A Discrete Anatomy of the Neutrino mass matrix

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A few known facts

- Fermion mass terms are complex symmetric 3×3 matrices M.
- The Hermitean combination $M^2 = MM^*$can be diagonalised by means of a unitary transformation U to produce real eigenvalues.

Take U_l and U_ν the corresponding diagonalising matrices for charged leptons and neutrinos respectively. These matrices are by no means unique since

$$U_l \longrightarrow U_l A_l$$

$$U_\nu \longrightarrow U_\nu B_\nu$$

where

$$A_l = \begin{bmatrix} e^{ia_1} & 0 & 0 \\ 0 & e^{ia_2} & 0 \\ 0 & 0 & e^{ia_3} \end{bmatrix}$$

$$B_\nu = \begin{bmatrix} e^{ib_1} & 0 & 0 \\ 0 & e^{ib_2} & 0 \\ 0 & 0 & e^{ib_3} \end{bmatrix}$$

are equally good choices.

- Define the lepton mixing matrix as

$$U_{PMNS} = U_l^\dagger U_\nu$$

- Parametrise the matrix U using the standard parametrisation

$$U_{PMNS} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{bmatrix}$$

where redundant phases can be removed by suitably choosing the matrices A_l and B_l .

- Experimental evidence shows that

$$\sin^2\theta_{12} \approx 0.312^{+0.019}_{-0.018},\ \sin^2\theta_{23} \approx 0.466^{+0.073}_{-0.058},\ \sin^2\theta_{13} \approx 0.126^{+0.053}_{-0.049}$$

leading remarkably close to the matrix

$$U_{TB} = \begin{bmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{bmatrix}.$$

This is the so called tri-bimaximal (TB) mixing.

The Problem

- Look for mass textures which can reproduce the tri-bimaximal pattern and generalise so that a small θ_{13} angle can be generated.
- Look for symmetric patterns so that the chosen textures can be incorporated into a viable model (possibly superstring inspired)

Formulation

A general Hermitean 3×3 matrix contains nine independent elements and can be written as

$$M = i \ln U$$

where U a unitary matrix. Using the Cayley-Hamilton theorem we may write

$$M = c_1 I + c_2 U + c_3 U^2$$

where c_1,c_2,c_3 are complex in general, U is a "generator" and $\det U = 1$.The standard form for U contains four independent elements. Adding six degrees of freedom from the c_i coefficients we have a total of ten, so one D.O.F. is redundant, and can be removed by requiring one eigenvalue of U to be equal to one.
The mass expression can be diagonalised by means of a similarity transformation. A diagonal unitary matrix is uniquely defined by

$$U_d = \begin{bmatrix} e^{ia_1} & 0 & 0 \\ 0 & e^{ia_2} & 0 \\ 0 & 0 & e^{ia_3} \end{bmatrix} .$$

One phase can be absorbed into a redefinition of the coefficients c_2 and c_3 while taking the determinant condition into account, we end up with three possible forms

$$D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{-i\alpha} \end{bmatrix}$$

$$D_2 = \begin{bmatrix} e^{i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\alpha} \end{bmatrix}$$

$$D_3 = \begin{bmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{-i\alpha} & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Denoting by m_1, m_2, m_3 the eigenvalues of M, and using i.e. D_1 we have

$$m_1 = c_1 + c_2 e^{i\alpha} + c_3 e^{2i\alpha}$$

$$m_2 = c_1 + c_2 + c_3$$

$$m_3 = c_1 + c_2 e^{-i\alpha} + c_3 e^{-2i\alpha}$$

or, inverting

$$c_1 = -\frac{1}{4}\left[\exp\left[-\frac{3}{2}i\alpha\right]\ m_1 + \frac{1}{4}\exp\left[\frac{3}{2}i\alpha\right]\ m_3\right]\csc\frac{\alpha}{2}\csc\alpha + \frac{1}{4}\csc^2\frac{\alpha}{2}\ m_2$$

$$c_2 = \frac{1}{4}\left[\exp\left[-i\alpha\right](m_1 - m_2) - \exp\left[i\alpha\right](m_2 - m_3)\right]\csc^2\frac{\alpha}{2}$$

$$c_3 = -\frac{1}{4}\left[\exp\left[-\frac{i}{2}\alpha\right](m_1 - m_2) - \exp\left[\frac{i}{2}\alpha\right](m_2 - m_3)\right]\csc\frac{\alpha}{2}\csc\alpha\ \cdot$$

Hence, the mass eigenvalues problem is essentially disentangled from the diagonalising matrix. In the case of complex symmetric mass matrices the relations above are valid for the squares of the respective mass eigenvalues.

Deforming the TB-mixing matrix

Now, suppose that the generators U of the mass matrices constitute elements of a discrete group. It follows that they must satisfy relations of the form

$$U^n = 1$$

for some integer value of n. Their eigenvalues will be

$$e^{\frac{2\pi i}{n}}, e^{-\frac{2\pi i}{n}}, 1$$

in some order and can be diagonalised by means of a unitary transformation to produce a diagonal matrix $D_{i,n}$ where the subscript i refers to the eigenvalues ordering. This way, for the charged leptons we have

$$U_l = V_l D_{i,n} V_l^\dagger$$

for he neutrinos

$$U_\nu = V_\nu D_{j,m} V_\nu^\dagger$$

and the mixing matrix is

$$U_{PMNS} = C = V_l^\dagger V_\nu \cdot$$

If U_l and U_ν belong to the same group they must satisfy a relation of the form

$$(U_l U_\nu)^p = 1 \cdot$$

Also,

$$U_l U_\nu = V_l D_n V_l^\dagger V_\nu D_m V_\nu^\dagger = V_l D_n C D_m C^\dagger V_l^\dagger = V_l \mathcal{T} V_l^\dagger$$

where

$$\mathcal{T} = D_n C D_m C^\dagger$$

This way,

$$(U_l U_\nu)^p = V_l \mathcal{T}^p V_l^\dagger$$

implying $\mathcal{T}^p = 1$.

The eigenvalues of $\mathcal{T}$ must be 1 , $e^{\frac{2\pi i}{p}}$, $e^{-\frac{2\pi i}{p}}$ respectively, since the determinant of $\mathcal{T}$ is 1. The trace of $\mathcal{T}$ equals $1 + 2\cos\frac{2\pi}{p}$. Therefore, we must seek solutions of the form

$$1 + 2\cos\frac{2\pi}{p} = \mathrm{Tr}\mathcal{T}$$

for given values of m and n. In the cases where $\mathrm{Tr}\mathcal{T}$ takes complex values, there is no solution. Let us now define

$$D_{1,m} = \mathrm{Diagonal}\left[1, e^{\frac{2\pi i}{m}}, e^{-\frac{2\pi i}{m}}\right]$$

$$D_{2,m} = \mathrm{Diagonal}\left[e^{\frac{2\pi i}{m}}, 1, e^{-\frac{2\pi i}{m}}\right]$$

$$D_{3,m} = \mathrm{Diagonal}\left[e^{\frac{2\pi i}{m}}, e^{-\frac{2\pi i}{m}}, 1\right] \cdot$$

A subsequent search leads to a limited number of viable solutions shown below. We use a five integer notation (i, j, n, m, p) where i, n refer to the matrix $D_{i,n}$ for the charged leptons, j, m refer to the matrix $D_{j,m}$ for the neutrinos and p as defined previously. For the moment we assume that $\delta = 0$.

Viable Models

- Case $(2, 2, 3, 2, p)$
  Here, θ_{12} is given by

$$\sin^2\theta_{12} = \frac{1}{6\cos^2\theta_{13}}\left(1 - 2\cos\frac{2\pi}{p}\right).$$

  The only acceptable value is $p = 3$ giving

$$\sin\theta_{12}\,\cos\theta_{13} = -\frac{1}{\sqrt{3}}$$

$$\tan 2\theta_{23} = -\frac{2\cot 2\theta_{13}}{\sqrt{3 - \sec^2\theta_{13}}}.$$

- Case $(3, 2, 3, 2, p)$
  Here, θ_{12} is given by

$$\sin\theta_{12} = -\frac{2\cos\frac{\pi}{p}}{\sqrt{3}\cos\theta_{13}}.$$

Hence, as above, the only acceptable value is $p = 3$, leading to the constraint

$$\sin\theta_{12} = -\frac{1}{\sqrt{3}\cos\theta_{13}}$$

as in $(2, 2, 3, 2, p)$. For $p = 3$ the angle θ_{23} is given by

$$\tan 2\theta_{23} = -\frac{2\cot 2\theta_{13}}{\sqrt{3 - \sec^2\theta_{13}}}.$$

Setting $\sin\theta_{13} = s$ the mixing matrix becomes

$$C = \begin{bmatrix} \sqrt{\frac{2}{3} - s^2} & -\frac{1}{\sqrt{3}} & s \\ \frac{\sqrt{3}}{2}s + \frac{1}{2}\sqrt{\frac{2}{3} - s^2} & \frac{1}{\sqrt{3}} & \frac{1}{2}s - \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3} - s^2} \\ -\frac{\sqrt{3}}{2}s + \frac{1}{2}\sqrt{\frac{2}{3} - s^2} & \frac{1}{\sqrt{3}} & \frac{1}{2}s + \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3} - s^2} \end{bmatrix}.$$

Since the only allowed generalization requires that $n = 3$, $m = 2$, $p = 3$, for the present approach, we are led to the conclusion that the only finite symmetry groups that can connect the charged lepton and the neutrino mass matrices are either the discrete group A_4 or a group containing an A_4 subgroup and possessing a 3-dimensional representation (i.e. S_4). Observe that in the adopted formalism the middle column of C remains unchanged i.e. given by the column vector $\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}^T$. It turns out that the so constructed generalization of the TB mixing matrix utilizes the freedom of making linear transformations inside the degenerate neutrino subspace to create a non vanishing value for the θ_{13} angle. This subspace is obviously orthogonal to the $\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}$ axis.

As an example, we compute the mixing matrix for $\theta_{13} = \pi/20$, where the elements although exhibiting significant deviations from the TB case, are still consistent with data.

$$C = \begin{bmatrix} 0.801371 & -0.57735 & 0.156434 \\ 0.536162 & 0.57735 & -0.61579 \\ 0.265209 & 0.57735 & 0.772225 \end{bmatrix}.$$

The mass spectrum

We have found two models with identical predictions with respect to the mixing. The models differ only in the structure of the charged lepton mass matrix implying different coefficients $c_{1,2,3}$ for the two cases.

The results found show that for leptons $n = 3$ while for neutrinos we get $m = 2$. The corresponding coefficient functions are:

- Leptons

$$c_1^2 = \frac{1}{3}(m_e + m_\mu + m_\tau)$$
$$c_2^2 = \frac{1}{6}\left(2m_\mu - m_e - m_\tau\right) - \frac{i}{2\sqrt{3}}(m_e - m_\tau)$$
$$c_3^2 = \frac{1}{6}\left(2m_\mu - m_e - m_\tau\right) + \frac{i}{2\sqrt{3}}(m_e - m_\tau)$$

and

$$c_1^3 = \frac{1}{3}(m_e + m_\mu + m_\tau)$$
$$c_2^3 = \frac{1}{6}\left(2m_\tau - m_e - m_\mu\right) - \frac{i}{2\sqrt{3}}(m_e - m_\mu)$$
$$c_3^3 = \frac{1}{6}\left(2m_\tau - m_e - m_\mu\right) + \frac{i}{2\sqrt{3}}(m_e - m_\mu)$$

- Neutrinos

For the neutrinos $m = 2$. The D_2 and the corresponding neutrino matrices are given by:

$$D_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

In this case, it is clear that the neutrino mass spectrum turns out to be degenerate since ${D_2}^2 = 1$ implying $m_1 = m_3$. In order to establish a breaking pattern we must write

$$M = d_1 I + d_2 D_2 + R_2$$

where R_2 is the remainder term to be determined.

$$R_2 = \begin{bmatrix} m_1 - d_1 + d_2 & 0 & 0 \\ 0 & m_1 - d_1 - d_2 & 0 \\ 0 & 0 & m_3 - d_1 + d_2 \end{bmatrix} .$$

Each non vanishing element must be proportional to the same mass difference in order to have a breaking pattern.

Since

$$(R_1)_{11} - (R_1)_{33} = m_1 - m_3$$

this mass difference is $m_1 - m_3$ if no special relations between neutrino masses are assumed. So we have

$$\begin{aligned} m_1 - d_1 + d_2 &= r_1 (m_1 - m_3) \\ m_2 - d_1 - d_2 &= r_2 (m_1 - m_3) \\ m_3 - d_1 + d_2 &= r_3 (m_1 - m_3) \end{aligned}$$

with $r_1 - r_3 = 1$. Solving for d_1, d_2 we get

$$\begin{aligned} d_1 &= \frac{1}{2}(m_1 + m_2) - \frac{1}{2}(r_1 + r_2)(m_1 - m_3) \\ d_2 &= -\frac{1}{2}(m_1 - m_2) + \frac{1}{2}(r_1 - r_2)(m_1 - m_3) \end{aligned}$$

for arbitrary r_1 and r_2.

A different breaking pattern would require invariant relations between the neutrino masses. For instance if we require that

$$m_1 - d_1 + d_2 = r_1 \left(m_1 - m_2\right)$$
$$m_2 - d_1 - d_2 = r_2 \left(m_1 - m_2\right)$$
$$m_3 - d_1 + d_2 = r_3 \left(m_1 - m_2\right)$$

consistency implies that

$$r_3 = r_1 + \frac{m_3 - m_1}{m_1 - m_2}$$

independent of the masses i.e. $\frac{m_3 - m_1}{m_1 - m_2} = \mu$, and

$$d_1 = \frac{1}{2}\left(m_1 + m_2\right) - \frac{1}{2}\left(r_1 + r_2\right)\left(m_1 - m_2\right)$$

$$d_2 = \frac{1}{2}\left(r_1 - r_2 - 1\right)\left(m_1 - m_2\right) \cdot$$

Conclusions

- We have examined the structure of the lepton and neutrino mass matrices assuming that they can be expanded as polynomials of finite group elements which act as generators.
- This procedure has been proven useful for putting some order on the enormous number of possibilities given in the literature in a systematic and mathematically consistent way since it does not involve the mass eigenvalues.
- Various models are classified by means of three integer numbers which define the group.
- The number of finite groups that can reproduce current data by allowing a non zero value for the θ_{13} mixing angle is restricted. The groups allowed are either A_4 or a group containing an A_4 subgroup and possessing a 3-dimensional representation (i.e.S_4).
- Exact group symmetry introduces a degeneracy in the neutrino spectrum which has to lifted by means of an external breaking mechanism.

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- Work currently in progress