GW propagation in the inhomogeneous universe

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Outline

♦ Inspiral **binary** system

♦ Large-scale **structure** affect the wave-form and the amplitude of the wave

♦ Two main effects
  • Change in the **redshift** → frequency and mass
  • Change in the luminosity **distance** → amplitude
Inspiral binary system

euclidian universe expanding perturbed universe

First stage:

- We solve the trajectory of the two masses assuming a quasi-circular orbit. The system looses energy through GW emission, leading to an inspiral.

- Assuming that the change in radius is slow, we calculate the energy momentum tensor and find the two polarisations.
Polarisations

\[ h_+ (t) = \frac{2}{r} M_c^{5/3} \left( \frac{\pi f(t_{\text{ret}})}{r} \right)^{2/3} \left( 1 + \cos^2 \varphi \right) \cdot \cos \Phi(t_{\text{ret}}) \]

\[ h_\times (t) = \frac{4}{r} M_c^{5/3} \left( \frac{\pi f(t_{\text{ret}})}{r} \right)^{2/3} \cos \varphi \cdot \sin \Phi(t_{\text{ret}}) \]

Chirp mass

\[ M_c = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \quad t_{\text{ret}} = t - \frac{r}{c} \]

The frequency evolves

\[ \dot{f} = \frac{96}{5} \pi^{8/3} \left( \frac{G M_c}{c^3} \right)^{5/3} f^{11/3} \]

\[ f(\tau) = \frac{1}{\pi} \left( \frac{5}{256 \tau} \right)^{3/8} \left( \frac{G M_c}{c^3} \right)^{-5/8} \quad \tau = t - t_{\text{coal}} \]

\[ \Phi(\tau) = \int_{\tau}^{\tau} d\tau' f(\tau') = -2 \left( \frac{5G M_c}{c^3} \right)^{-5/8} \tau^{5/8} + \Phi_0 \]

see e.g. Maggiore, Gravitational Waves, 2008
Second stage: propagation

euclidian universe ↔ expanding perturbed universe

GWs propagate on null geodesics, that depend on the metric.

Three effects:

\[ h_+ (\tau) = \frac{2}{r} M_c^{5/3} \left( \pi f(\tau) \right)^{2/3} \left( 1 + \cos^2 \varphi \right) \cdot \cos \Phi(\tau) \]

- frequency \[ f_S \rightarrow f_O \]
- phase \[ \Phi_S \rightarrow \Phi_O \]
- distance \[ 1/r \] governs the dilution of energy, which is different in a non-euclidian universe.
Frequency and phase

♦ Frequency

\[ 1 + z \equiv \frac{E_S}{E_O} = \frac{f_S}{f_O} \quad \Rightarrow \quad f_O = \frac{f_S}{1 + z} \]

♦ Phase

\[ \Phi_S(\tau_S) = \int_{\tau_S}^{\tau_S} d\tau'_S f_S(\tau'_S) = \int_{1+z}^{1+z} \frac{d\tau'_O}{1 + z} (1 + z)f_O(\tau'_O) = \Phi_O(\tau_O) \]

The phase is \textbf{constant} on null geodesics:

\[ k^\mu = -\partial_\mu \Phi \]

\[ k^\mu k_\mu = 0 = -k^\mu \partial_\mu \Phi = -\frac{d\Phi}{d\lambda} \quad \Rightarrow \quad \Phi = \text{const} \]
Distance

♦ In an euclidian universe, $1/r$ comes from the wave propagation equation

$$\Box h_+ = \Box h_\times = 0 \quad \text{with} \quad \Box = \partial_r^2 - \partial_t^2$$

♦ In a non-euclidian universe

$$\Box = g^{\mu \nu} D_\mu D_\nu$$

\[ \frac{1}{r} \rightarrow \frac{1 + z}{d_L} \]

Luminosity distance: \[ d_L \equiv \sqrt{\frac{L}{4\pi F}} \]

The **luminosity distance** tells us how the energy emitted by the source is spread during propagation in a generic universe.

Schutz, 1986

Laguna, Larson, Spergel and Yunes, 2009
At the observer

\[ h_+ = \frac{2}{d_L} (1 + z)^{5/3} M_c^{5/3} \left( \pi f_O(\tau_O) \right)^{2/3} (1 + \cos^2 \varphi) \cdot \cos \Phi_O(\tau_O) \]

\[ h_\times = \frac{4}{d_L} (1 + z)^{5/3} M_c^{5/3} \left( \pi f_O(\tau_O) \right)^{2/3} \cos \varphi \cdot \sin \Phi_O(\tau_O) \]

Redshifted chirp mass: \[ M_c = (1 + z) M_c \]

\[ \frac{df_S}{d\tau_S} = \frac{96}{5} \pi^{8/3} \left( \frac{GM_c}{c^3} \right)^{5/3} f_S^{11/3} \]

\[ f_S = (1 + z) f_O \]

\[ d\tau_S = \frac{d\tau_O}{1 + z} \]

\[ \frac{df_O}{d\tau_O} = \frac{96}{5} \pi^{8/3} \left( \frac{GM_c}{c^3} \right)^{5/3} f_O^{11/3} \]

assuming a constant redshift
At the observer

\[ h_+ = \frac{2}{d_L} \mathcal{M}_c^{5/3} \left( \pi f_O(\tau_O) \right)^{2/3} \left( 1 + \cos^2 \varphi \right) \cdot \cos \Phi_O(\tau_O) \]

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assuming a constant redshift

Schutz, 1986
At the observer

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h_+ = \frac{2}{d_L} M_c^{5/3} \left( \pi f_O(\tau_O) \right)^{2/3} \left( 1 + \cos^2 \varphi \right) \cdot \cos \Phi_O(\tau_O)
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Redshifted chirp mass: \( M_c = (1 + z) M_c \)

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\]

\[
f_S = (1 + z) f_O
\]

\[
d\tau_S = \frac{d\tau_O}{1 + z}
\]

assuming a constant redshift.
The effect of structures: redshift

\[ ds^2 = -a^2(\eta) \left[ (1 + 2\Psi(x, \eta)) d\eta^2 + (1 - 2\Phi(x, \eta)) \delta_{ij} dx^i dx^j \right] \]

Perturbations affect: ♦ the redshift ➔

- frequency \( f_O = \frac{f_S}{1 + z} \)
- mass \( M_c = (1 + z) M_c \)
- ♦ the luminosity distance

We solve the null geodesic equation at linear order

\[ \frac{dk^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta = 0 \]

\[ 1 + z = \frac{E_S}{E_O} = \left( \frac{g_{\mu\nu} k^\mu v^\nu}{(g_{\mu\nu} k^\mu v^\nu)_O} \right)_S \]
The effect of structures: redshift

\[ ds^2 = -a^2(\eta) \left[ (1 + 2\Psi(x, \eta)) d\eta^2 + (1 - 2\Phi(x, \eta)) \delta_{ij} dx^i dx^j \right] \]

\[ 1 + z = \frac{a_O}{a_S} \left[ 1 + \mathbf{v}_S \cdot \mathbf{n} - \mathbf{v}_O \cdot \mathbf{n} + \Psi_O - \Psi_S - \int_0^{r_S} dr (\dot{\Phi} + \dot{\Psi}) \right] \]

Doppler \hspace{1cm} Gravitational redshift \hspace{1cm} Integrated Sachs-Wolfe

Gravitational redshift:
The effect of structures: redshift

\[ ds^2 = -a^2(\eta) \left[ (1 + 2\Psi(x, \eta)) d\eta^2 + (1 - 2\Phi(x, \eta)) \delta_{ij} dx^i dx^j \right] \]

\[ 1 + z = \frac{a_O}{a_S} \left[ 1 + \mathbf{v}_S \cdot \mathbf{n} - \mathbf{v}_O \cdot \mathbf{n} + \Psi_O - \Psi_S - \int_0^{r_S} dr \left( \dot{\Phi} + \dot{\Psi} \right) \right] \]

- Doppler
- Gravitational redshift
- Integrated Sachs-Wolfe

ISW: integrated along the trajectory, sensitive to dark energy.
The effect of structures: distance

\[ d_L = \sqrt{\frac{L}{4\pi F}} = (1 + z) \sqrt{\frac{dA_O}{d\Omega_S}} \]

\[ \frac{D^2 \xi^\alpha (\lambda)}{D \lambda^2} = R^\alpha_{\beta\mu\nu} k^\beta k^\mu \xi^\nu \]

\[ k^\alpha \text{ momentum} \]

\[ \xi^\alpha \text{ connection vector} \]

In a homogeneous and isotropic expanding universe:

\[ d_L(z_S) = (1 + z_S) r_S = (1 + z_S) \int_0^{z_S} \frac{dz}{H(z)} \]
Luminosity distance perturbations

Bonvin, Durrer and Gasparini, 2006
Hui and Greene, 2006

\[ d_L = (1 + z_S) r_S \left\{ 1 + \frac{1}{H_S r_S} \mathbf{v}_O \cdot \mathbf{n} + \left( 1 - \frac{1}{H_S r_S} \right) \mathbf{v}_S \cdot \mathbf{n} \right\} \]

- **Doppler**

\[ \text{Doppler} \]

- **Gravitational potential**

\[ \text{Gravitational potential} \]

- **Time delay**

\[ + \frac{2}{r_S} \int_0^{r_S} \text{d}r \, \dot{\Psi} - 2 \left( 1 - \frac{1}{H_S r_S} \right) \int_0^{r_S} \text{d}r \, \dot{\Psi} \]

- **Integrated Sachs-Wolfe**

\[ \text{Integrated Sachs-Wolfe} \]

- **Lensing**

\[ - \int_0^{r_S} \text{d}r \, \frac{(r_S - r)}{r_S r} \Delta \Psi \]

source

observer
Impact for observations: redshift

The **frequency** and the redshifted chirp **mass** are perturbed.  

- If we don’t know the chirp mass, we measure it through:

\[
\frac{df_O}{d\tau_O} = \frac{96}{5} \pi^{8/3} \left( \frac{G M_c}{c^3} \right)^{5/3} f_O^{11/3}
\]

and then we plug the **perturbed** mass into the polarisations

\[
h_+ = \frac{2}{d_L} M_c^{5/3} \left( \pi f_O(\tau_O) \right)^{2/3} (1 + \cos^2 \varphi) \cdot \cos \Phi_O(\tau_O)
\]

- If we know the chirp mass (e.g. neutron stars binaries), and we look at close-by binaries, for which the expansion is negligible, then the perturbations induce an **error** on the measured mass.
Error on the chirp mass

\[ M_{c}^{\text{meas.}} = M_{c}^{N-N} + \delta M_{c} \]

Dominant effect due to velocities

\[ \frac{\delta M_{c}}{M_{c}} \approx \frac{\delta z}{1 + z} = (v_{S} - v_{O})n \]
Impact for observations: distance

- We measure the perturbed luminosity distance

\[ h_+ = \frac{2}{d_L} \mathcal{M}_c^{5/3} \left( \pi f_O(\tau_O) \right)^{2/3} (1 + \cos^2 \varphi) \cdot \cos \Phi_O(\tau_O) \]

- If we want to extract cosmological parameters, the perturbations will induce an error.

- Two dominant contributions:
  - At small redshift: peculiar velocities
  - At large redshift: lensing

Holz and Linder, 2004
\[ d_L = d_{L}^{\text{FRW}} + \delta d_L \]

with non-linearities

5 \(- 10\%\) error

Holz and Hughes, 2005
Other effects

- The deflection has an impact on the **position** of the source in the sky.

\[ \phi = \frac{\pi}{2} \]

- The deflection has an impact on the **orientation** of the binary system with respect to us: even if we see the system edge-on, we will observe a small component \( h_x \).
Conclusion

- Large-scale structure affect both the redshift and the luminosity distance.

- The redshift perturbations change the observed frequency and the redshifted chirp mass. The effect is however negligible, less than 0.1 percent.

- The luminosity distance perturbations on the other hand can reach a few percents.

- At small redshift, the dominant contribution is due to velocities.

- At large redshift, the dominant contribution is due to gravitational lensing.