.




From A. Taffard

**In our analysis**, with $\tan\beta = 10$, $A_0 = 100$ GeV and $m_0 = M_{1/2} = 800$ GeV, for $M_{GUT} = 2 \cdot 10^{16}$ GeV:

$m_h = 125.55 \pm 0.05$ GeV (with $\sigma = 1.8$ GeV) and

$m_{\tilde{g}} \sim 1700$ GeV and the masses of the 1st and 2nd generation of squarks occur at the same level. The **LSP** is always the lightest neutralino and is a pure bino.

CMSSM: with $\tan\beta = 30$, $A_0 = -m_0$ and $\mu > 0$ and $m_0 = M_{1/2} = 800$ GeV:

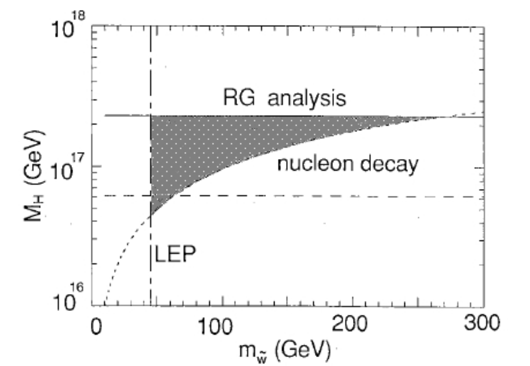
gluino and squark masses at the experimental exclusion limits. The lightest neutral Higgs has an average mass of 125 GeV.

# Conclusions

- We have presented a new approach towards treating, in a collective way, the large number of HET encountered in superheavy GUT quantities by mapping them to a few properly defined free parameters.
- This methodology is simple, less time-consuming and can be applied in any GUT model.
- The agreement with the whole of the limitations we have imposed is favoured from small to central values of $m_0$ and $M_{1/2}$ (500 GeV to 1.5 TeV) and from small, positive values of $A_0$.
- The constraints set by the LHC experiment, regarding the mass of the neutral, lightest Higgs and the masses of the supersymmetric particles, support the choice of the aforesaid parameter region.
- Large values of $tan\beta$ make the satisfaction of proton decay constraint difficult.
- The adopted supersymmetric $SO(10)$ model can fulfil all the set criteria, for soft masses with values at the limit of supersymmetry discovery in the next run of LHC.

# Backup slides

Hisano, Murayama, Yanagida, '92, '93




- $\cdots$ Constraint from proton lifetime experimental bounds.
- — Constraint from gauge coupling constants RGEs with $a_3 = 0.118 \pm 0.007$
- – – Constraint from gauge coupling constants RGEs with $\Delta a_3 = 0.0035$
- $\cdot - \cdot$ $m_{\tilde{w}} > 45$GeV bound from LEP.

# Low energy boundary conditions

At the lower scale $M_Z$: During the running of RGEs of gauge coupling constants, the effect of the low energy thresholds are encoded within $a_i$ from the low energy boundary conditions that are imposed at $M_Z$

$$\hat{a}_1^{-1}(M_Z) = \frac{3}{5}\, a_{em}^{-1}\, cos^2\partial\,(1 - \Delta_\gamma + \frac{a_{em}}{2\pi}\ \ln\frac{M_S}{M_Z})$$

$$\hat{a}_2^{-1}(M_Z) = a_{em}^{-1}\, sin^2\partial\,(1 - \Delta_\gamma + \frac{a_{em}}{2\pi}\ \ln\frac{M_S}{M_Z})$$

$\boldsymbol{\Delta_\gamma}$: contributions from leptons and light quarks.

$\boldsymbol{\partial}$: the weak mixing angle in $\overline{DR}$, which comprises boundary conditions at $M_Z$.

$\boldsymbol{M_S}$: mass parameter representing the contribution of all the heavy particles of the SM $(W,\ t,\ H^+)$ and of the sparticles.

**The strong coupling constant $a_{strong}\ (M_Z)$ in $\overline{MS}$: output in our analysis.**

$$
\begin{aligned}
a_1'^{-1}(M_Z) = \quad & a_{GUT}^{-1} + (2\,N_g - 6)\,\overline{\Lambda} - (\frac{2}{3}\,N_g + \frac{1}{2})\,SUSY \\
& + \frac{12}{5}\,\overline{M}_{V_3} + \frac{16}{5}\,\overline{M}_{V_1} + \frac{2}{5}\,\overline{M}_{V_2} + 10\,\overline{M}_{V_4} \\
& - \frac{3}{5}\,\overline{M}_{C_1} - \frac{3}{5}\,\overline{M}_{\overline{C}_1} - \frac{2}{5}\,\overline{M}_{C_2} - \frac{2}{5}\,\overline{M}_{\overline{C}_2} \\
& - \frac{2}{5}\,\overline{M}_3 - \frac{3}{5}\,\overline{M}_2 - \frac{2}{5}\,\overline{M'}_3 \\
& - \frac{6}{5}\,(\overline{\jmath}_{G_{3,1}} + \overline{\jmath}_{G_{3,2}}) - \frac{8}{5}\,(\overline{\jmath}_{G_{1,1}} + \overline{\jmath}_{G_{1,2}}) \\
& - \frac{1}{5}\,(\overline{\jmath}_{G_{2,1}} + \overline{\jmath}_{G_{2,2}})
\end{aligned}
$$

$$
\begin{aligned}
a_2'^{-1}(M_Z) = \quad & a_{GUT}^{-1} + (2\,N_g - 6)\,\overline{\Lambda} - (\frac{2}{3}\,N_g + \frac{13}{6})\,SUSY \\
& + 6\,\overline{M}_{V_2} + 6\,\overline{M}_{V_4} - 2\,\overline{M}_{A_2} - \overline{M}_{C_1} - \overline{M}_{\overline{C}_1} \\
& - \overline{M}_2 - 2\,(\overline{\jmath}_{G_{2,1}} + \overline{\jmath}_{G_{2,2}})
\end{aligned}
$$

$$
\begin{aligned}
a_3'^{-1}(M_Z) = \quad & a_{GUT}^{-1} + (2\,N_g - 6)\,\overline{\Lambda} - (\frac{2}{3}\,N_g + 2)\,SUSY \\
& + 2\,\overline{M}_{V_1} + 4\,\overline{M}_{V_2} + 4\,\overline{M}_{V_4} - 3\,\overline{M}_{A_3} \\
& - \overline{M}_{C_2} - \overline{M}_{\overline{C}_2} - \overline{M}_3 - \overline{M'}_3 \\
& - (\overline{\jmath}_{G_{1,1}} + \overline{\jmath}_{G_{1,2}}) - 2\,(\overline{\jmath}_{G_{2,1}} + \overline{\jmath}_{G_{2,2}})
\end{aligned}
$$

$$
\overline{M}_{GUT} = \frac{1}{2\,\pi}\ln\frac{M_{GUT}}{M_z}, \qquad \overline{M}_{SUSY} = \frac{1}{2\,\pi}\ln\frac{M_{SUSY}}{M_z}, \qquad \overline{M}_i = \frac{1}{2\,\pi}\ln\frac{M_{GUT}}{M_i},
$$

$M_{SUSY}$ a common mass scale for the supersymmetric particles
$M_i$ the mass of every superheavy particle $i$ which decouples.

# The ratio $x$