Holographic fluids and integrability

Marios Petropoulos

CPHT – Ecole Polytechnique – CNRS

EESFYE 2015 – University of Athens

April 2015

Highlights

Foreword

Riemann, Weyl and Cotton

Fluids and gravity

Holographic fluids

Integrability and resummation

Outlook

Holography

Originally: microscopic correspondence

$$\text{type II string theory on } AdS_5 \times S^5$$
$$\updownarrow$$
$$N = 4 \text{ YM on } D = 4 \text{ conformal boundary of } AdS_5$$

with $g_{string} \leftrightarrow {}^1/g_{YM}$

Later: macroscopic extension

$$\text{gravity plus matter on } d = D + 1 \text{ asymptotically AdS background}$$
$$\updownarrow$$
$$\text{phenomenological description of states of a } D\text{-dim boundary CFT}$$

Examples: AdS/QCD, AdS/CMT (superconductors, superfluids, ...)

Holography is a tool

- *Ultimate goal: compute correlation functions in strongly coupled (conformal) field theories*
- *Alternatively: compute transport coefficients in fluids*[1] *– fluid: hydrodynamic approximation of finite-T and finite-μ states of the (C)FT*

Holographic fluid: hydrodynamic approximation of finite-T and finite-μ states of a boundary CFT holographically dual to some bulk gravitational set-up

[1] Kubo-like formulas: correlation functions $\leftrightarrow$ transport coefficients

Fluid/gravity correspondence

Profound relationship

- *Originally in a classical framework: relationship between two sets of non-linear equations, Einstein's Eqs. and Navier–Stokes Eqs.* [Damour 1979; also Eling, Lysov, Oz, Strominger, … since 2009]
- *More recently within the holographic correspondence with a quantum perspective* [Bhattacharyya, Hubeny, Loganayagam, Minwalla, Rangamani, … since 2007]

Connected via renormalization-group flow from the boundary (UV) to the horizon (IR) where the former correspondence takes place

[Kuperstein, Mukhopadhyay, … since 2012]

Holographic fluids

Original motivation: determine transport properties

- Start with some bulk gravitational background related to some boundary fluid in local thermodynamic equilibrium
- Perturb and analyse the response using the bulk-boundary dictionary

$\longrightarrow$ perturbative response methods

Intriguing observation: some exact bulk solutions of Einstein Eqs. describe [Leigh, Petkou, Petropoulos '10, '11; Caldarelli, Leigh, Petkou, Petropoulos, Pozzoli, Siampos '12]

- *non-trivial fluid stationary states*
- *on non-trivial boundary backgrounds*

$\longrightarrow$ enable to probe substantially transport properties without response analysis [Mukhopadhyay, Petkou, Petropoulos, Pozzoli, Siampos '13]

Here

The question: can one exhibit more systematically exact bulk Einstein solutions that would produce appropriately designed fluid states – and provide more information on transport?

The spirit: find an integrable phase subspace corresponding to some first integral – effective reduction from 2nd- to 1st-order equations

The answer: yes

Highlights

Curvature decomposition in 4 dim

Metric $ds^2 = g_{AB}\, dx^A dx^B$, connection Γ and curvature $\mathbf{R}$

- The Riemann R_{ABCD} has 20 independent components

  1 the scalar R
  9 the traceless Ricci $S_{AB} = R_{AB} - \frac{R}{4} g_{AB}$
  10 the Weyl W_{ABCD} (conformal properties)

- The Weyl can be split into 5 self-dual W^+ and 5 anti-self-dual W^- components

Atiyah–Hitchin–Singer packaging in 3×3 matrices [Cahen, Debever, Defise '67; Atiyah, Hitchin, Singer '78]

$\lambda, \mu, \nu, \ldots = 0, 1, 2$

- traceless Ricci $\to 9 \to$ generic matrix $S_{\mu\nu}$
- self-dual Weyl tensor $\to 5 \to$ symmetric and traceless $W^+_{\mu\nu}$
- anti-sd Weyl tensor $\to 5 \to$ symmetric and traceless $W^-_{\mu\nu}$

Here the signature is Lorentzian : $(+-++)$

- W^+ and W^- are complex-conjugate
- The 10 independent components are captured in 5 complex functions $\Psi_a, \; a = 0, \ldots, 4$ projections of W onto a null tetrad

The existence of 4 principal null directions, potentially degenerate with higher multiplicity, translates into special algebraic relationships among the Ψ*s: Petrov type I, II, III, D, N, O*

Curvature decomposition in 3 dim

The Riemann has 6 independent components

- All Riemann components are in the Ricci
- The Weyl tensor vanishes

The conformal properties are captured by the Cotton tensor

$$C^{\mu\nu} = \frac{\epsilon^{\mu\rho\sigma}}{\sqrt{|g|}} \nabla_\rho \left(R^\nu{}_\sigma - \frac{R}{4} \delta^\nu{}_\sigma \right)$$

- symmetric
- traceless
- identically conserved: $\nabla_\mu C^{\mu\nu} = 0$

Highlights

Foreword

Riemann, Weyl and Cotton

Fluids and gravity

Holographic fluids

Integrability and resummation

Outlook

Relativistic fluid dynamics

Fluids in dim D gravitational backgrounds are described in terms of $g_{\mu\nu}$ and $D+1$ independent quantities u, ε, p all inside $T^{\mu\nu}$

$$T^{\mu\nu} = T^{\mu\nu}_{perf} + T^{\mu\nu}_{visc}$$

- $T^{\mu\nu}_{perf} = \varepsilon u^\mu u^\nu + p h^{\mu\nu}$ *($h_{\mu\nu}$: metric on $\Sigma \perp u$)*
- $T^{\mu\nu}_{visc}$ *as expansion in $\nabla^n u \rightarrow$ transport coefficients*

Landau frame: all corrections are transverse wrt u – built on *shear* σ, *expansion* Θ, *acceleration* a, *vorticity* ω and higher derivatives

The Eqs. are $\nabla_\mu T^{\mu\nu} = 0$ (D) plus an Eq. of state (1)

Fluids and gravity

Originally: black-hole horizon responds to perturbations as a viscous fluid *[Damour 1979]*

- damped shear waves
- viscosity $\eta = {}^{1}/_{16\pi G}$
- Bekenstein–Hawking entropy $s = {}^{1}/_{4G} \Rightarrow {}^{\eta}/_{s} = {}^{1}/_{4\pi}$

Origin? Deeper and more general relationship between Einstein's Eqs. and fluid dynamics *[Eling, Lysov, Oz, Strominger, ... since 2009]*

Gravity in 4 dim: 10 Einstein's Eqs. involving G_{AB} ($\nabla_A G^{AB} = 0$)

- 6 evolution: $G_{\mu\nu}$ (2nd order)
- 4 constraint: G_{rr}, $G_{r\mu}$ (1st order)

Initial-value formulation (Cauchy problem): Σ_t, $g_{\mu\nu}$ (6), $K_{\mu\nu}$ (6) *with*

- Hamiltonian constraint (1): $R^{(3)} - 2\Lambda + K^2 - K_{\mu\nu}K^{\mu\nu} = 0$
- momentum constraint (3): $\nabla_\mu \left(K^{\mu\nu} - g^{\mu\nu}K\right) = 0$

constraints in 4-dim $\mathcal{M} \leftrightarrow$ dynamics for $K_{\mu\nu}$ on 3-dim Σ_t

Imposing algebraic Petrov in 4 dim: $K_{\mu\nu}$ (6) $\to$ (4) *as for a fluid on* Σ_t

- $K_{\mu\nu} \leftrightarrow \epsilon, p, \mathrm{u}$
- $R^{(3)} - 2\Lambda + K^2 - K_{\mu\nu}K^{\mu\nu} = 0$: Eq. of state
- $\nabla_\mu \left(K^{\mu\nu} - g^{\mu\nu}K\right) = 0$: energy-momentum conservation

incompressible Navier–Stokes appear e.g. on black-hole horizons and conformal fluids appear on the conformal boundary $\Sigma_\mathrm{t}\big|_{r\to\infty}$

Highlights

Foreword

Riemann, Weyl and Cotton

Fluids and gravity

Holographic fluids

Integrability and resummation

Outlook

The boundary geometry

Work in the phenomenological i.e. gravity approximation – here: pure gravitational backgrounds → neutral boundary fluids

- Bulk Einstein space with $\Lambda = -3k^2$: asymptotically AdS $d = D + 1$-dim geometry
- Conformal boundary at $r \to \infty$: D-dim geometry

$$\boxed{\mathrm{d}s^2_{\text{bulk}} \approx \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu}$$

A primer: Schwarzschild AdS_4 black hole

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1+k^2r^2-\frac{2M}{r}} - \left(1 + k^2 r^2 - \frac{2M}{r}\right)\mathrm{d}t^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)$$

$$\mathrm{d}s^2_{\text{bry.}} = -\mathrm{d}t^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)$$

The boundary fluid

Holography: Hamiltonian evolution from data on the boundary – captured in Fefferman–Graham expansion for large r [Fefferman, Graham '85]

Two independent boundary Cauchy data:

- metric: "generalized coordinate" – leading term
- fundamental form: "conjugate momentum" – subleading term

$$\boxed{\mathrm{d}s^2_{\text{bulk}} = \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 \mathrm{d}s^2_{\text{bry.}} + \cdots + \frac{16\pi G}{3k(kr)^{D-2}} T_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \cdots}$$

$T_{\mu\nu}$ is traceless and conserved: interpreted as the stress–energy tensor of the boundary fluid in the hydrodynamic regime

The boundary fluid

- *is related to the constant-r fluid via holographic RG flow*
- *is not perfect and satisfies* $\eta/s \geq 1/4\pi$ *[Policastro, Son, Starinets '01]*

Holographic approach is superior – it allows

- *to determine more transport coefficients*
- *to integrate the boundary data into exact bulk solutions*

Back to the primer: Schwarzschild AdS_4 black hole ($D = 3$)

$$T_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\varepsilon}{2}\left(2\mathrm{d}t^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)\right) = \frac{\varepsilon}{2}\left(3\mathrm{u}^2 + \mathrm{d}s^2_{\text{bry.}}\right)$$

perfect-fluid stress tensor with $\mathrm{u} = \partial_t$ and $\varepsilon = 2p = {}^{Mk^2}/_{4\pi G}$

- $T^{\mu\nu} = T^{\mu\nu}_{\text{perf}}$ because of the kinematic state (fluid at rest)
- no information on any transport coefficient

<u>Goal:</u> find bulk Einstein spaces with

- *involved $ds^2_{bry.}$ and $u \to$ non-vanishing transverse $\nabla^n u$*
- *simple $T^{\mu\nu} \to$ extract information on transport properties*

exact bulk $\to$ exact transport coefficients: integrability properties

Highlights

Foreword

Riemann, Weyl and Cotton

Fluids and gravity

Holographic fluids

Integrability and resummation

Outlook

The philosophy

The question: given a boundary geometry $ds^2_{bry.}$ can one determine

- *the conditions it should satisfy*
- *the stress tensor it should be accompanied with*

for the FG expansion to be exactly resummable?

Focus on the sd and asd components of the Weyl $W^{\pm}$ – FG expansion:

$$W^{\pm}_{\mu\nu} = \frac{8\pi G}{k^2 r^3} T^{ref\pm}_{\mu\nu} + \cdots$$

Here

$$\boxed{T^{\mathrm{ref}\pm}_{\mu\nu} = T^{\mu\nu} \pm \frac{i}{8\pi G k^2} C^{\mu\nu}}$$

symmetric, traceless and conserved

The answer

The metric $ds^2_{bry.}$ must admit 2 symmetric, traceless and exactly conserved rank-2 tensors $T^{ref\pm}$ related by complex conjugation

The pattern: scan classes of $\mathrm{d}s^2_{\mathrm{bry.}}$ admitting exact $\mathrm{T}^{\mathrm{ref}\pm}$ and

- further impose on $\mathrm{d}s^2_{\mathrm{bry.}}$ the condition

$$\boxed{\mathrm{C} = 8\pi G k^2 \,\mathrm{Im}\mathrm{T}^{\mathrm{ref}+}} \tag{C}$$

- build the bulk with the resulting $\mathrm{d}s^2_{\mathrm{bry.}}$ and the stress tensor

$$\boxed{\mathrm{T} = \mathrm{Re}\mathrm{T}^{\mathrm{ref}+}} \tag{T}$$

The reference tensors $T^{ref\pm}$

Integrability in Einstein spaces is tight to Petrov special types

$\Longrightarrow$ $W^{\pm}$ are remarkably simple and so must be $\mathsf{T}^{\mathsf{ref}\pm}$

simpler to scan for $\mathsf{T}^{\mathsf{ref}\pm}$ than for C and T

Boundary geometries expected to lead to resummable series should have canonical $T^{ref\pm}$ i.e.

- *either possess complex-conjugate time-like geodesic congruences associated with perfect-fluid-form $T^{ref\pm}$*
- *or admit null congruences associated with pure-radiation $T^{ref\pm}$*
- *or a combination of both*

$\Longrightarrow$ $\mathsf{T}^{\mathsf{ref}\pm}$ follow the *Segre* classification of the 3-dim Cotton

the right integrability recipe

Results

Using the boundary data $ds^2_{bry.}$ and T constructed in this way, the derivative expansion

- *is exactly resummable $\rightarrow$ all Petrov-algebraic Einstein spaces: Kundt, Robinson–Trautman, Plebański–Demiański, . . .*
- *gives access to transport properties $\rightarrow$ the fluid is in non-trivial kinematical configurations and the stress–energy tensor is not perfect: $T = T^{perf} + \Pi$*

Highlights

Foreword

Riemann, Weyl and Cotton

Fluids and gravity

Holographic fluids

Integrability and resummation

Outlook

Fluid/gravity correspondence

- *constant-r slices Σ_t inside Petrov-algebraic bulk: Cauchy data plus constraints $\leftrightarrow$ fluid plus dynamics*
- *on the conformal boundary of asymptotically AdS spaces – macroscopic holography*
  - *Cauchy data: $2+1$-dim $ds^2_{bry.}$ and T with fluid dynamics*
  - *physical content: transport properties*

In the latter: *bottom-up approach based on integrability*

- *Idea: shape $ds^2_{bry.}$ and T for exact ascendent*
- *Pattern: design conserved $T^{ref\pm}$ of perfect fluid or radiation*
- *Integrability: guaranteed by the "1st order equation" $C = 8\pi G k^2 \, \mathrm{Im} T^{ref+}$ – again Petrov-algebraic bulk*

More general questions

- *Corners of integrability of Einstein's Eqs.*
- *Solution-generating patterns à la Ehlers and Geroch in AdS spaces*
- *Higher-dimensional Einstein spaces: Spin(7) bulks and G_2 boundaries*

Highlights

The ancestor of holography

Gravitational duality

Gravity, holography and the Fefferman–Graham expansion

Fluids and resummation

The Robinson–Trautman

The Plebański–Demiański type D class

LeBrun's filling-in

The "filling-in" problem – 1982

- A round S^3 can be "filled-in" by H_4

$$\mathrm{d}s^2_{H_4} = \frac{\mathrm{d}r^2}{1+r^2} + r^2 \mathrm{d}\Omega^2_{S^3} \to r^2 \mathrm{d}\Omega^2_{S^3}$$

- How to fill-in *analytically* a Berger sphere?

$$\mathrm{d}\Omega^2_{\mathrm{Berger}} = \left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \gamma\left(\sigma^3\right)^2$$

(σ^i: Maurer–Cartan forms of $SU(2)$)

Answer: Einstein space with self-dual Weyl tensor – quaternionic space *[LeBrun '82; Pedersen '86; Pedersen, Poon '90; Tod '94; Hitchin '95]*

A classic example

Bianchi IX AdS Schwarzschild–Taub–NUT

- Einstein space with $\Lambda = -3k^2$, mass M, nut charge n

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}r^2}{V(r)} + \left(r^2 - n^2\right)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right) \\ &\quad + V(r)\left(\mathrm{d}\tau + 4n\sin^2\frac{\vartheta}{2}\mathrm{d}\varphi\right)^2 \end{aligned}$$

$$V(r) = \tfrac{1}{r^2-n^2}\left[r^2 + n^2 - 2Mr + k^2\left(r^4 - 6n^2r^2 - 3n^4\right)\right]$$

- Weyl (anti-)self-dual (*i.e.* quaternionic) iff

$$M = \pm n(1 - 4k^2n^2)$$

$\Longleftrightarrow$ no conical singularity at $r = n$

The boundary geometry: $ds^2 \underset{r\to\infty}{\longrightarrow} \frac{dr^2}{k^2r^2} + k^2r^2 ds^2_{bry.}$

$$\begin{aligned} \mathrm{d}s^2_{\text{bry.}} &= \left(\mathrm{d}\tau + 4n\sin^2\frac{\vartheta}{2}\mathrm{d}\varphi\right)^2 + \frac{1}{k^2}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \\ &= \frac{1}{k^2}\left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2\right) + 4n^2\left(\sigma^3\right)^2 \end{aligned}$$

with $\tau = -2n(\psi+\varphi)$ and $0 \le \vartheta \le \pi, 0 \le \varphi \le 2\pi, 0 \le \psi \le 4\pi$

$$\begin{cases} \sigma^1 = \sin\vartheta\sin\psi\,\mathrm{d}\varphi + \cos\psi\,\mathrm{d}\vartheta \\ \sigma^2 = \sin\vartheta\cos\psi\,\mathrm{d}\varphi - \sin\psi\,\mathrm{d}\vartheta \\ \sigma^3 = \cos\vartheta\,\mathrm{d}\varphi + \mathrm{d}\psi. \end{cases}$$

Conclusion: $\mathrm{d}s^2_{\text{bry.}}$ is a Berger sphere

Highlights

Gravitational duality

Similar to electric–magnetic duality in general relativity – Euclidean regime

- Solve Einstein's Eqs. – self-dual gravitational instantons
  [Newman, Tamburino, Unti '63; Eguchi, Hanson '78]
- Provide another handle for understanding the theory
  - linear regime [works by Bunster, Julia, Henneaux. . .]
  - mass and nut as electric and magnetic charges [Dowker '74]

Self-duality deeply related with integrability – in the '70 all integrable systems were thought to be SDYM reductions [Ward, '85]

Curvature decomposition

Metric $ds^2 = \delta_{ab}\theta^a\theta^b$, connection one-form ω_{ab} and curvature two-form $\mathcal{R}_{ab} \in \mathbf{6}$ of $SO(4) \cong SO(3)_{sd} \otimes SO(3)_{asd}$

- Reducible under $SO(3)_{\mathrm{sd}}$ and $SO(3)_{\mathrm{asd}}$: $\mathbf{6} = (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$
- Curvature two-form $(\lambda, \mu \ldots = 1, 2, 3)$

  $$(\mathbf{3},\mathbf{1}) \;\; \mathcal{S}_\lambda = \frac{1}{2}\left(\mathcal{R}_{0\lambda} + \frac{1}{2}\epsilon_{\lambda\mu\nu}\mathcal{R}^{\mu\nu}\right)$$
  $$(\mathbf{1},\mathbf{3}) \;\; \mathcal{A}_\lambda = \frac{1}{2}\left(\mathcal{R}_{0\lambda} - \frac{1}{2}\epsilon_{\lambda\mu\nu}\mathcal{R}^{\mu\nu}\right)$$

  and similarly for the connection one-form
- Basis for the space of two-forms $\wedge^2$

  $$(\mathbf{3},\mathbf{1}) \;\; \phi^\lambda = \theta^0 \wedge \theta^\lambda + \frac{1}{2}\epsilon^\lambda{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$$
  $$(\mathbf{1},\mathbf{3}) \;\; \chi^\lambda = \theta^0 \wedge \theta^\lambda - \frac{1}{2}\epsilon^\lambda{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$$

Atiyah–Hitchin–Singer decomposition of $\mathcal{S}_\mu$, $\mathcal{A}_\mu$ *[Cahen, Debever, Defise '67; Atiyah, Hitchin, Singer '78]*

$$
\begin{aligned}
\mathcal{S}_\mu &= \tfrac{1}{2} W^+_{\mu\nu}\phi^\nu + \tfrac{1}{12} s\phi_\mu + \tfrac{1}{2} C^+_{\mu\nu}\chi^\nu \\
\mathcal{A}_\mu &= \tfrac{1}{2} W^-_{\mu\nu}\chi^\nu + \tfrac{1}{12} s\chi_\mu + \tfrac{1}{2} C^-_{\nu\mu}\phi^\nu
\end{aligned}
$$

with $W^\pm$ and $C^\pm$ 3×3 matrices, and s a function encoding the 20 components of the Riemann

- $s = {}^R\!/_2$ scalar curvature $\to 1$
- $C^\pm_{\mu\nu}$ traceless Ricci $\to 9$
- $W^+_{\mu\nu}$ self-dual Weyl tensor symmetric and traceless $\to 5$
- $W^-_{\mu\nu}$ anti-self-dual Weyl tensor symmetric and traceless $\to 5$

Quaternionic spaces: $C^\pm = 0 \quad s = 2\Lambda \quad W^- = 0$ *or* $W^+ = 0 \Leftrightarrow$ *Einstein & Weyl (anti-)self-dual*

Highlights

Gravity in $d = 4$

Palatini formulation and $3+1$ split [Leigh, Petkou '07; Mansi, Petkou, Tagliabue '08]

$$I_{\mathrm{EH}} = -\frac{1}{32\pi G}\int_{\mathcal{M}} \epsilon_{abcd}\left(\mathcal{R}^{ab} + \frac{k^2}{2}\theta^a \wedge \theta^b\right) \wedge \theta^c \wedge \theta^d$$

θ^a an orthonormal frame $\mathrm{d}s^2 = \eta_{ab}\theta^a\theta^b$ $(\eta : +\varepsilon++)$
gauge: no lapse, no shift

- Coframe: $\theta^r = \frac{\mathrm{d}r}{kr}$ and θ^μ

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{k^2 r^2} + \eta_{\mu\nu}\theta^\mu\theta^\nu$$

- Connection: $\omega^{r\mu} = \mathcal{K}^\mu$ and $\omega^{\mu\nu} = -\epsilon^{\mu\nu\rho}\mathcal{B}_\rho$ or (a)sd combination $1/2(\mathcal{K}^\mu \pm \mathcal{B}^\mu)$ for $\varepsilon = +$

Hamiltonian evolution of $\theta^\mu, \mathcal{K}^\mu, \mathcal{B}_\rho$ from boundary data – what are the independent boundary data? Answer in asymptotically AdS: Fefferman–Graham expansion for large r *[Fefferman, Graham '85; subtleties: de Haro, Skenderis, Solodukhin, '00]*

$$
\begin{aligned}
\theta^\mu(r,x) &= kr\, E^\mu(x) + \frac{1}{kr} F^\mu_{[2]}(x) + \frac{1}{k^2 r^2} F^\mu_{[3]}(x) + \cdots \\
\mathcal{K}^\mu(r,x) &= -k^2 r\, E^\mu(x) + \frac{1}{r} F^\mu_{[2]}(x) + \frac{2}{kr^2} F^\mu_{[3]}(x) + \cdots \\
\mathcal{B}^\mu(r,x) &= B^\mu(x) + \frac{1}{k^2 r^2} B^\mu_{[2]}(x) + \cdots
\end{aligned}
$$

Independent $2+1$ boundary data: E^μ and $F^\mu_{[3]}$

The holographic fluid

Interpretation of the boundary data

- E^{μ}: boundary orthonormal coframe – allows to determine
$$\mathrm{d}s^2_{\text{bry.}} = \eta_{\mu\nu} E^{\mu} E^{\nu} = g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}$$
  - $F^{\mu}_{[2]} = -{}^1\!/_{2k^2} S^{\mu\nu} e_{\nu}$: Schouten
  - $B^{\mu}_{[2]} = {}^1\!/_{2k^2} C^{\mu\nu} e_{\nu}$: Cotton
  - ...
- $F^{\mu}_{[3]}$: stress current one-form – allows to construct the vev of the boundary stress tensor
$$\mathrm{T} = \frac{3k}{8\pi G} F^{\mu}_{[3]} e_{\mu} = T^{\mu}{}_{\nu} E^{\nu} \otimes e_{\mu}$$

Macroscopic object carrying microscopic data from the bulk

Bulk Weyl self-duality and its boundary manifestation

Expanding $W^{\pm} = 0$ *leads to* $B_{[2]} = \pm(i)\frac{3k}{2}F_{[3]}$ *i.e.*

$$8\pi G k^2 T_{\mu\nu} \pm (i) C_{\mu\nu} = 0$$

[Leigh, Petkou '07; de Haro '08; Mansi, Petkou, Tagliabue '08; Miskovic, Olea '09]

Key property: C *and* T are

- traceless
- conserved

Away from the self-dual point, so is

$$\boxed{T^{\text{ref}\pm}_{\mu\nu} = T^{\mu\nu} \pm \frac{(i)}{8\pi G k^2} C^{\mu\nu}}$$

reflecting $W^{\pm}_{\mu\nu} = \frac{8\pi G}{k^2 r^3} T^{\text{ref}\pm}_{\mu\nu} + \cdots \neq 0$

Highlights

The ancestor of holography

Gravitational duality

Gravity, holography and the Fefferman–Graham expansion

Fluids and resummation

The Robinson–Trautman

The Plebański–Demiański type D class

Vector field u with $u_\mu u^\mu = -1$ and space–time variation $\nabla_\mu u_\nu$

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \sigma_{\mu\nu} + \frac{1}{D-1}\Theta h_{\mu\nu} + \omega_{\mu\nu}$$

- $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$: projector/metric on the orthogonal space
- $a_\mu = u^\nu \nabla_\nu u_\mu$: acceleration
- $\sigma_{\mu\nu}$: symmetric traceless part – shear
- $\Theta = \nabla_\mu u^\mu$: trace – expansion
- $\omega_{\mu\nu}$: antisymmetric part – vorticity

In $2+1$ dimensions

$$T^{\mu\nu}_{visc} = -\left(2\eta\sigma^{\mu\nu} + \zeta h^{\mu\nu}\Theta + \zeta_H \epsilon^{\rho\lambda(\mu} u_\rho \sigma_\lambda{}^{\nu)}\right) + O\left(\nabla^2 u\right)$$

Conformal fluids (tracelessness): $\varepsilon = 2p, \zeta = 0, \ldots$

On conformal perfect fluids with some time-like velocity field u

- $\mathsf{T}^{\text{perf}} = p\left(3\mathsf{u}^2 + ds^2_{\text{bry.}}\right)$
- Euler equations $\begin{cases} \nabla_{\mathsf{u}} \log p + {}^3\!/_2\, \Theta = 0 \\ \nabla_{\perp} \log p + 3\mathsf{a} = 0 \end{cases}$
- Integrability criterion: $d\mathsf{A} = 0$ with $\mathsf{A} = \mathsf{a} - \frac{\Theta}{2}\mathsf{u}$

$\Longrightarrow$ geodesic and expansionless u solve them with constant p

On the actual stress tensor $T = Re\, T^{ref+}$

- Not expected to be perfect: $\mathsf{T} = \mathsf{T}^{\text{perf}} + \Pi$
- The fluid congruence u is read off from the perfect piece
- T^{perf} and Π *are not* separately conserved

The series expansion

Using the boundary data $ds^2_{bry.}$ *and* T *as well as* C *and* u *the* *partly resummed* *derivative expansion reads* [Bhattacharyya et al '08; Caldarelli et al '12]

$$\boxed{\begin{aligned} ds^2_{bulk} &= -2u(dr + rA) + r^2k^2 ds^2_{bry.} + \frac{1}{k^2}\Sigma \\ &+ \frac{u^2}{\rho^2}\left(\frac{8\pi G T_{\lambda\mu}u^\lambda u^\mu}{k^2} r + \frac{C_{\lambda\mu}u^\lambda \eta^{\mu\nu\sigma}\omega_{\nu\sigma}}{2k^6}\right) + h.d. \end{aligned}} \qquad (R)$$

- $A = a - \frac{\Theta}{2}u \quad \omega = \frac{1}{2}(du + u \wedge a)$
- $\Sigma = -2u\nabla_\nu \omega^\nu{}_\mu dx^\mu - \omega_\mu{}^\lambda \omega_{\lambda\nu} dx^\mu dx^\nu - \frac{1}{2}u^2\left(R + 4\nabla_\mu A^\mu - 2A_\mu A^\mu\right)$
- $\rho^2 = r^2 + \frac{1}{2k^4}\omega_{\mu\nu}\omega^{\mu\nu} \quad \eta^{\mu\nu\sigma} = \epsilon^{\mu\nu\sigma}/\sqrt{-g_{bry.}}$

Using Eqs. (C) and (T) the first terms of (R) are exact Einstein

Output:

- *Integration achieved: limited derivative expansion is exact Einstein (Plebański–Demiański, Robinson–Trautman, Kundt…)*
- *Remarkable form of $T^{ref\pm}$ $\Rightarrow$ special form of $W^{\pm}$: algebraic Petrov type (Kerr, Taub–NUT, C-metric, pp-waves…)*

Consequence for holographic fluids: transport properties

- *Status: exact solutions provide rich information on transport coefficients (in particular when T is non-perfect)* [Mukhopadhyay et al '13; de Freitas, Reall '14; Bakas, Skenderis '14]
- *Next: perturbation of exact Einstein spaces as a deeper probe for transport can be made more systematic – captured in the known h.d. terms of the ds^2_{bulk} expansion*

Highlights

The ancestor of holography

Gravitational duality

Gravity, holography and the Fefferman–Graham expansion

Fluids and resummation

The Robinson–Trautman

The Plebański–Demiański type D class

Examples without vorticity

$$\boxed{ds^2_{bry.} = -dt^2 + \frac{2}{k^2P^2}d\zeta d\bar{\zeta}} \qquad (nv)$$

$P(t,\zeta,\bar{\zeta})$ real & *a priori* arbitrary – define $K = 2P^2\partial_\zeta\partial_{\bar{\zeta}}\log P$

- Cotton-tensor components $C_{\mu\nu}$:

$$-i\begin{pmatrix} 0 & -\frac{k^2}{2}\partial_\zeta K & \frac{k^2}{2}\partial_{\bar{\zeta}}K \\ -\frac{k^2}{2}\partial_\zeta K & -\partial_t\left(\frac{\partial^2_\zeta P}{P}\right) & 0 \\ \frac{k^2}{2}\partial_{\bar{\zeta}}K & 0 & \partial_t\left(\frac{\partial^2_{\bar{\zeta}} P}{P}\right) \end{pmatrix}$$

- Complex-conjugate geodesic & expansionless congruences $\mathrm{u}^+ = -\mathrm{d}t + \frac{\alpha^+}{P^2}\mathrm{d}\zeta$ and c.c.: $\alpha^\pm(\zeta,\bar{\zeta})$ satisfy

$$\boxed{k^2P\partial_\zeta\alpha^- = 2\left(k^2\alpha^-\partial_\zeta P + \partial_t P\right) \quad \text{plus c.c.}} \qquad (\mathrm{h})$$

- With M constant $\mathrm{T}^{\mathrm{ref}\pm} = \frac{Mk^2}{8\pi G}\left(3\left(\mathrm{u}^{\pm}\right)^2 + \mathrm{d}s^2_{\mathrm{bry.}}\right)$ is conserved
- Requiring $\mathrm{C} = 8\pi G k^2\,\mathrm{Im}\mathrm{T}^{\mathrm{ref}+}$ sets 1 constraint on P

$$\boxed{\left(\partial_\zeta K\right)^2 + 6M\partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) = 0} \tag{D}$$

plus 1 constraint on α^- $\partial_{\bar{\zeta}} K = 3Mk^2\frac{\alpha^-}{P^2}$ – combined with (h) gives

$$\boxed{P^2\partial_{\bar{\zeta}}\partial_\zeta K - 6M\partial_t \log P = 0} \tag{E}$$

(plus c.c.)

The stress tensor T

- Using $\mathsf{T} = \mathrm{Re}\,\mathsf{T}^{\mathrm{ref}+}$ one finds the *non-perfect* ${}^{8\pi G}/_{k^2}\mathsf{T}$

$$\begin{pmatrix} 2M & -\frac{1}{2k^2}\partial_\zeta K & -\frac{1}{2k^2}\partial_{\bar\zeta} K \\ -\frac{1}{2k^2}\partial_\zeta K & -\frac{1}{k^4}\partial_t\left(\frac{\partial_\zeta^2 P}{P}\right) & \frac{M}{k^2P^2} \\ -\frac{1}{2k^2}\partial_{\bar\zeta} K & \frac{M}{k^2P^2} & -\frac{1}{k^4}\partial_t\left(\frac{\partial_{\bar\zeta}^2 P}{P}\right) \end{pmatrix}$$

- The *perfect part* is $\mathsf{T}^{\mathrm{perf}} = \frac{Mk^2}{8\pi G}\left(3\mathsf{u}^2 + \mathrm{d}s^2_{\mathrm{bry.}}\right)$ with $\mathsf{u} = -\mathrm{d}t$ a geodesic expanding congruence with zero shear and zero vorticity – *not conserved*

Resummation: using $ds^2_{bry.}$, C, T and u in Eq. (R)

$$\boxed{ds^2_{bulk} = 2dt\,dr - 2Hdt^2 + 2\frac{r^2}{P^2}d\zeta d\bar{\zeta} + h.d.} \qquad \text{(RT)}$$

with

$$2H = K + 2r\partial_t \log P - \frac{2M}{r} + k^2 r^2$$

The displayed part without h.d. is

- exact Einstein thanks to Eq. (E) $\to$ integrability condition
- Petrov type D thanks to Eq. (D) $\Leftrightarrow 3\Psi_2\Psi_4 = 2\Psi_3^2$

Robinson–Trautman type D class

- u $\longleftarrow$ 2 multiplicity-2 bulk principle null directions
- $u_\pm \longleftarrow$ 2/4 bulk tetrad elements

Now pure-radiation reference tensor

$$4\pi G k^2 \, T^{ref+} = F(t,\zeta)\, d\zeta^2$$

arbitrary $F(t,\zeta) \Rightarrow T^{ref\pm}$ *conserved*

- Requiring $\mathsf{C} = 8\pi G k^2 \, \mathrm{Im} \mathrm{T}^{\mathrm{ref}+}$ sets 1 constraint on P

$$\boxed{\partial_\zeta K = 0} \tag{N}$$

plus

$$\partial_t \left(\frac{\partial_\zeta^2 P}{P} \right) + F(t,\zeta) = 0 \tag{F}$$

(plus c.c.)

- Eq. (N) sets $K = K(t)$ and determines $P(t,\zeta,\bar\zeta)$
- Eq. (F) determines $F(t,\zeta)$ – *no constraint*

- Using $\mathrm{T} = \mathrm{Re}\mathrm{T}^{\mathrm{ref}+}$ one finds the *non-perfect* stress tensor

$$8\pi G k^2\, \mathrm{T} = F(t,\zeta)\, \mathrm{d}\zeta^2 + \bar{F}(t,\bar{\zeta})\, \mathrm{d}\bar{\zeta}^2$$

Using $ds^2_{bry.}$, C, T and $u = -dt$ in Eq. (R) gives (RT) with $M = 0$

- Petrov type N thanks to

$$\begin{aligned} M = 0 \ &\Leftrightarrow\ \Psi_2 = 0 \\ (N) \ &\Leftrightarrow\ \Psi_3 = 0 \end{aligned}$$

- Always exact Einstein

Note: $P(t,\zeta,\bar{\zeta}) = \frac{1+\varepsilon/2\, g\,\bar{g}}{\sqrt{2f\,\partial_\zeta g\,\partial_{\bar{\zeta}}\bar{g}}}$ with $\varepsilon = 0, \pm 1$ and $f(t), g(t,\zeta)$ arbitrary functions – $F(t,\zeta)$ expressed in terms of $g(t,\zeta)$ and its derivatives

Robinson–Trautman type N class

u $\longleftarrow$ 1 multiplicity-4 bulk principle null direction

Highlights

The ancestor of holography

Gravitational duality

Gravity, holography and the Fefferman–Graham expansion

Fluids and resummation

The Robinson–Trautman

The Plebański–Demiański type D class

Examples with vorticity

$$\boxed{ds^2_{bry.} = -Q^2\left(dt - b\right)^2 + \frac{2}{k^2P^2}d\zeta d\bar{\zeta}} \qquad (nv)$$

P, Q real fcts and $\mathrm{b} = b_\zeta \mathrm{d}\zeta + b_{\bar{\zeta}}\mathrm{d}\bar{\zeta}$ a real form – *a priori* arbitrary

- Impose $\exists$ 1 Killing $\Rightarrow$ 2nd one [Mukhopadhyay *et al* '13]
- Impose $\exists$ 2 c.c. accelerating non-expanding congruences $\mathrm{u}_\pm \Rightarrow$ perfect-fluid conserved $\mathrm{T}^{\mathrm{ref}\pm}$ (non-constant pressure)
- Impose $\mathrm{C} = 8\pi G k^2\, \mathrm{Im}\mathrm{T}^{\mathrm{ref}+} \Rightarrow$ solve for P, Q and $\mathrm{b} \Rightarrow \mathrm{d}s^2_{\mathrm{bry.}}$
- Extract $\mathrm{T} = \mathrm{Re}\mathrm{T}^{\mathrm{ref}+} = \mathrm{T}^{\mathrm{perf}} + \Pi$
- $\mathrm{T}^{\mathrm{perf}}$ generally non-conserved – aligned with $\mathrm{u} = -\mathrm{d}t + \mathrm{b}$ shearless, expanding accelerating congruence with vorticity
- Resum – Eq. (R): exact Petrov type D Plebański–Demiański familly (mass, rotation, nut, "twist", acceleration)