Overview of detection methods for stochastic GW backgrounds

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eLISA Cosmology Workshop, CERN

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Motivation / Context

1. Why do we care about detecting stochastic backgrounds?
2. Why is detection challenging?
3. What detection methods does one use?
4. What are the prospects for detection?

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3. What detection methods does one use?
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4. What are the prospects for detection?
   - depends on source of background (e.g., astrophysical or cosmological); specified by detection sensitivity curves

Plan of talk

1. Characterizing stochastic backgrounds
2. Cross-correlation methods
    a. simple example; frequentist and Bayesian approaches
3. Response functions and overlap functions for cross-correlations
4. Single-detector data analysis methods for (e)LISA
5. Detection sensitivity curves

1. Characterizing GW backgnds

- Random GW signal; "confusion noise" from a large number of weak, independent, unresolved sources
- Cosmological or astrophysical in origin
- Characterized statistically in terms of the moments

$$\langle h_{ab}(t)\rangle\,,\ \langle h_{ab}(t)h_{cd}(t')\rangle\,,\ \langle h_{ab}(t)h_{cd}(t')h_{ef}(t'')\rangle\,,\cdots$$

- Plane-wave expansion:

Fourier components

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} df \int_{S^2} d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, e^A_{ab}(\hat{k}) e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

- Statistical properties encoded in:

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(no loss of generality)

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0

(no loss of generality)

in terms of quadratic expectation values

(if Gaussian)

Isotropic, unpolarized Gaussian-stationary background

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$$\langle h_A(f,\hat{k})h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{16\pi}\delta(f-f')\delta_{AA'}\delta^2(\hat{k},\hat{k}')S_h(f)$$

strain power spectral density (Hz^{-1})

energy density spectrum (dimensionless)

$$S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_{\rm gw}(f)}{f^3}\,, \qquad \Omega_{\rm gw}(f) = \frac{1}{\rho_c}\frac{d\rho_{\rm gw}}{d\ln f}$$

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Power-law background:

$$h_c(f) = A_\alpha \left(\frac{f}{f_{\rm ref}}\right)^\alpha \quad \begin{cases} \alpha = -1\,, & \Omega_{\rm gw}(f) = \text{const} \\ \alpha = -2/3\,, & \Omega_{\rm gw}(f) \propto f^{2/3} \end{cases}$$

cosmological background

binary inspiral

Other types of backgrounds

1. Anisotropic, polarized, and/or non-Gaussian backgrounds, ... are also specified in terms of the expectation values of the Fourier components

2. E.g., anisotropic, unpolarized background:

$$\langle h_A(f,\hat{k}) h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{4}\delta(f-f')\delta_{AA'}\delta^2(\hat{k},\hat{k}')\mathcal{P}(f,\hat{k})$$

where $$\int_{S^2} d^2\Omega_{\hat{k}}\, \mathcal{P}(f,\hat{k}) = S_h(f)$$

3. Although early analyses (before ~2000) focused on isotropic unpolarized backgrounds, more recent analyses have considered anisotropic, polarized, non-Gaussian backgrounds

4. This talk will focus on isotropic, unpolarized, Gaussian-stationary backgrounds

2. Cross-correlation method

1. A stochastic GW background is correlated across multiple detectors in ways that differ from instrumental noise
2. Cross-correlation methods basically use the random output of one detector as a template for the other, taking into account the physical separation and relative orientation of the two detectors
3. Frequentist and Bayesian methods both start with a likelihood; use detection statistics or model selection to search for signals

Example of cross-corr method

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Single sample of data in two detectors; uncorrelated noise, common GW signal:

$$d_1 = h + n_1$$

$$d_2 = h + n_2$$

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Example of cross-corr method

Single sample of data in two detectors; uncorrelated noise, common GW signal:

$$d_1 = h + n_1$$

$$d_2 = h + n_2$$

$$\langle d_1 d_2 \rangle = \langle h^2 \rangle + \langle n_1 n_2 \rangle + \cancelto{0}{\langle h n_2 \rangle} + \cancelto{0}{\langle n_1 h \rangle} = \langle h^2 \rangle + \langle n_1 n_2 \rangle$$

If $\langle n_1 n_2 \rangle = 0$ then $\boxed{\langle d_1 d_2 \rangle = \langle h^2 \rangle \equiv \sigma_h^2}$

cross-correlation

variance of random GW signal

Likelihood function

Likelihood function

If the noise and GW signal are described by multivariate Gaussian distributions with covariance matrices:

$$C_n = \begin{pmatrix} \sigma_{n_1}^2 & 0 \\ 0 & \sigma_{n_2}^2 \end{pmatrix} \qquad C_h = \sigma_h^2$$

Likelihood function

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Likelihood function:

$$p(\mathbf{d}|\boldsymbol{\theta}_n, \boldsymbol{\theta}_h) = \frac{1}{\sqrt{\det(2\pi \mathbf{C})}} e^{-\frac{1}{2}\mathbf{d}^T \mathbf{C}^{-1} \mathbf{d}}$$

data vector: (d_1, d_2)
noise params: $(\sigma_{n1}^2, \sigma_{n2}^2)$
signal parameters: σ_h^2

$$C = \begin{pmatrix} \sigma_{n_1}^2 + \sigma_h^2 & \sigma_h^2 \\ \sigma_h^2 & \sigma_{n_2}^2 + \sigma_h^2 \end{pmatrix}$$

Frequentist analysis

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Maximum likelihood detection statistic:

signal+noise model

$$\Lambda_{\rm ML} \equiv \frac{\max_{\boldsymbol{\theta}_n} \max_{\boldsymbol{\theta}_h} p(\mathbf{d}|\boldsymbol{\theta}_n, \boldsymbol{\theta}_h)}{\max_{\boldsymbol{\theta}_n} p_n(\mathbf{d}|\boldsymbol{\theta}_n)} = \frac{\hat{\sigma}_h^2}{\hat{\sigma}_{n_1}\hat{\sigma}_{n_2}}$$

noise-only model

Frequentist analysis

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noise-only model

Maximum likelihood estimators:

$$\boxed{\hat{\sigma}_h^2 = d_1 d_2}$$

cross-correlation statistic

$$\hat{\sigma}_{n_1}^2 = d_1^2 - d_1 d_2$$
$$\hat{\sigma}_{n_2}^2 = d_2^2 - d_1 d_2$$

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NOTE: If we treat the noise variances as known (i.e., vary only σ_h^2):

$$\hat\sigma_h^2 = \frac{1}{(\sigma_{n_1}^2 + \sigma_{n_2}^2)^2}\left[2\sigma_{n_1}^2\sigma_{n_2}^2 d_1 d_2 + \sigma_{n_2}^4(d_1^2 - \sigma_1^2) + \sigma_{n_1}^4(d_2^2 - \sigma_2^2)\right]$$

single detector 'excess power' statistics

Bayesian analysis

Bayesian analysis

Use Bayes' theorem to calculate posterior distributions:

joint posterior:

$$p(\boldsymbol{\theta}_n, \boldsymbol{\theta}_h|\mathbf{d}) = \frac{\overbrace{p(\mathbf{d}|\boldsymbol{\theta}_n, \boldsymbol{\theta}_h)}^{\text{likelihood}}\overbrace{\pi(\boldsymbol{\theta}_n, \boldsymbol{\theta}_h)}^{\text{prior}}}{\underbrace{p(\mathbf{d})}_{\text{normalization factor}}}$$

marginalized posterior:

$$p(\boldsymbol{\theta}_h|\mathbf{d}) = \int d\boldsymbol{\theta}_n\, p(\boldsymbol{\theta}_n, \boldsymbol{\theta}_h|\mathbf{d})$$

etc.

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Use Bayes' theorem to calculate posterior distributions:

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likelihood

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normalization factor

marginalized posterior:

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etc.

Compare competing models by calculating odds ratio:

signal+noise model

Bayes factor *B*

$$\frac{p(M_1|\mathbf{d})}{p(M_0|\mathbf{d})} = \boxed{\frac{p(\mathbf{d}|M_1)}{p(\mathbf{d}|M_0)}} \frac{p(M_1)}{p(M_0)}$$

noise-only model

Bayesian analysis

Use Bayes' theorem to calculate posterior distributions:

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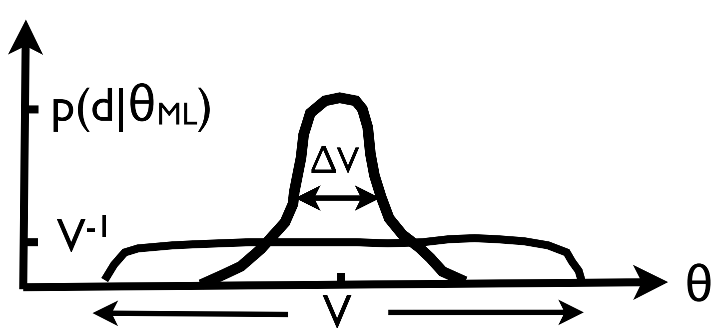
noise-only model



$$\mathcal{B} \equiv \frac{p(\mathbf{d}|M_1)}{p(\mathbf{d}|M_0)} = \frac{\int d\boldsymbol{\theta}_n \int d\boldsymbol{\theta}_h \, p(\mathbf{d}|\boldsymbol{\theta}_n, \boldsymbol{\theta}_h)\pi(\boldsymbol{\theta}_n, \boldsymbol{\theta}_h)}{\int d\boldsymbol{\theta}_n \, p_n(\mathbf{d}|\boldsymbol{\theta}_n)\pi_n(\boldsymbol{\sigma}_n)} \simeq \Lambda_{\mathrm{ML}} \frac{\Delta V_1/V_1}{\Delta V_0/V_0}$$

3. Response functions

Response function converts metric perturbations to detector output

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convolution

$$r = Rh = (R^{+} * h_{+} + R^{\times} * h_{\times})(t, \vec{x})$$

detector response

impulse response

detector location

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detector response

impulse response

detector location

$$r(t) = \int d\tau \int d^3x \, h_{ab}(t-\tau, \vec{x}-\vec{y}) R^{ab}(\tau, \vec{y})$$

$$= \int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f, \hat{k}) \boxed{R^A(f, \hat{k})} e^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

transfer function for a plane-wave
with frequency f, direction k, polarization A

Example: Single-link response function (e.g., pulsar timing)

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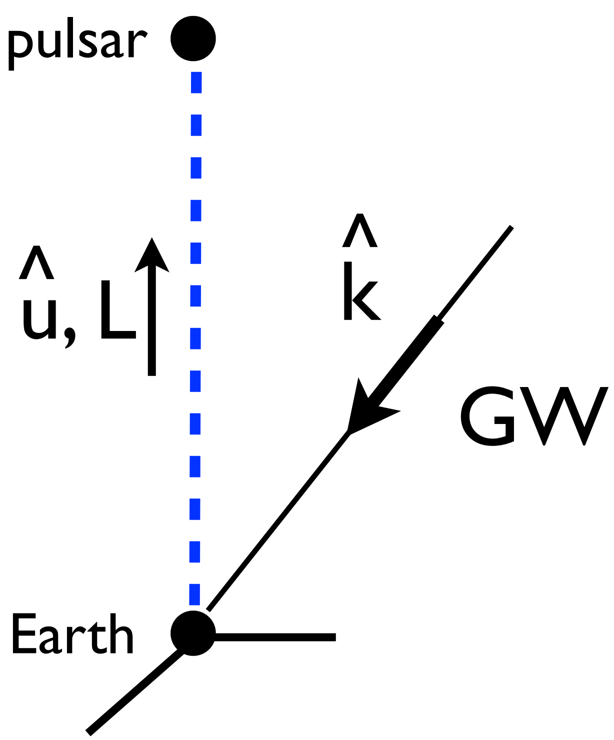

$$r(t) = \Delta T(t) = \frac{1}{2c} u^a u^b \int_{s=0}^{L} ds\, h_{ab}(t(s), \vec{x}(s))$$

where

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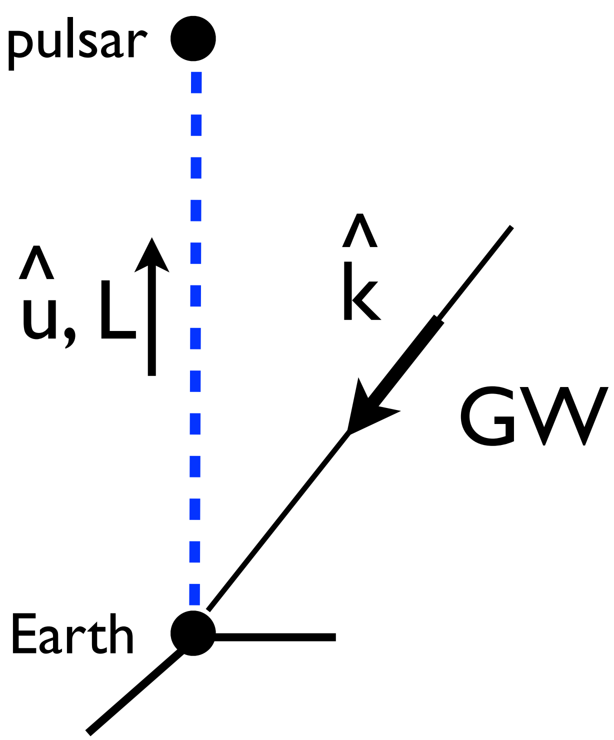

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$$R^A(f, \hat{k}) = \underbrace{\frac{1}{i2\pi f}}_{=1 \text{ for doppler freq measurement}} \underbrace{\boxed{\frac{1}{2} u^a u^b e^A_{ab}(\hat{k}) \frac{1}{1+\hat{k}\cdot\hat{u}}}}_{F^A(\hat{k},\hat{u})} \left[\overbrace{1}^{\text{earth term}} - \overbrace{e^{-i\frac{2\pi f L}{c}(1+\hat{k}\cdot\hat{u})}}^{\text{pulsar term}}\right]$$

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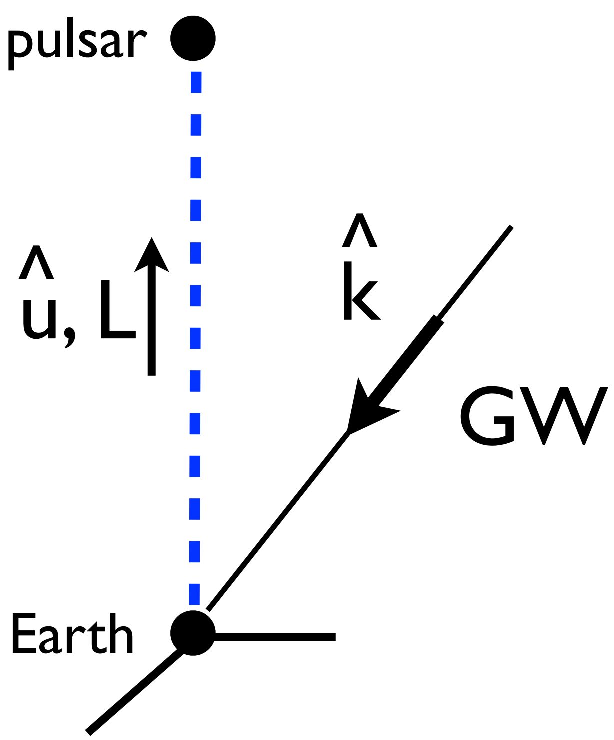

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earth term

pulsar term

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ignore

=1 for doppler
freq measurement

$F^A(\hat{k},\hat{u})$

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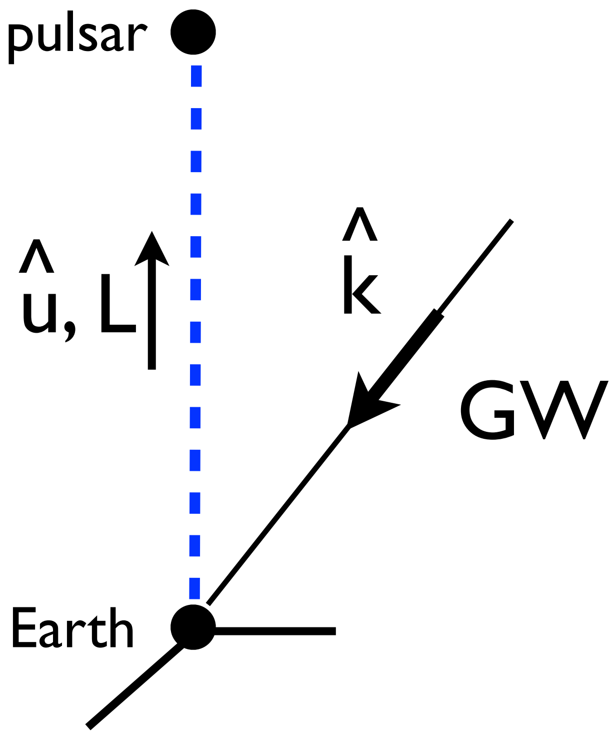

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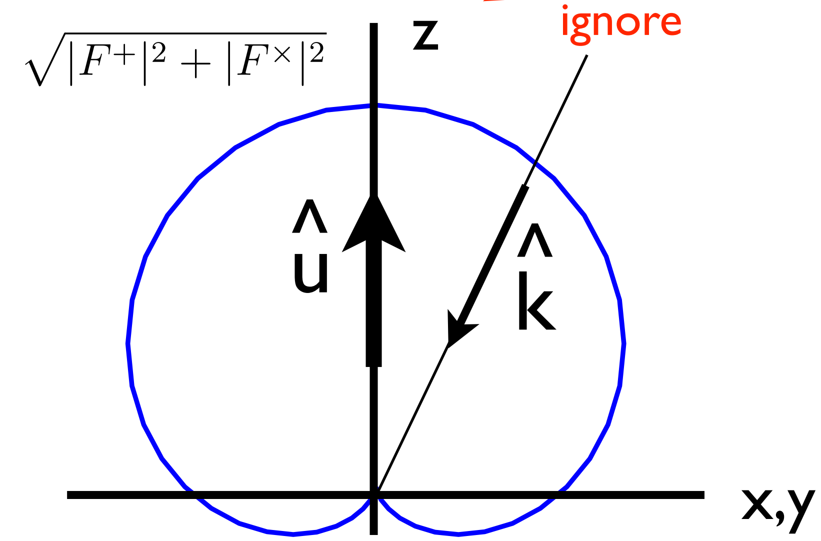

Example: Response function for equal-arm interferometer in long-wavelength limit (e.g., LIGO, Virgo, ...)

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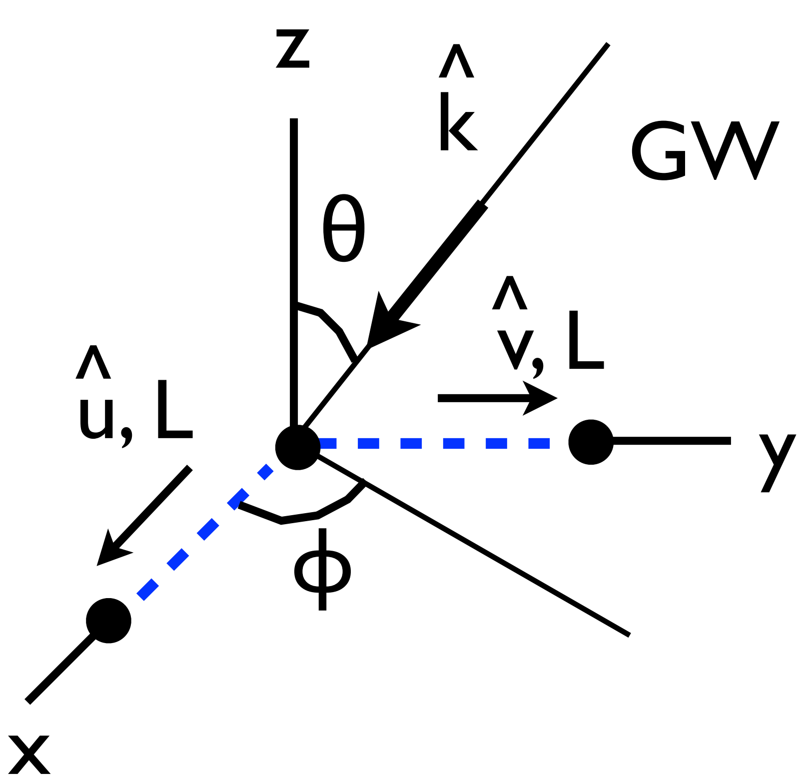
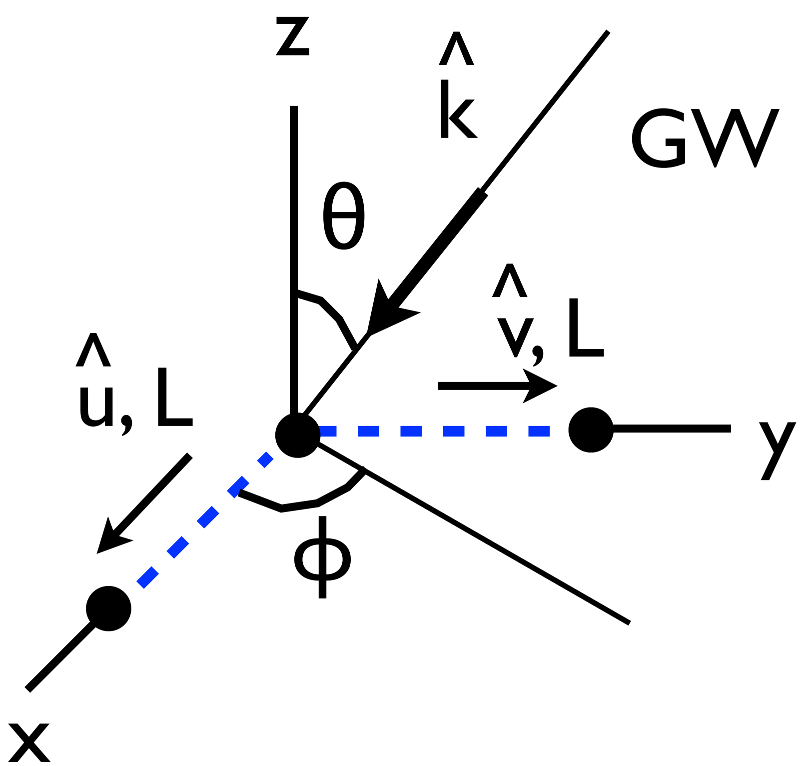
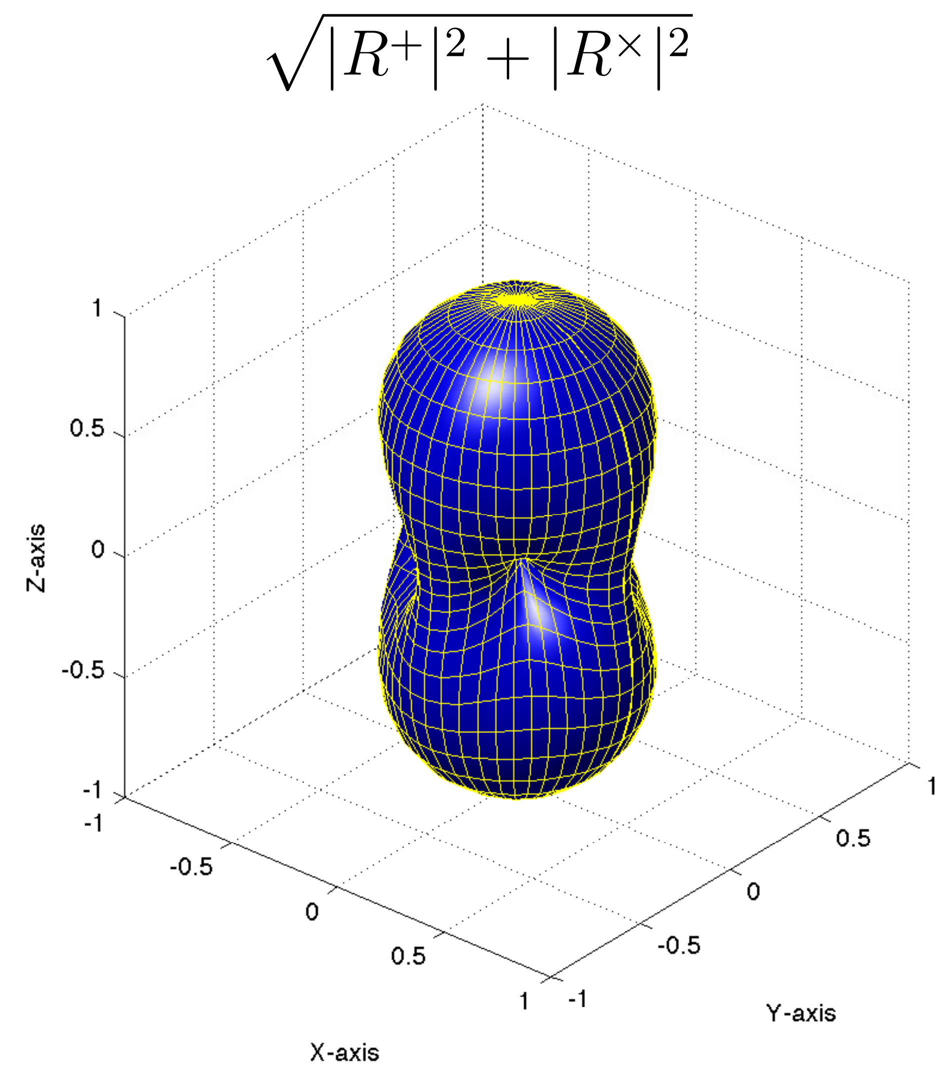


$$r(t) = \frac{1}{2}\left(\frac{\Delta T_{u,\,\mathrm{round-trip}}(t)}{T} - \frac{\Delta T_{v,\,\mathrm{round-trip}}(t)}{T}\right)$$

$$R^A(f,\hat{k}) \simeq \underbrace{\boxed{\frac{1}{2}(u^a u^b - v^a v^b)}}_{\text{detector tensor}} e^A_{ab}(\hat{k})$$

Example: Response function for equal-arm interferometer in long-wavelength limit (e.g., LIGO, Virgo, ...)





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Overlap functions

Transfer function between GW power and detector cross-power

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cross-correlation of detector responses

$$\langle \tilde{r}_I(f)\tilde{r}_J(f')\rangle = \frac{1}{2}\delta(f-f')\Gamma_{IJ}(f)S_h(f)$$

labels detector

$$\boxed{\Gamma_{IJ}(f) = \frac{1}{8\pi}\int_{S^2} d^2\Omega_{\hat{k}} \sum_A R_I^A(f,\hat{k})R_J^{A*}(f,\hat{k})e^{-i2\pi f\hat{k}\cdot(\vec{x}_I-\vec{x}_J)/c}}$$

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1. Encodes reduction in sensitivity to a stochastic GW background due to separation and relative orientation of two detectors
2. For I=J, represents the transfer function from GW power to detector response power in a single detector
3. For anisotropic backgrounds, the relevant quantity is the integrand

Pulsar timing overlap function

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Pulsar timing overlap function

$$\Gamma_{IJ}(f) = \frac{1}{(2\pi f)^2}\frac{1}{3}\zeta_{IJ}$$

simple function of angle between 2 Earth-pulsar baselines

$$\zeta_{IJ} = \frac{3}{2}\left(\frac{1-\cos\psi_{IJ}}{2}\right)\log\left(\frac{1-\cos\psi_{IJ}}{2}\right) - \frac{1}{4}\left(\frac{1-\cos\psi_{IJ}}{2}\right) + \frac{1}{2} + \frac{1}{2}\delta_{IJ}$$

pulsar-pulsar contribution for autocorrelation

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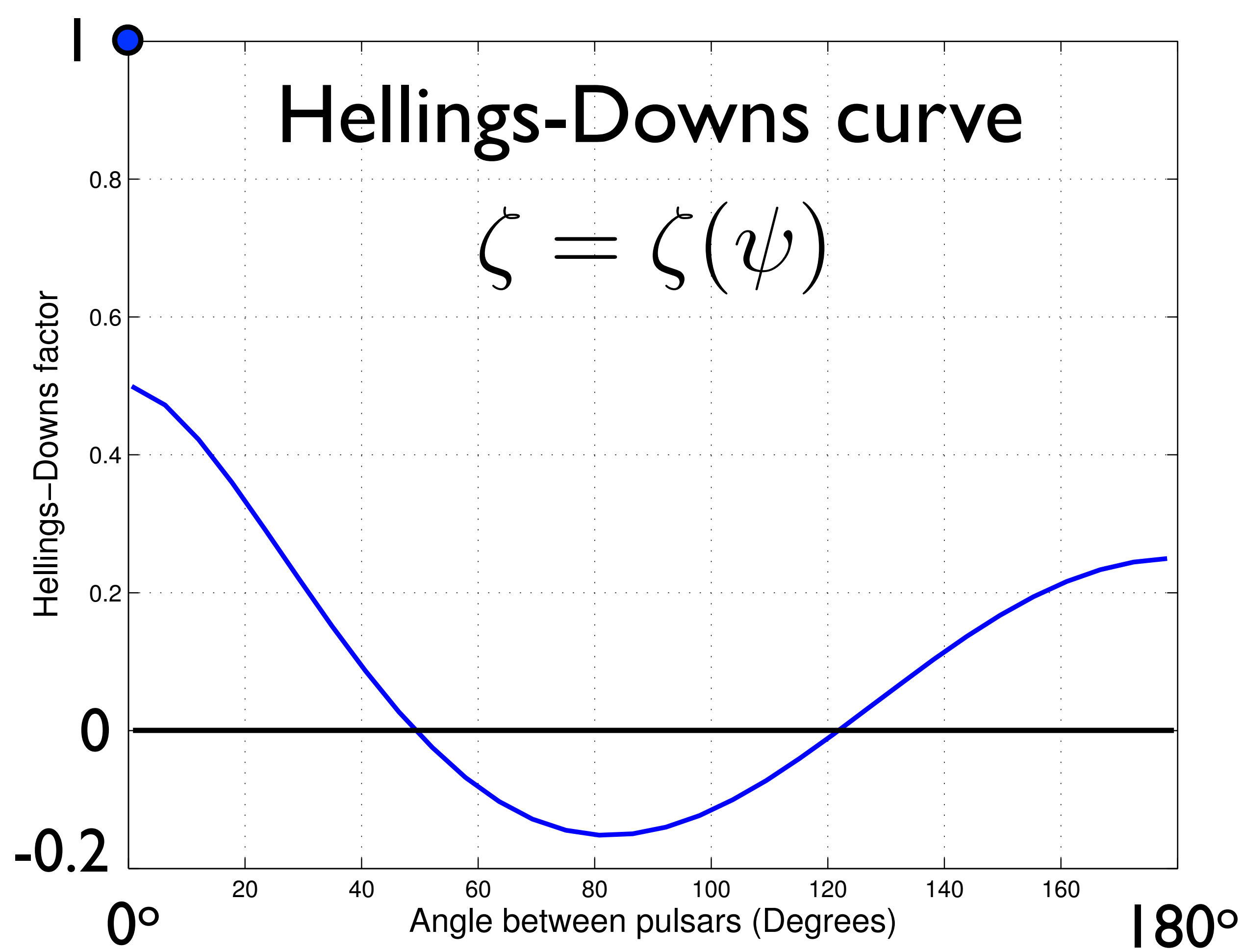
pulsar-pulsar contribution for autocorrelation

fixed frequency, different pulsar pairs



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LIGO Hanford-LIGO Livingston overlap function (long-wavelength approx)

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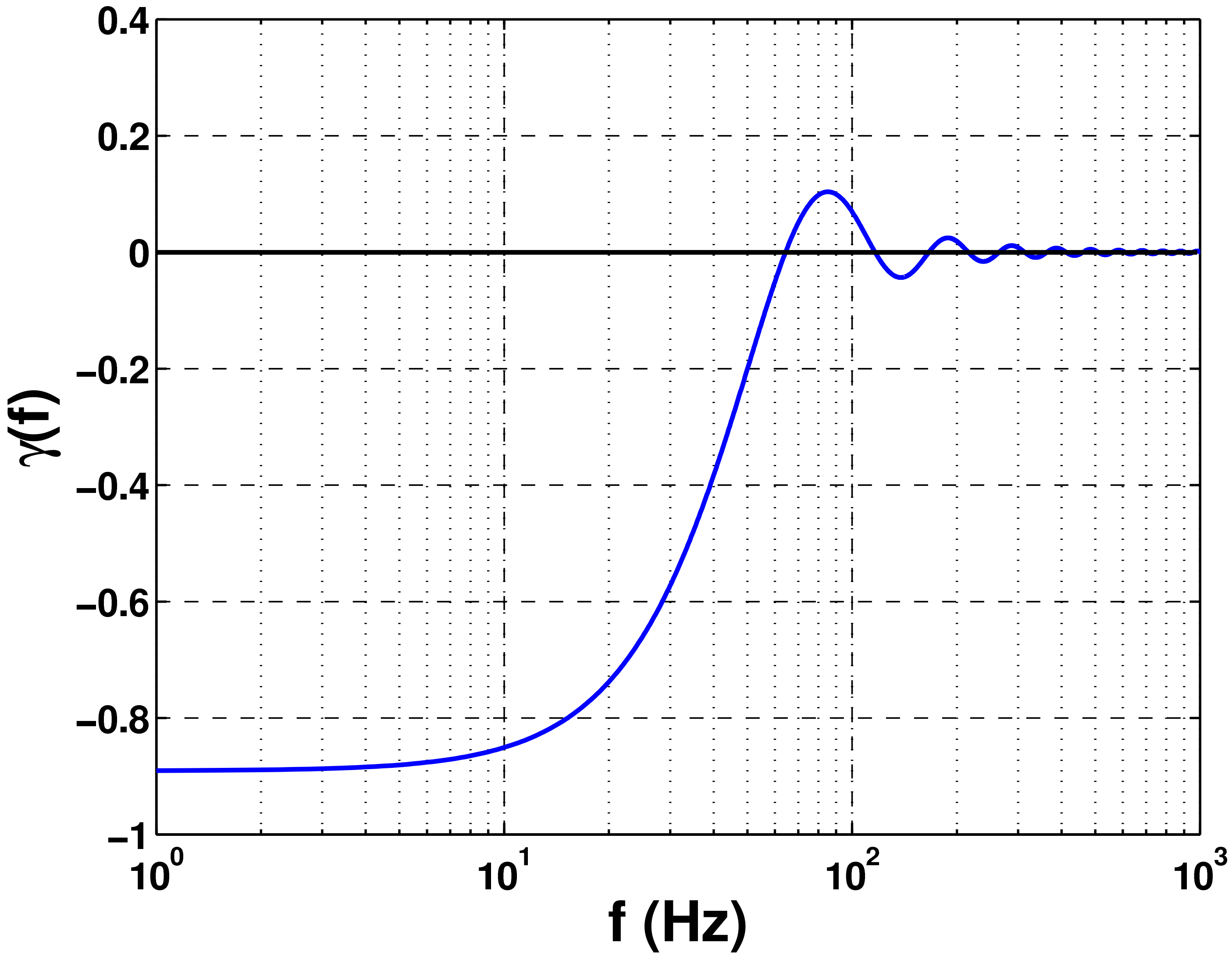

$\Gamma_{IJ}(f)$ = sum of spherical Bessel functions $j_0(\alpha)$, $j_1(\alpha)$, $j_2(\alpha)$, where $\alpha=2\pi|\Delta\vec{x}|f/c$, with coeffs depending on separation vector $\Delta\vec{x}$, and the detector tensors

variable frequency, fixed detector pair

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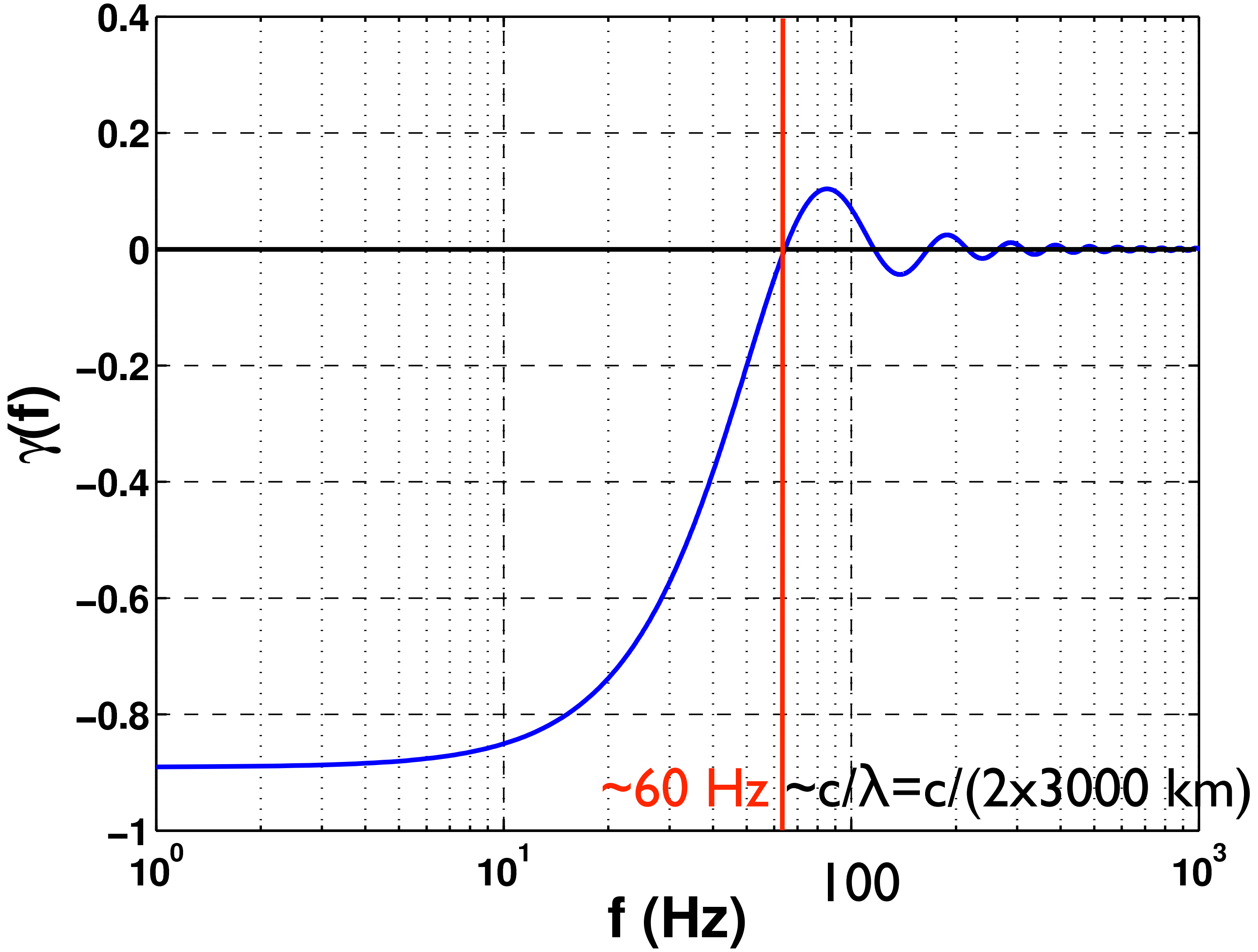

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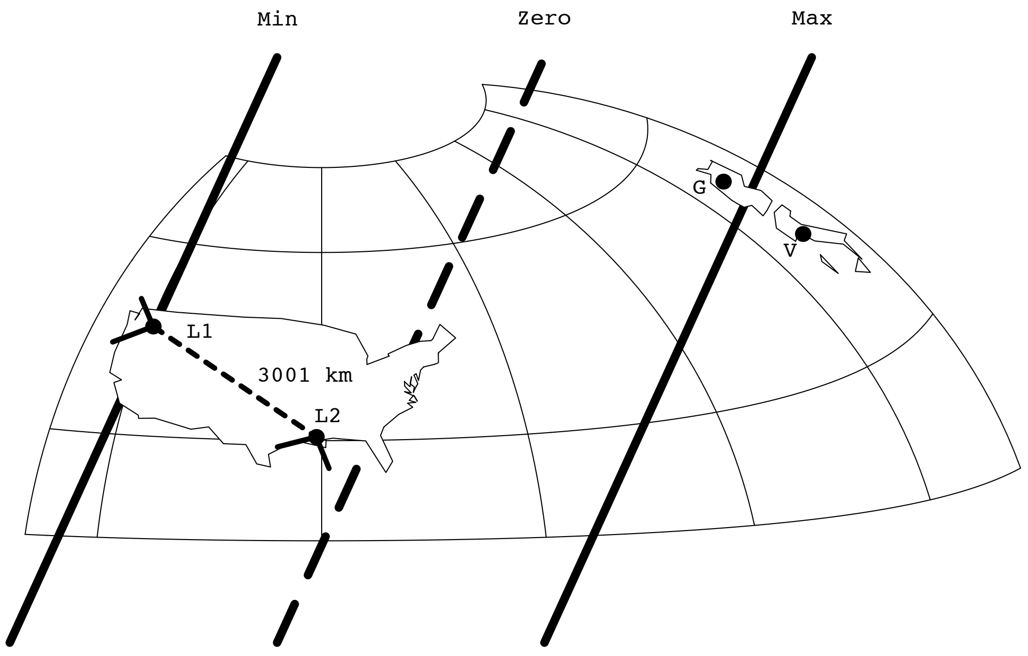
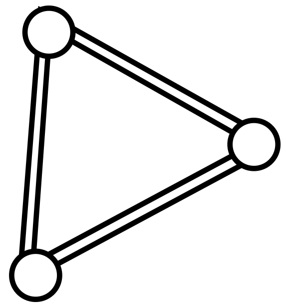
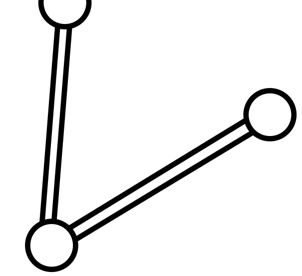




4. (e)LISA data analysis



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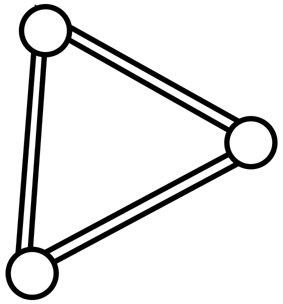

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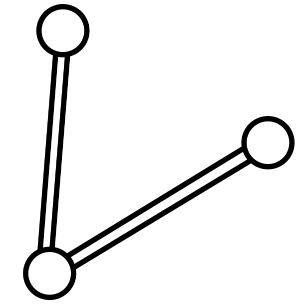

1. Cross-correlation is not an option for (e)LISA; have to resort to single-detector methods to discriminate signal from noise

   a. LISA: 6 links, 3 michelsons (X,Y,Z), 3 noise and signal orthogonal channels (A,E,T) (T is a null channel - insensitive to GW at low f)

   b. eLISA: 4 links, single michelson X (no null channel)

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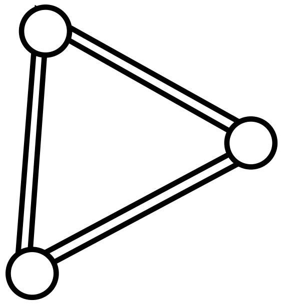

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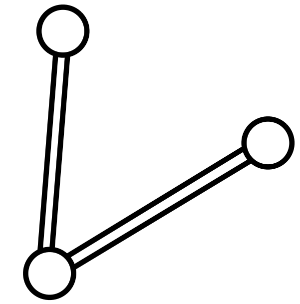

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2. Proper modeling of instrumental noise, astrophysical foregrounds (galactic white dwarf binaries), and GW background allows you to discriminate all three components (Adams & Cornish, 2010, 2014):
   a. $\Omega_{gw}(f) \sim$ few x 10^{-13} with 1-year data
   b. Reduction from 6 to 4 links increases $\Omega_{gw}(f)$ by only ~3x
   c. Null channel not crucial; becomes important if noise is not well understood (e.g., non-Gaussian or non-stationary)

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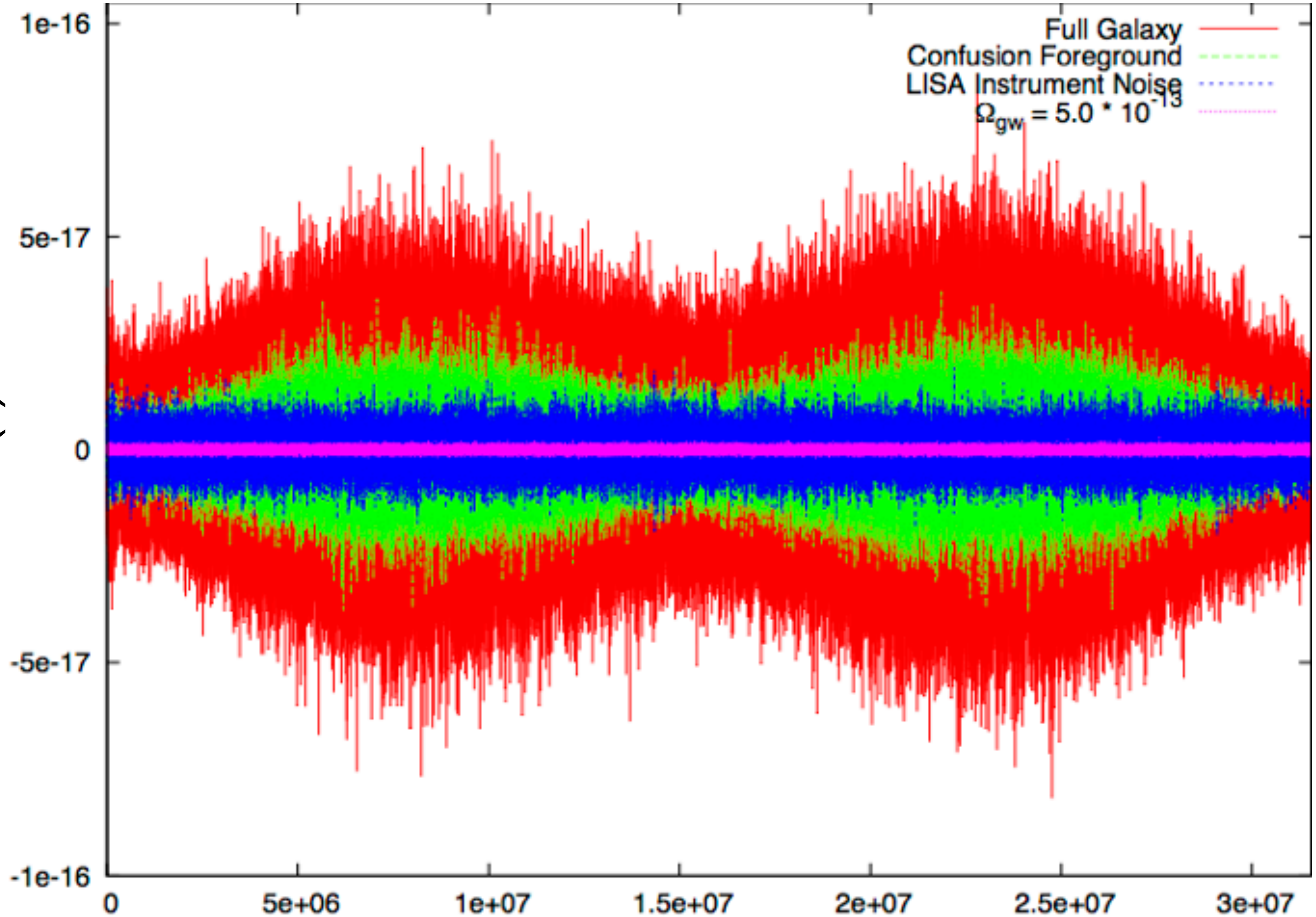
Astrophysical foreground

yearly modulation of foreground discriminates it from other noise components



X(t)

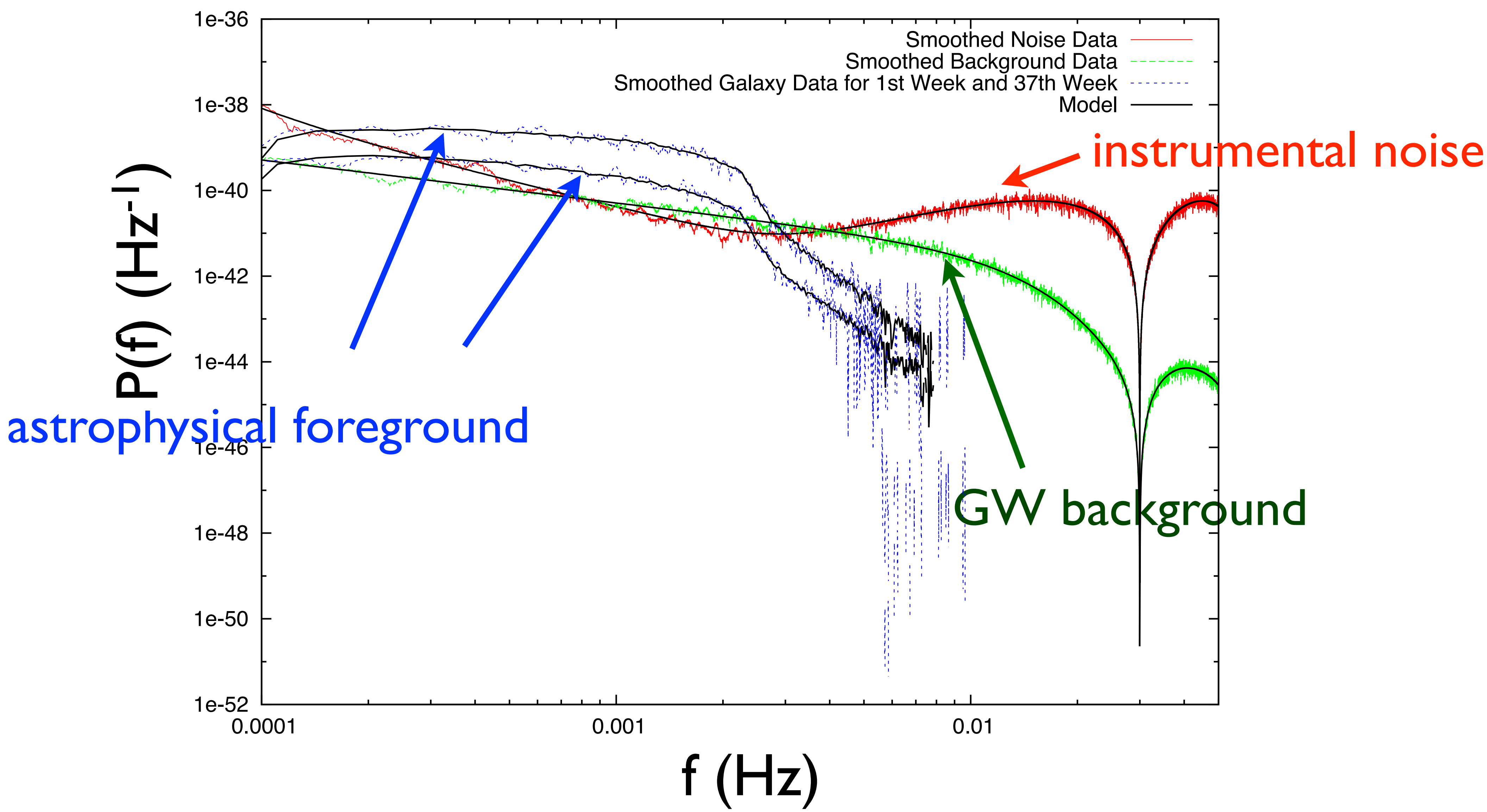
$\Omega_{gw}(f)=5 \times 10^{-13}$

t (sec)

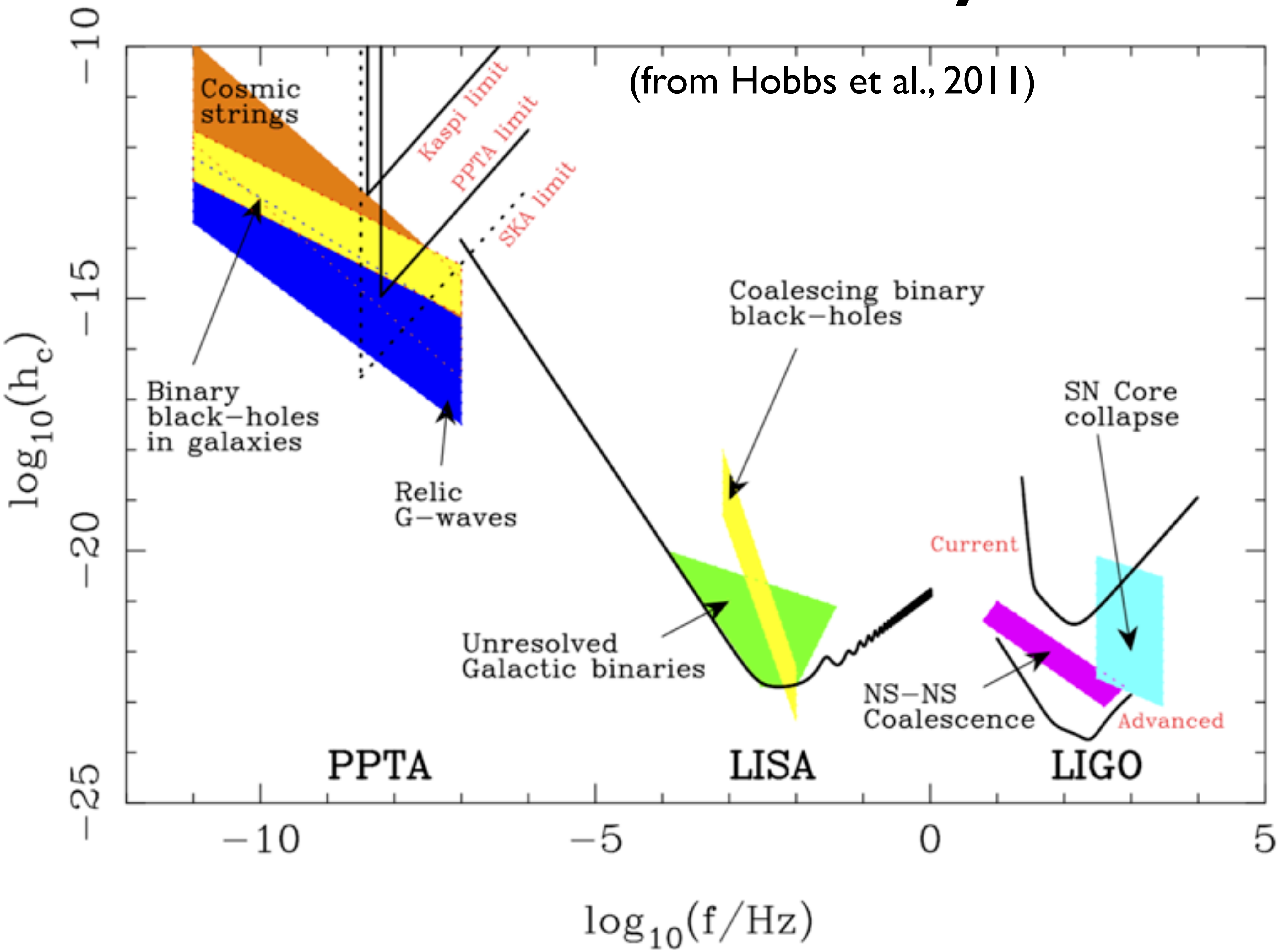
1 year

Model spectra

different spectral shapes allow to differentiate the different noise components



5. Detection sensitivity curves



Remarks about sensitivity curves

1. Can also plot $S_h(f)$ or $\Omega(f)$ versus frequency for different detectors
2. Detection sensitivity curves should be different for different sources:
    a. transient sources (e.g., binary coalescence, supernovae)
    b. long-lived sources (continuous waves or stochastic background)
3. Sensitivity curves should reflect details of the data analysis method (e.g. cross-corr or single-detector; integration over time and freq)

Power-law integrated sensitivity curves for cross-corr searches

For a set of power-law indices, find the amplitude of the background that make the network SNR = 1. Sensitivity curve is envelope of these power-law curves.

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Network signal-to-noise ratio

$$\rho = \sqrt{2T}\left[\int_{f_{\min}}^{f_{\max}} df \sum_{I=1}^{M}\sum_{J>I}^{M} \frac{\Gamma_{IJ}^2(f) S_h^2(f)}{P_{nI}(f) P_{nJ}(f)}\right]^{1/2}$$

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Effective strain noise spectral density

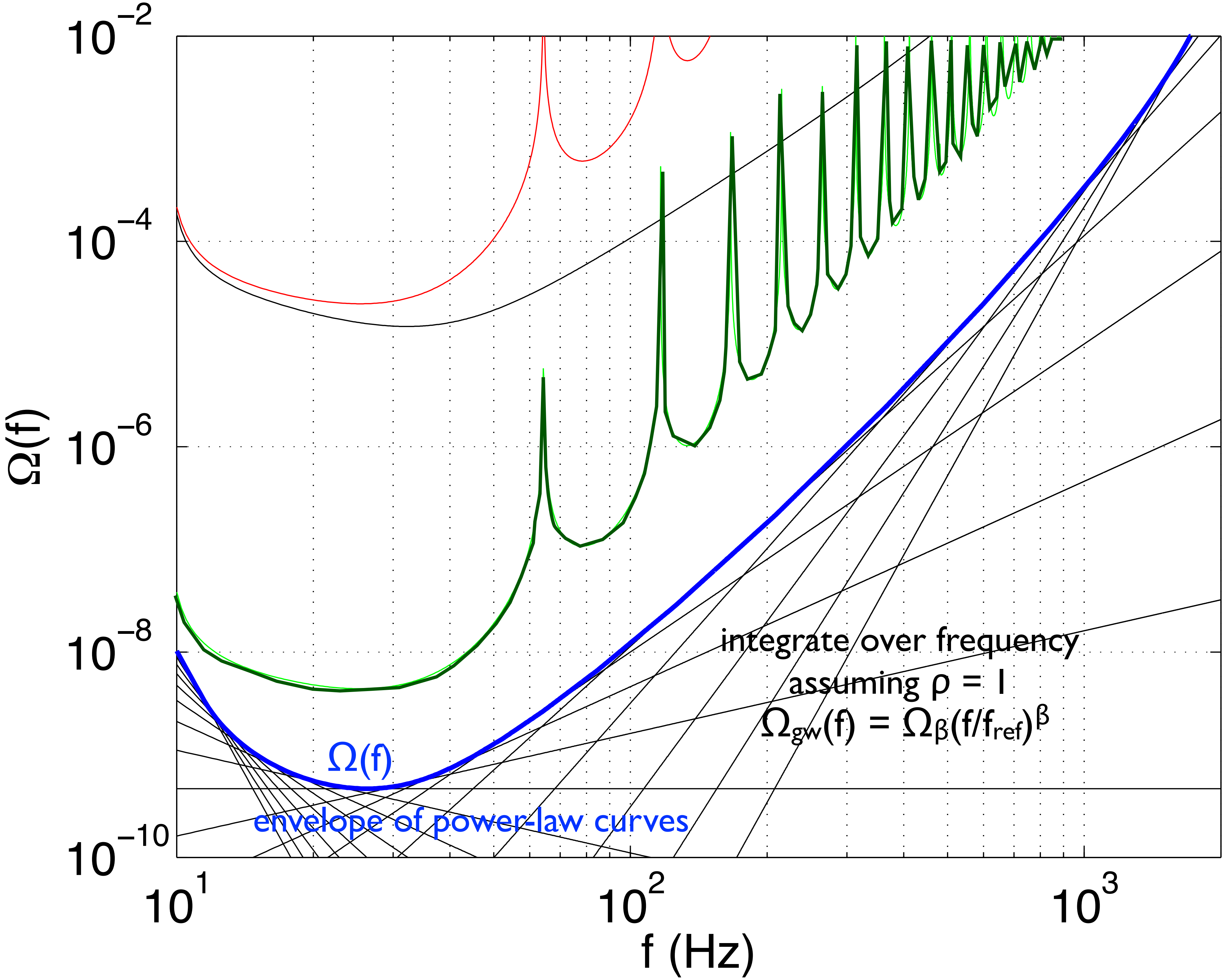
$$S_{\rm eff}(f) = \left[\sum_{I=1}^{M}\sum_{J>I}^{M} \frac{\Gamma_{IJ}^2(f)}{P_{nI}(f)P_{nJ}(f)}\right]^{-1/2}$$

$S_{\rm eff}(f) = P_{nI}(f)/\Gamma_{II}(f)$ (for a single detector)

$S_{\rm eff}(f) = \sqrt{P_{nI}(f)P_{nJ}(f)}/|\Gamma_{IJ}(f)|$ (for a pair of detectors)

Power-law integrated sensitivity curves

(for LIGO-Hanford, LIGO-Livingston)



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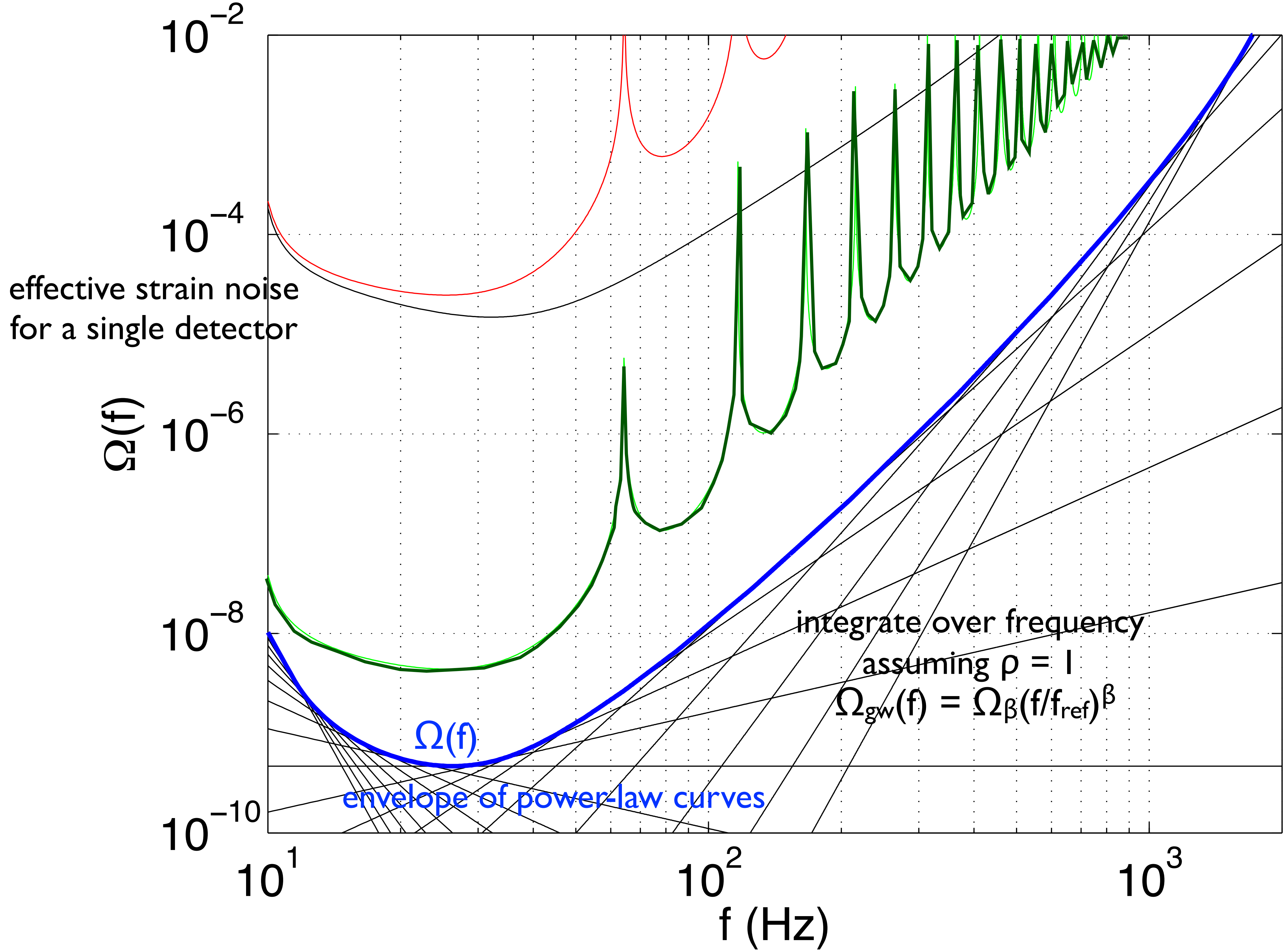

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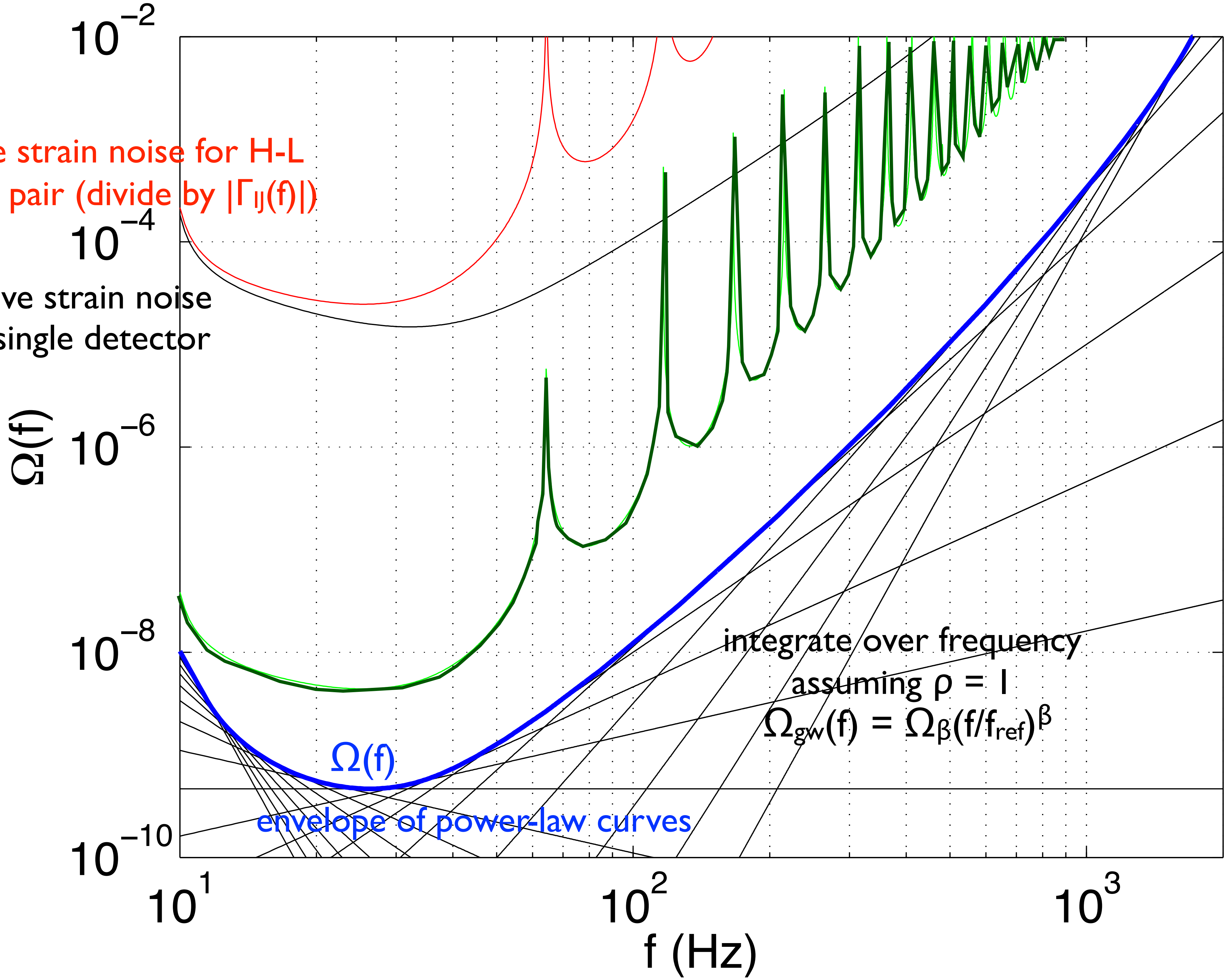
Power-law integrated sensitivity curves
(for LIGO-Hanford, LIGO-Livingston)



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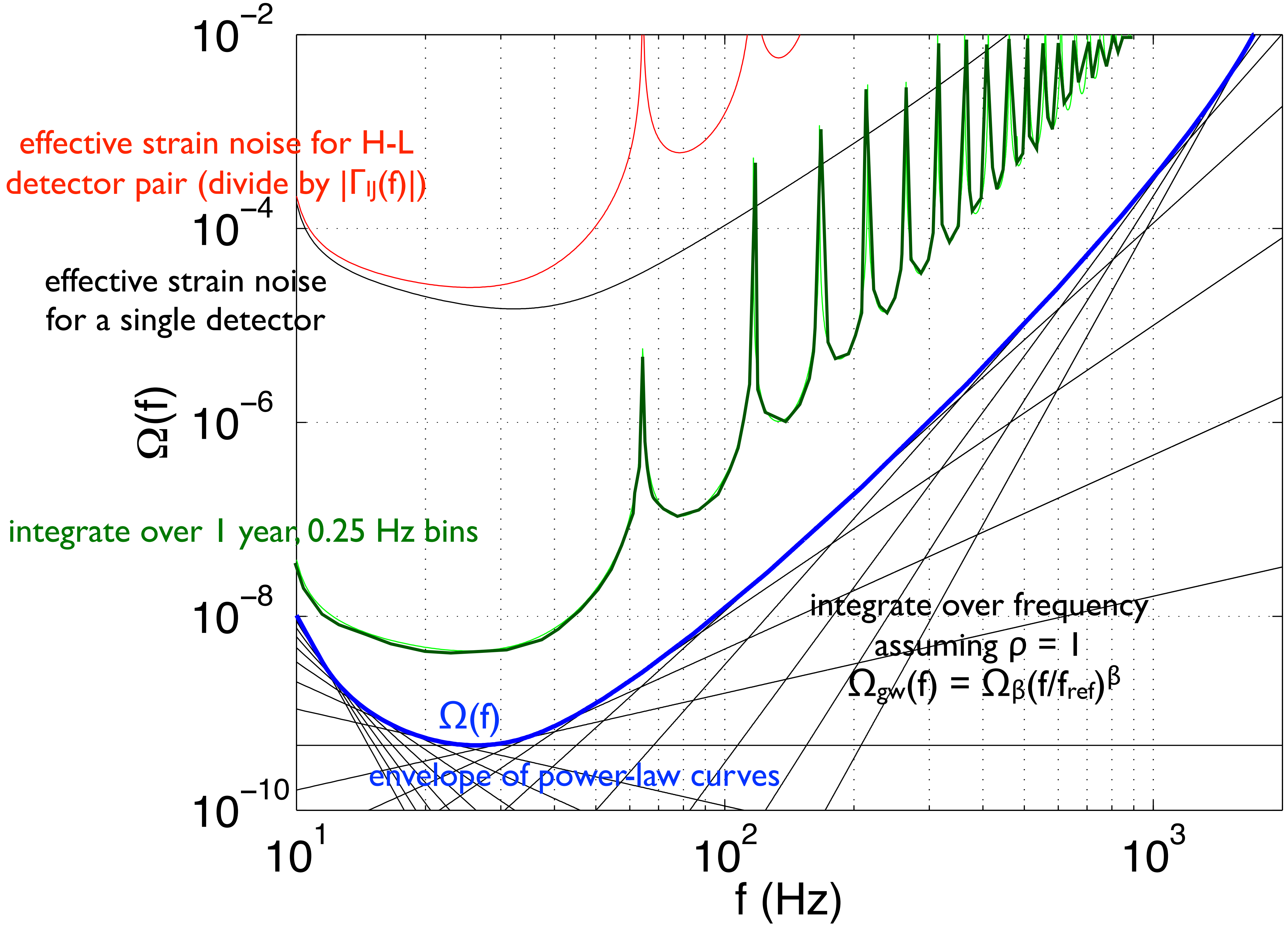
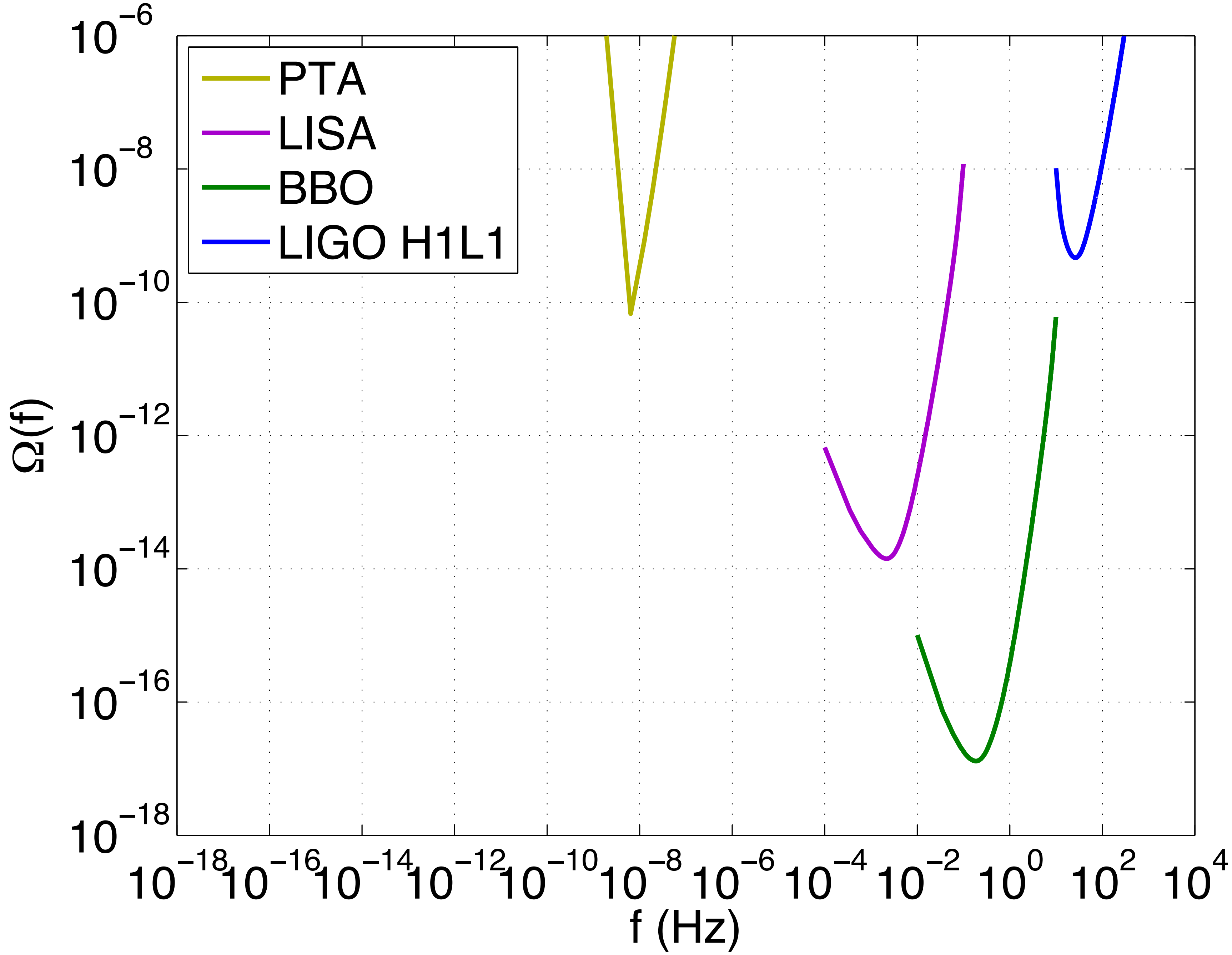
Power-law integrated sensitivity curves

(for LIGO-Hanford, LIGO-Livingston)



Projected power-law integrated sensitivity curves



Summary

1. Detecting a stochastic GW background is important because it can provide info about both astrophysical source populations and the very early Universe, which are inaccessible by other means

2. Detection is challenging because a stochastic GW signal it is just another source of noise in a single detector

3. Cross-correlation methods can be used whenever you have multiple detectors that all respond to the common GW background (e.g., LIGO, Virgo, ... pulsar timing)

4. Proper modeling of instrument noise and GW signal can be used to discriminate between signal and noise if the frequency spectra or time-domain behavior are different (e.g., (e)LISA)

extra slides

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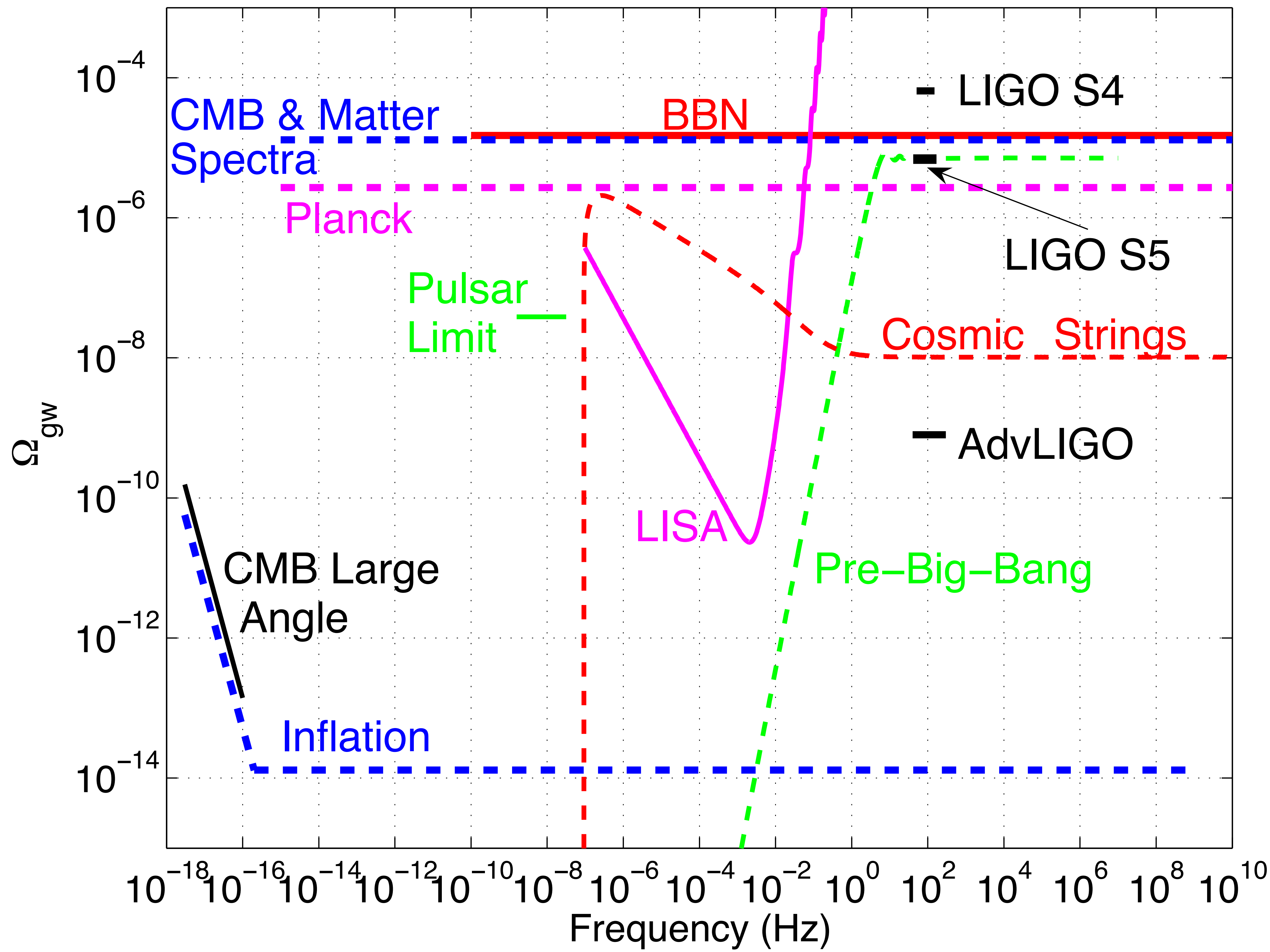
(from Abbott et al., 2009)



Different types of response

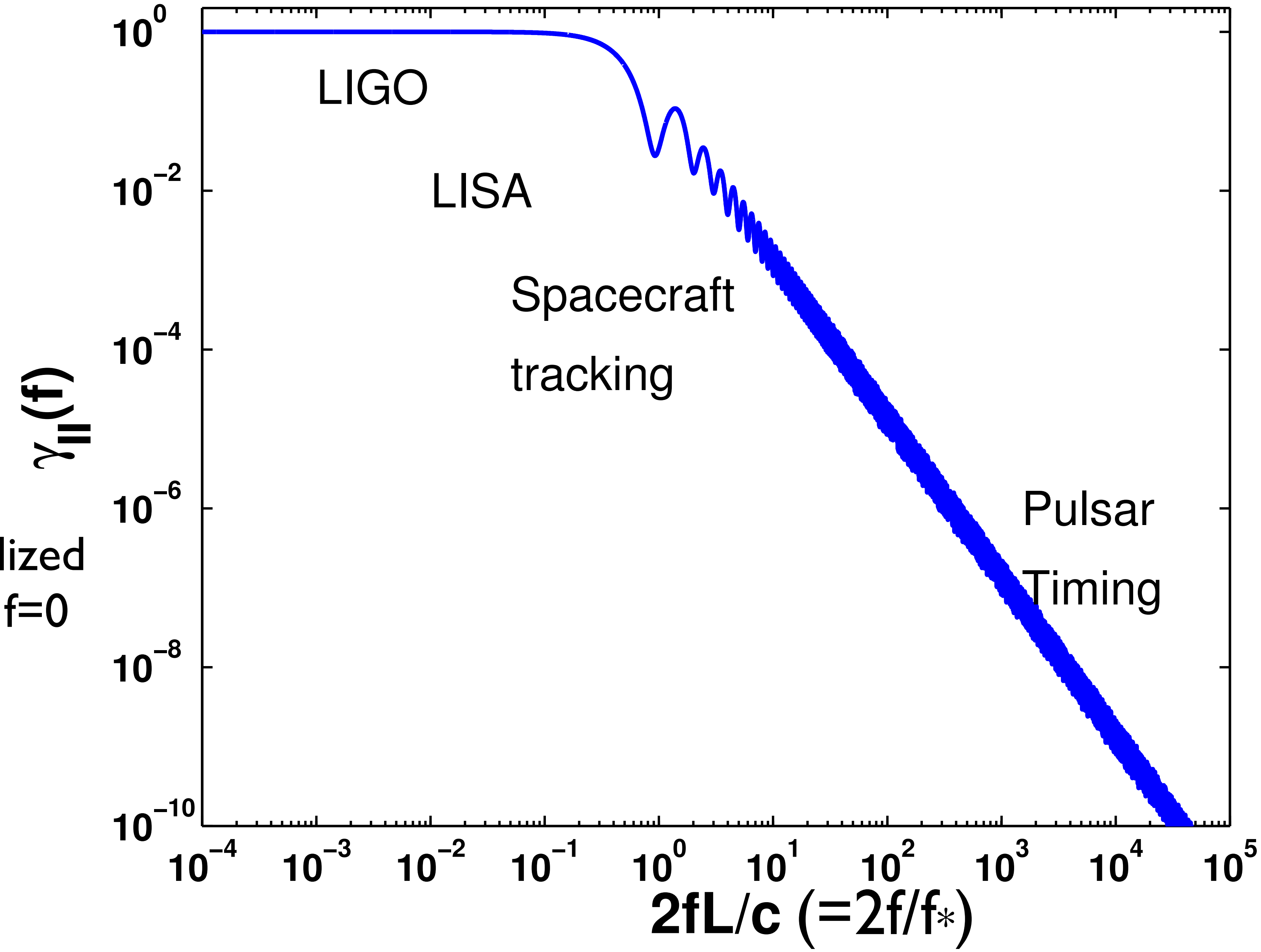
$$r_{\text{strain}}(t) \equiv \frac{\Delta L(t)}{L} = \frac{\Delta T(t)}{2L/c}$$

$$r_{\text{phase}}(t) \equiv \Delta\Phi(t) = 2\pi\nu_0 \Delta T(t)$$

$$r_{\text{doppler}}(t) \equiv \frac{\Delta\nu(t)}{\nu_0} = \frac{d\Delta T(t)}{dt}$$

$$r_{\text{timing}}(t) = \Delta T(t)$$

$\Gamma_{II}(f)$ normalized to unity at f=0



Beam detector	L (km)	f_* (Hz)	f (Hz)	f/f_*	Relation
Ground-based interferometer	~ 1	$\sim 10^5$	$10 - 10^4$	$10^{-4} - 10^{-1}$	$f \ll f_*$
Space-based interferometer	$\sim 10^6$	$\sim 10^{-1}$	$10^{-4} - 10^{-1}$	$10^{-3} - 1$	$f \lesssim f_*$
Spacecraft Doppler tracking	$\sim 10^9$	$\sim 10^{-4}$	$10^{-6} - 10^{-3}$	$10^{-2} - 10$	$f \sim f_*$
Pulsar timing	$\sim 10^{17}$	$\sim 10^{-12}$	$10^{-9} - 10^{-7}$	$10^3 - 10^5$	$f \gg f_*$

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Acts as a noise calibrator for the system if you how it is related to the noise in the other channel.

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Example:

$$d_1 = h + n_1$$

$$d_2 = n_2$$

Covariance matrix:

$$C = \begin{pmatrix} \sigma_{n_1}^2 + \sigma_h^2 & 0 \\ 0 & \sigma_{n_2}^2 \end{pmatrix}$$

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transfer function
(estimate or model)

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then

$$\hat{\sigma}_{n_2}^2 = d_2^2$$
$$\hat{\sigma}_{n_1}^2 = \alpha\, d_2^2$$
$$\hat{\sigma}_h^2 = d_1^2 - \alpha\, d_2^2$$

transfer function
(estimate or model)

Likelihood function

- Frequentists and Bayesians both assume that the measured data are drawn from an underlying probability distribution given a particular hypothesis or model (called the likelihood function)

- For the case of an additive signal in noise d = Rh + n we have:

noise and signal parameters detector response

$$p(\mathbf{d}|\boldsymbol{\theta}_n, \boldsymbol{\theta}_h) = \int p_n(\mathbf{d} - \mathbf{R}\mathbf{h}|\boldsymbol{\theta}_n)p_h(\mathbf{h}|\boldsymbol{\theta}_h)\, d\mathbf{h}$$

- For stochastic signals, h is random, so we marginalize (i.e., integrate) over it

- For multivariate Gaussian distributions for the noise and signal with covariance matrices C_n and C_h, the likelihood function is also multivariate Gaussian with covariance matrix $C = C_n + R\, C_h\, R^T$

- But we can consider other probability distributions as well -- e.g., for non-Gaussian noise and non-Gaussian stochastic background analyses

Overview of analysis methods

EARLY ANALYSES (before 2000)	MORE RECENT ANALYSES
used frequentist statistics	have used Bayesian inference
used cross-correlation methods	typically use cross-correlation methods, but use null channel or knowledge about instrument noise when cross-corr not available
assumed stationary, Gaussian noise	have allowed non-Gaussian noise
assumed stationary, Gaussian, unpolarized, and isotropic stochastic GW backgrounds	have allowed non-Gaussian, polarized, and anisotropic stochastic GW backgrounds
were done in the context of ground-based detectors (e.g., resonant bars and LIGO-like interferometers) where the long-wavelength approximation is valid	have been done in the context of space-based detectors (e.g., spacecraft tracking, LISA) and pulsar timing arrays for which the long-wavelength approximation is no longer valid

Despite apparent differences, ALL analyses use a likelihood function (e.g., as a sampling distribution for frequentist statistics or for calculating posterior distributions for Bayesian inference) and take advantage of cross-correlation if multiple detectors are available

FREQUENTIST	BAYESIAN
probabilities assigned only to propositions about outcomes of repeatable experiments (i.e., *random variables*), not to hypotheses or parameters, which have *fixed but unknown* values	probabilities can be assigned to hypotheses and parameters, since probability is degree of belief (or confidence, plausibility) in any proposition
assumes measured data are drawn from an underlying probability distribution, which assumes the truth of a particular hypothesis or model (likelihood function)	same
constructs a statistic to estimate a parameter, or to decide whether or not to claim a detection	needs to specify prior degree of belief in a particular hypothesis or parameter
calculates the probability distribution of the statistic (sampling distribution)	uses Bayes' theorem to update prior degree of belief in light of new data (likelihood + prior $\rightarrow$ posterior)
constructs confidence intervals and p-values (for parameter estimation and hypothesis testing)	constructs posteriors and odds ratios (for parameter estimation and hypothesis testing)

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Likelihood function is starting point for both frequentist and Bayesian analyses!!	
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Example: Frequentist optimally-filtered cross-correlation statistic

Generalization of the simple cross-correlation statistic to include a filter function Q(t-t') chosen to maximize the SNR of the statistic

$$Y \equiv \int_0^T dt \int_0^T dt' \, d_1(t) d_2(t') Q(t-t') = \int df \, \tilde{d}_1(f) \tilde{d}_2^*(f) \tilde{Q}(f)$$

where $$\mathrm{SNR}_Y = \frac{\langle Y \rangle}{\sqrt{\langle Y^2 \rangle - \langle Y \rangle^2}}$$

Typical assumptions:
1) Stationary data (Q(t,t') = Q(t-t'))
2) Uncorrelated noise (Y is unbiased)
3) Weak signal (noise power is just the measured auto-correlated power)
4) Only unknown is the overall strength of the GW background

Optimally-filtered statistic

Filtered cross-correlation:

stationary

$$\boxed{Y_{IJ} := \int_0^T dt \int_0^T dt'\; x_I(t)x_J(t')Q(t-t') = \int df\; \tilde{x}_I(f)\tilde{x}_J^*(f)\tilde{Q}(f)}$$

Expected value and variance:

$$\langle Y_{IJ}\rangle = T\int df\;\gamma_{IJ}(f)H(f)\tilde{Q}(f) \equiv T\left(\tilde{Q}, \frac{\gamma_{IJ}H}{P_I P_J}\right)$$

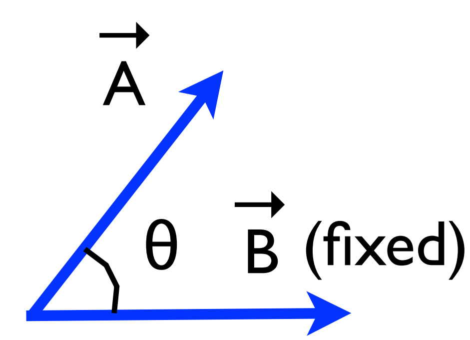
$$\sigma^2_{Y_{IJ}} \approx T\int df\; P_I(f)P_J(f)|\tilde{Q}(f)|^2 \equiv T(\tilde{Q},\tilde{Q})$$

inner product

weak signal approx

Choose Q to maximize signal-to-noise ratio:

$$\mathrm{SNR}^2 \equiv \frac{\langle Y_{IJ}\rangle^2}{\sigma^2_{Y_{IJ}}} \propto \frac{\left(\tilde{Q}, \frac{\gamma_{IJ}H}{P_I P_J}\right)^2}{(\tilde{Q},\tilde{Q})} \quad \Rightarrow \quad \boxed{\tilde{Q}(f) \propto \frac{\gamma_{IJ}(f)H(f)}{P_I(f)P_J(f)}}$$



$$\frac{|\vec{A}\cdot\vec{B}|^2}{|\vec{A}|^2} = |\vec{B}|^2\cos^2\theta$$