On-Shell Diagrammatics and the Perturbative Structure of Planar Gauge Theories

Paolo Benincasa

IFT-UAM/CSIC

May 29th – Iberian Strings 2015, Salamanca

P.B. – *to appear*

(Very) minimal bibliography

On-shell diagrams and Grassmannian for $\mathcal{N} = 4$ SYM:

- Arkani-Hamed et al. – arXiv:1212.5605 [hep-th]

Forward amplitudes ($\mathcal{N} \geq 1$) and all-loop RR ($\mathcal{N} = 4$):

- Caron-Huot – arXiv:1007.3224 [hep-th]
- Arkani-Hamed et al. – arXiv:1008.2958 [hep-th]

Mom-twistor/on-shell diagrams/Grassmannian regularisation:

- Bourjaily, Caron-Huot, Trnka – arXiv:1303.4734 [hep-th]
- Ferro, Lukowski, Staudacher – arXiv:1407.6736 [hep-th]
- P.B., Conde, Gordo – arXiv:1411.7987 [hep-th]

Multivariate residues and unitarity:

- Sogaard, Zhang – arXiv:1310.6006 [hep-th]

Reviews:

- Elvang, Huang – arXiv:1308.1697 [hep-th]
- P.B. – arXiv:1312.5583 [hep-th]

How do we understand perturbative field theory?

Textbook formulation: Lagrangian $\longleftrightarrow$ Feynman diagrammatics.

Manifest properties:

- Poincaré invariance
- Unitarity
- Locality of the interactions

Drawbacks:

- Field redefinitions
- Gauge redundancies
- Gauge invariance broken at intermediate stages
- Further structures and symmetries hidden

Constructive approach to perturbation theory

Object of investigation: Scattering amplitudes $\mathcal{M}_n$

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}, \quad \langle i,j \rangle = \epsilon^{ab} \lambda_a^{(i)} \lambda_b^{(j)} \quad [i,j] = \epsilon^{\dot{a}\dot{b}} \tilde{\lambda}_{\dot{a}}^{(i)} \tilde{\lambda}_{\dot{b}}^{(j)}$$

$$P_{ij}^2 \equiv \left(p^{(i)} + p^{(j)}\right)^2 = \langle i,j \rangle [i,j]$$

Constructive approach to perturbation theory

Object of investigation: Scattering amplitudes $\mathcal{M}_n$

$$\text{Poincaré invariance} \begin{cases} \text{Momentum conservation:} \\ \mathcal{M}_n \;=\; \delta^{(2\times 2)}\left(\sum_{i=1}^{n}\lambda^{(i)}\tilde{\lambda}^{(i)}\right) M_n\left(\{\lambda^{(i)},\tilde{\lambda}^{(i)};h_i\}\right) \\ \\ \text{Little group covariance:} \\ \hat{\mathcal{H}}^{(i)}\mathcal{M}_n \;=\; -2h_i\,\mathcal{M}_n \end{cases}$$

Constructive approach to perturbation theory

Object of investigation: Scattering amplitudes $\mathcal{M}_n$

Analytic function ⇓

Poincaré invariance:

- Momentum conservation:
$$\mathcal{M}_n = \delta^{(2\times 2)}\left(\sum_{i=1}^{n} \lambda^{(i)}\tilde{\lambda}^{(i)}\right) M_n\left(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i\}\right)$$

- Little group covariance:
$$\hat{\mathcal{H}}^{(i)}\mathcal{M}_n = -2h_i\,\mathcal{M}_n$$

Constructive approach to perturbation theory

Object of investigation: Scattering amplitudes $\mathcal{M}_n$

Analytic function

Super-Poincaré:

$$\mathcal{M}_n = \delta^{(2\times 2)}\left(\sum_{i=1}^{n} \lambda^{(i)}\tilde{\lambda}^{(i)}\right)\delta^{(2\times\mathcal{N})}\left(\sum_{i=1}^{n} \lambda^{(i)}\tilde{\eta}^{(i)}\right) M_n(\{\lambda^{(i)},\, \tilde{\lambda}^{(i)};\, \tilde{\eta}^{(i)}\})$$

Grassmann coherent states ($\mathcal{N} < 4$):

$$|\lambda,\, \tilde{\lambda};\, \tilde{\eta},\, +\rangle = \sum_{s=0}^{\mathcal{N}} \frac{1}{s!}\left(\prod_{r=0}^{s} \tilde{\eta}^{I_r}\right)|\lambda,\, \tilde{\lambda};\, +(1-s/2)\rangle_{I_1\ldots I_s}$$

$$|\lambda, \tilde{\lambda};\, \tilde{\eta},\, -\rangle = \sum_{s=0}^{\mathcal{N}} \frac{1}{s!}\left(\prod_{r=0}^{s} \tilde{\eta}^{I_r}\right)|\lambda, \tilde{\lambda};\, -(1-(\mathcal{N}-s)/2)\rangle_{I_1\ldots I_s}$$

Building blocks: 3-particle amplitudes

(Super)-Poincaré invariance:

- momentum conservation: $(p^{(1)} + p^{(2)})^2 = (-p^{(3)})^2 = 0$

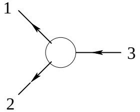
  $$\langle i, j\rangle [i, j] = 0, \forall i \quad \Longrightarrow \quad M_3 = M_3^{(1)}([i, j]) + M_3^{(2)}(\langle i, j\rangle)$$

- little group $\Longrightarrow$ functional form of $M_3^{(k)}$, $k = 1, 2$

Building blocks: 3-particle amplitudes

(Super)-Poincaré invariance:

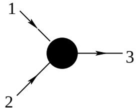
$$\mathcal{M}_3^{(1)}(1^+,2^+,3^-) \;=$$



$$=$$

$$= \delta^{(2\times 2)}\left(\sum_{i=1}^{3}\lambda^{(i)}\tilde{\lambda}^{(i)}\right)\delta^{(1\times\mathcal{N})}\left(\sum_{i=1}^{3}[i+1,i-1]\tilde{\eta}^{(i)}\right)\frac{[1,2]^{4-\mathcal{N}}}{[1,2][2,3][3,1]}$$

$$\mathcal{M}_3^{(2)}(1^-,2^-,3^+) \;=$$



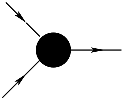
$$=$$

$$= \delta^{(2\times 2)}\left(\sum_{i=1}^{3}\lambda^{(i)}\tilde{\lambda}^{(i)}\right)\delta^{(2\times\mathcal{N})}\left(\sum_{i=1}^{3}\lambda^{(i)}\tilde{\eta}^{(i)}\right)\frac{\langle 1,2\rangle^{4-\mathcal{N}}}{\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle}$$

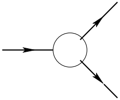
Decorated on-shell diagrammatics

On-shell rules:

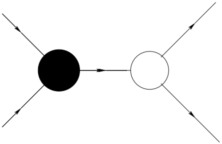
- Gluing $\longrightarrow$ On-shell processes



$$\sum_{h=\pm} \int \frac{d^2\lambda^{(I)} d^2\tilde{\lambda}^{(I)}}{\text{Vol}\{GL(1)\}} d\tilde{\eta}^{(I)}$$



$$\Downarrow$$

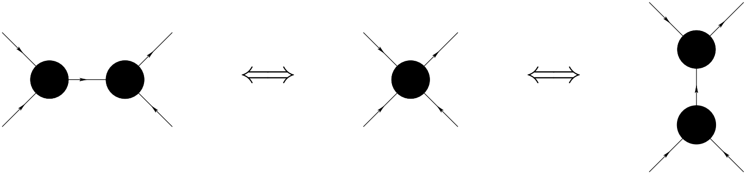


- More constraints than degrees of freedom: singularity
- Same number of contraints as dofs: localised diagram
- Less constraints than dofs: differential forms

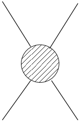
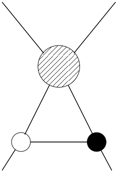
Decorated on-shell diagrammatics

On-shell rules

- Mergers



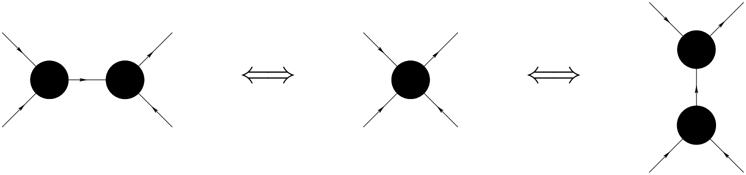
- BCFW bridges

$\mathcal{M}_n =$  $\Longrightarrow$  $= dz\,\mu(z)\,\mathcal{M}_n(z)$

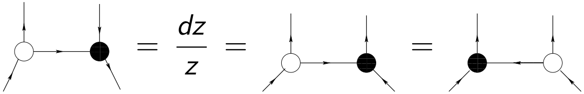
Decorated on-shell diagrammatics

On-shell rules

- Mergers



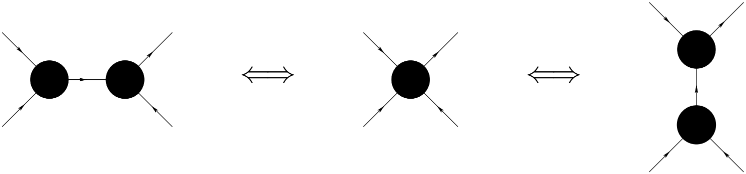
- BCFW bridges



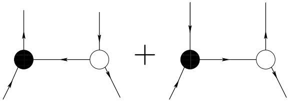
Decorated on-shell diagrammatics

On-shell rules

- Mergers



- BCFW bridges



$$\sim \frac{dz}{z} + dz\, z^{3-\mathcal{N}}$$

Decorated on-shell diagrammatics

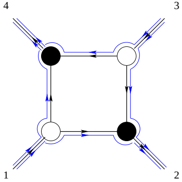
On-shell rules

- Helicity flows

Decorated on-shell diagrammatics

On-shell rules

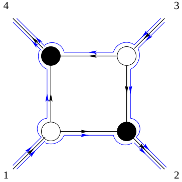
- Helicity flows



Decorated on-shell diagrammatics

On-shell rules

- Helicity flows

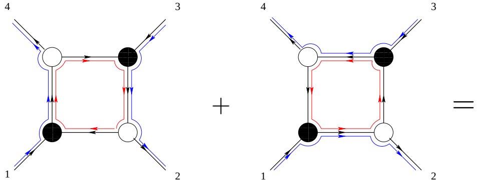


$$= \mathcal{M}_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+)$$

Decorated on-shell diagrammatics

On-shell rules

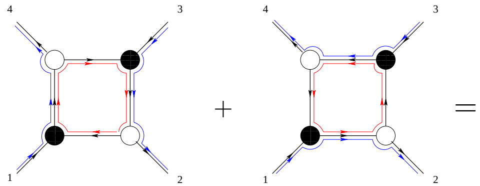
- Helicity flows



Decorated on-shell diagrammatics

On-shell rules

- Helicity flows

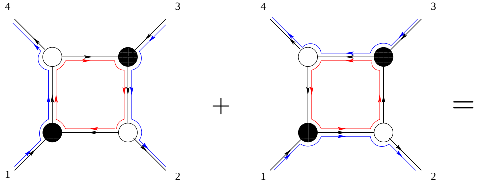


$$= \mathcal{M}_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+) \left[\left(-\frac{s}{u} \right)^{4-\mathcal{N}} + \left(-\frac{t}{u} \right)^{4-\mathcal{N}} \right]$$

Decorated on-shell diagrammatics

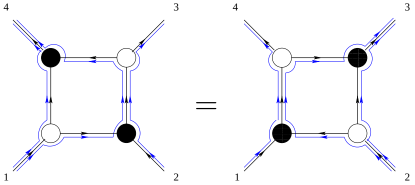
On-shell rules

- Helicity flows



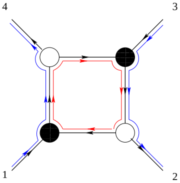
$$= \mathcal{M}_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+) \left[\left(-\frac{s}{u}\right)^{4-\mathcal{N}} + \left(-\frac{t}{u}\right)^{4-\mathcal{N}} \right]$$

- Square moves

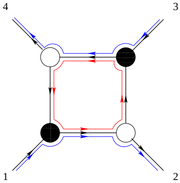


On-shell rules

- Helicity flows



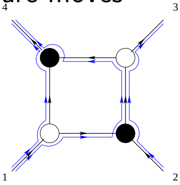
$+$



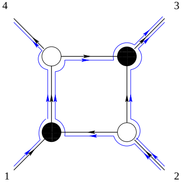
$=$

$$= \mathcal{M}_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+) \left[\left(-\frac{s}{u} \right)^{4-\mathcal{N}} + \left(-\frac{t}{u} \right)^{4-\mathcal{N}} \right]$$

- Square moves



$=$



$$= \mathcal{M}_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$$

Scattering amplitudes as on-shell forms

General structure of the perturbative expansion

$$\mathcal{M}_n(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i; \tilde{\eta}\}) = \sum_{L=0}^{\infty} \mathcal{M}_n^{(L)}(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i; \tilde{\eta}\}, \{z_l\}) \bigwedge_{l=1}^{4L} dz_l,$$

- $L = 0$: fully localised diagrams $\Longleftrightarrow$ Tree level amplitudes
- $L \geq 1$: $4L$-differential forms $\Longleftrightarrow$ L-loop *integrand*

Scattering amplitudes as on-shell forms

General structure of the perturbative expansion

$$\mathcal{M}_n(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i; \tilde{\eta}\}) = \sum_{L=0}^{\infty} \mathcal{M}_n^{(L)}(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i; \tilde{\eta}\}, \{z_l\}) \bigwedge_{l=1}^{4L} dz_l,$$

$\Uparrow$

Inequivalent on-shell diagrams with $4L$ dofs unfixed

- $L = 0$: fully localised diagrams $\Longleftrightarrow$ Tree level amplitudes
- $L \geq 1$: $4L$-differential forms $\Longleftrightarrow$ L-loop *integrand*

Scattering amplitudes as on-shell forms

General structure of the perturbative expansion

$$\mathcal{M}_n(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i; \tilde{\eta}\}) = \sum_{L=0}^{\infty} \mathcal{M}_n^{(L)}(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i; \tilde{\eta}\}, \{z_l\}) \bigwedge_{l=1}^{4L} dz_l,$$

Inequivalent on-shell diagrams with $4L$ dofs unfixed

- $L = 0$: fully localised diagrams $\Longleftrightarrow$ Tree level amplitudes
- $L \geq 1$: $4L$-differential forms $\Longleftrightarrow$ L-loop *integrand*

Scattering amplitudes as on-shell forms

General structure of the perturbative expansion

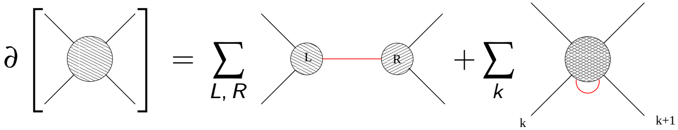
$$\mathcal{M}_n(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i; \tilde{\eta}\}) = \sum_{L=0}^{\infty} \mathcal{M}_n^{(L)}(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}; h_i; \tilde{\eta}\}, \{z_l\}) \bigwedge_{l=1}^{4L} dz_l,$$

Inequivalent on-shell diagrams with $4L$ dofs unfixed

- $L = 0$: fully localised diagrams $\Longleftrightarrow$ Tree level amplitudes
- $L \geq 1$: $4L$-differential forms $\Longleftrightarrow$ L-loop *integrand*

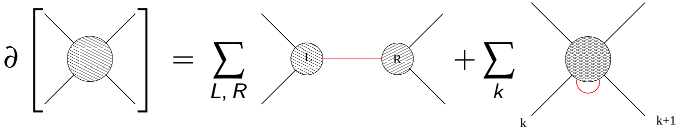
Scattering amplitudes as on-shell forms

Singularity equation:



Scattering amplitudes as on-shell forms

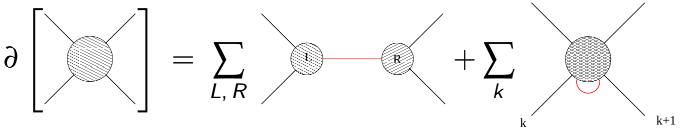
Singularity equation:



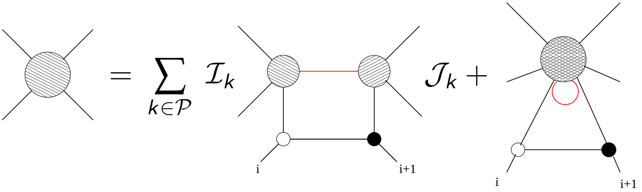
BCFW bridging as integration of the *singularity* equation

Scattering amplitudes as on-shell forms

Singularity equation:

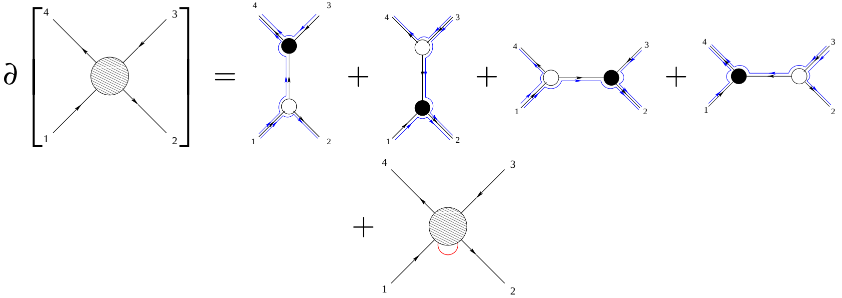


BCFW bridging as integration of the *singularity* equation



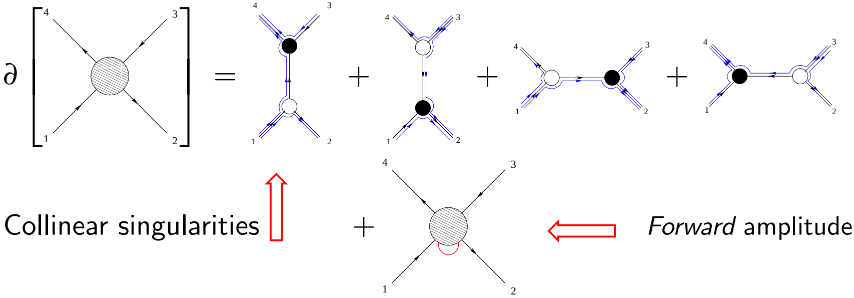
Scattering amplitudes as on-shell forms

Singularity equation:



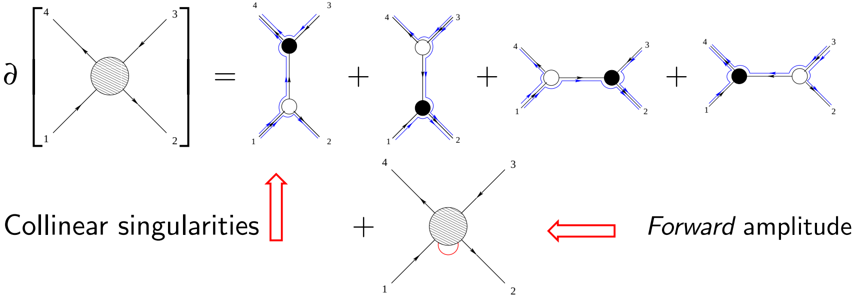
Scattering amplitudes as on-shell forms

Singularity equation:



Scattering amplitudes as on-shell forms

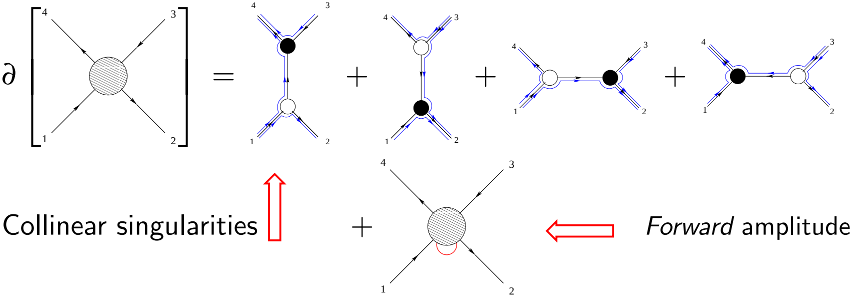
Singularity equation:



BCFW bridging as integration of the *singularity* equation

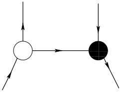
Scattering amplitudes as on-shell forms

Singularity equation:



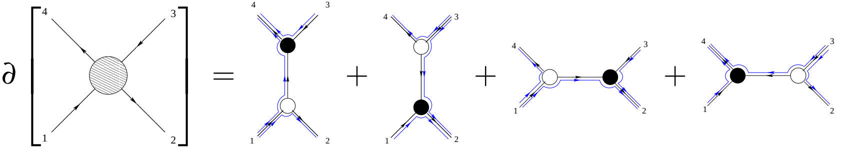
BCFW bridging as integration of the *singularity* equation

Tree-level

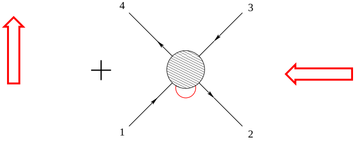
BCFW bridge 1: 

Scattering amplitudes as on-shell forms

Singularity equation:



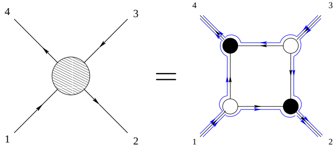
Collinear singularities



Forward amplitude

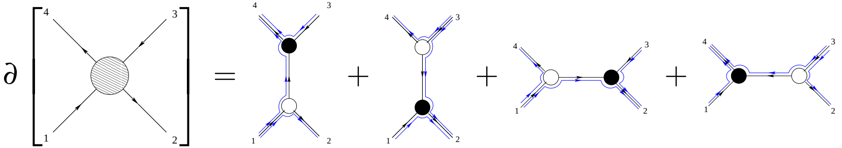
BCFW bridging as integration of the *singularity* equation

Tree-level

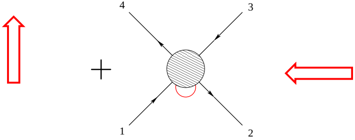


Scattering amplitudes as on-shell forms

Singularity equation:



Collinear singularities

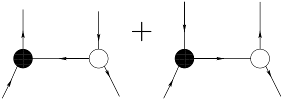


Forward amplitude

BCFW bridging as integration of the *singularity* equation

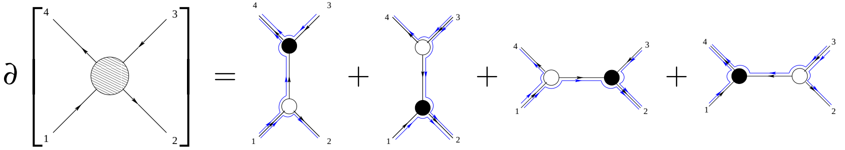
Tree-level

BCFW bridge 2:

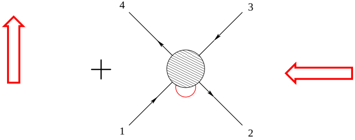


Scattering amplitudes as on-shell forms

Singularity equation:



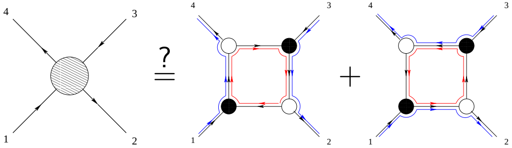
Collinear singularities



Forward amplitude

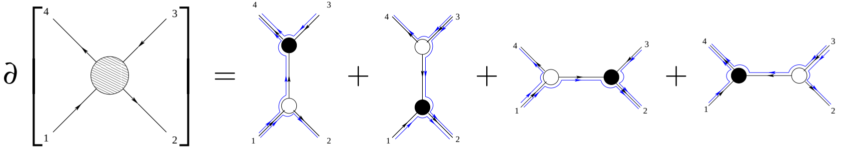
BCFW bridging as integration of the *singularity* equation

Tree-level

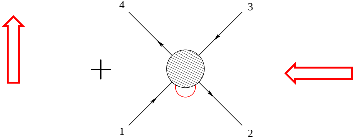


Scattering amplitudes as on-shell forms

Singularity equation:



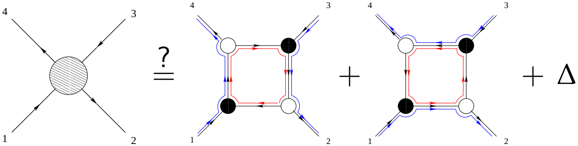
Collinear singularities



Forward amplitude

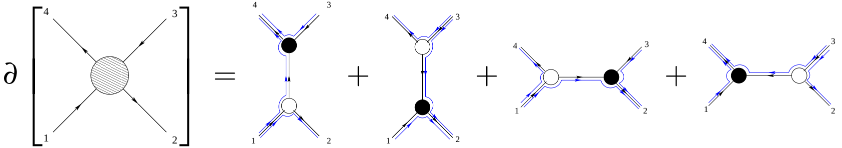
BCFW bridging as integration of the *singularity* equation

Tree-level

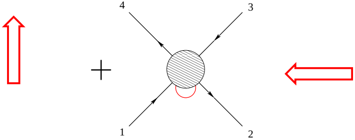


Scattering amplitudes as on-shell forms

Singularity equation:



Collinear singularities



Forward amplitude

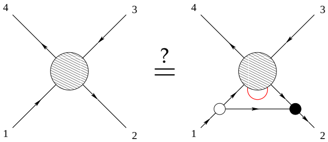
BCFW bridging as integration of the *singularity* equation

Tree-level

$$\Delta \;=\; \mathcal{M}_4^{\text{tree}}(-,+,-,+)\left[\varepsilon_{\mathcal{N},3}(4-\mathcal{N})\frac{st}{u^2} - \delta_{\mathcal{N},0}2\frac{s^2t^2}{u^4}\right]$$

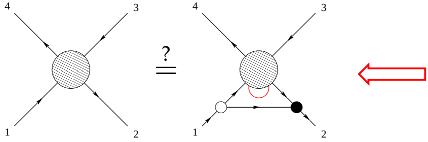
Scattering amplitudes as on-shell forms

One loop:



Scattering amplitudes as on-shell forms

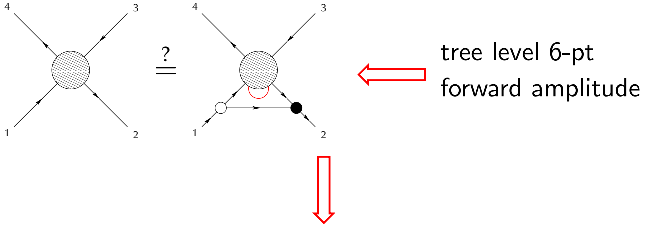
One loop:



tree level 6-pt
forward amplitude

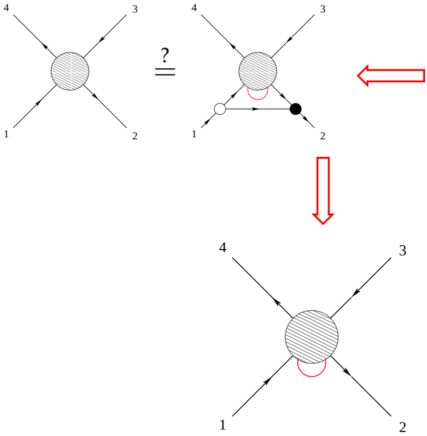
Scattering amplitudes as on-shell forms

One loop:



Scattering amplitudes as on-shell forms

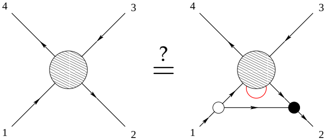
One loop:



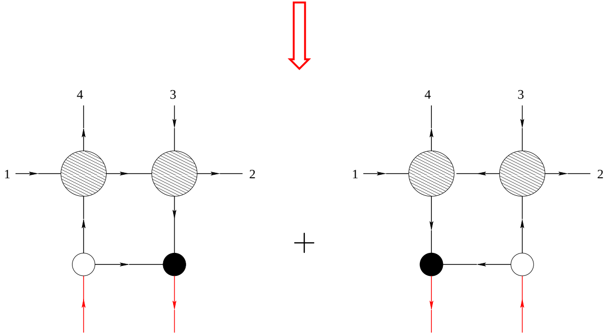
tree level 6-pt
forward amplitude

Scattering amplitudes as on-shell forms

One loop:

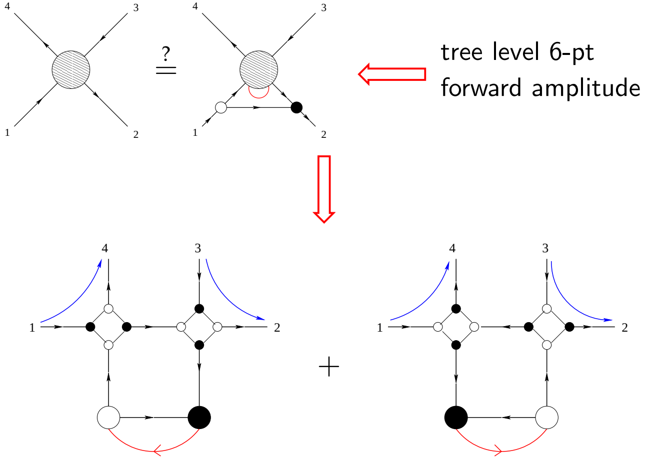


tree level 6-pt
forward amplitude



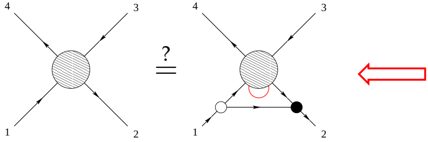
Scattering amplitudes as on-shell forms

One loop:

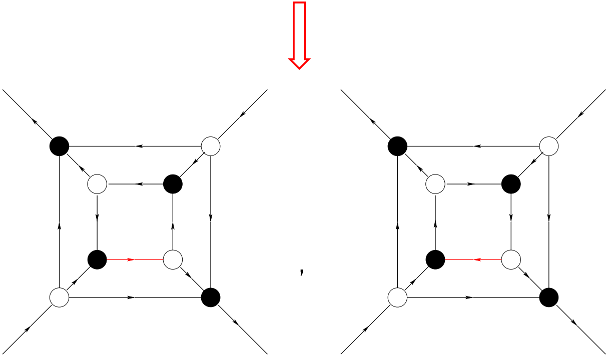


Scattering amplitudes as on-shell forms

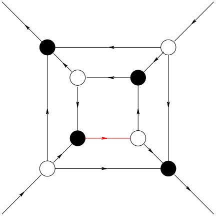
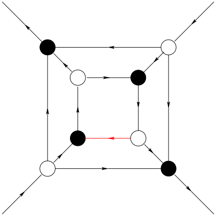
One loop:



tree level 6-pt
forward amplitude

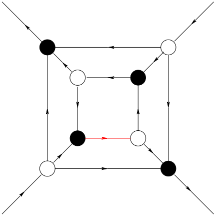
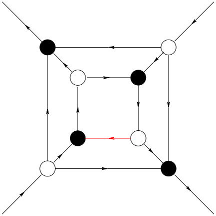


Scattering amplitudes as on-shell forms

$\tilde{\mathcal{M}}_4(\zeta) =$  $+$  $=$

$$= \bigwedge_{i=1}^{4} \frac{d\zeta_i}{\zeta_i} \frac{\left(-\frac{s}{u}\right)^{4-\mathcal{N}}(1-\zeta_1)^{4-\mathcal{N}}(1-\zeta_3)^{4-\mathcal{N}}\left(-\frac{t}{u}\right)^{4-\mathcal{N}}(1-\zeta_2)^{4-\mathcal{N}}(1-\zeta_4)^{4-\mathcal{N}}}{\left[-\frac{s}{u}(1-\zeta_1)(1-\zeta_3)-\frac{t}{u}(1-\zeta_2)(1-\zeta_4)\right]^{4-\mathcal{N}}}$$

Scattering amplitudes as on-shell forms

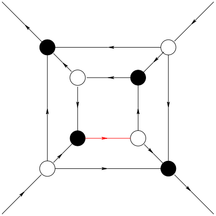
$\tilde{\mathcal{M}}_4(\zeta) =$  $+$  $=$

$$= \bigwedge_{i=1}^{4} \frac{d\zeta_i}{\zeta_i} \frac{\left(-\frac{s}{u}\right)^{4-\mathcal{N}} (1-\zeta_1)^{4-\mathcal{N}} (1-\zeta_3)^{4-\mathcal{N}} \left(-\frac{t}{u}\right)^{4-\mathcal{N}} (1-\zeta_2)^{4-\mathcal{N}} (1-\zeta_4)^{4-\mathcal{N}}}{\left[-\frac{s}{u}(1-\zeta_1)(1-\zeta_3) - \frac{t}{u}(1-\zeta_2)(1-\zeta_4)\right]^{4-\mathcal{N}}}$$

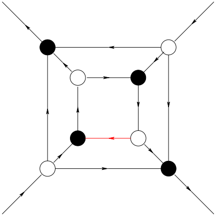
Extraction of physical info: Look at the poles

Scattering amplitudes as on-shell forms

$\tilde{\mathcal{M}}_4(\zeta) =$



$+$

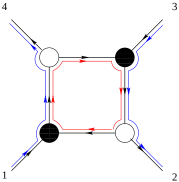


$$= \bigwedge_{i=1}^{4} \frac{d\zeta_i}{\zeta_i} \frac{\left(-\frac{s}{u}\right)^{4-\mathcal{N}} (1-\zeta_1)^{4-\mathcal{N}}(1-\zeta_3)^{4-\mathcal{N}} \left(-\frac{t}{u}\right)^{4-\mathcal{N}} (1-\zeta_2)^{4-\mathcal{N}}(1-\zeta_4)^{4-\mathcal{N}}}{\left[-\frac{s}{u}(1-\zeta_1)(1-\zeta_3) - \frac{t}{u}(1-\zeta_2)(1-\zeta_4)\right]^{4-\mathcal{N}}}$$

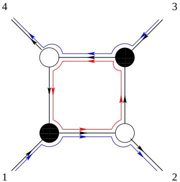
Extraction of physical info: Look at the poles

Leading singularity: $\gamma_{\mathrm{LS}} \equiv \{\zeta_i \in \hat{\mathbb{C}}^4 \,|\, \zeta_i = 0\}$

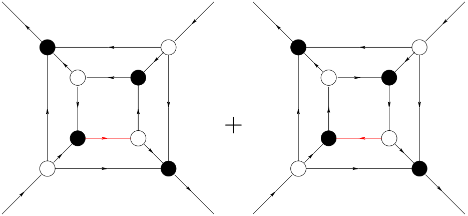
$\oint_{\gamma_{\mathrm{LS}}} \tilde{\mathcal{M}}_4(\zeta) =$



$+$



Scattering amplitudes as on-shell forms

$\tilde{\mathcal{M}}_4(\zeta) =$  $=$

$$= \bigwedge_{i=1}^{4} \frac{d\zeta_i}{\zeta_i} \frac{\left(-\frac{s}{u}\right)^{4-\mathcal{N}} (1-\zeta_1)^{4-\mathcal{N}} (1-\zeta_3)^{4-\mathcal{N}} \left(-\frac{t}{u}\right)^{4-\mathcal{N}} (1-\zeta_2)^{4-\mathcal{N}} (1-\zeta_4)^{4-\mathcal{N}}}{\left[-\frac{s}{u}(1-\zeta_1)(1-\zeta_3) - \frac{t}{u}(1-\zeta_2)(1-\zeta_4)\right]^{4-\mathcal{N}}}$$

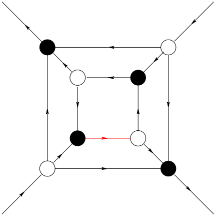
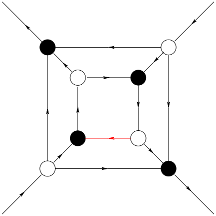
Extraction of physical info: Look at the poles

Triangle coefficients:

$\gamma_\triangle \equiv \{\zeta_i \in \hat{\mathbb{C}}^4 \,|\, \zeta_i = 0,\ i = 1, 3,\ \zeta_2 = 0,\ \zeta_4 = -u/t,\ \zeta_2 \succ \zeta_4\}$

$$\oint_{\gamma_\triangle} \tilde{\mathcal{M}}_4(\zeta) = \Delta$$

Scattering amplitudes as on-shell forms

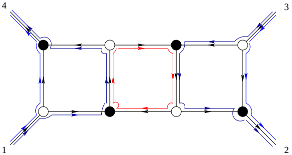
$$\tilde{\mathcal{M}}_4(\zeta) =$$  $+$  $=$

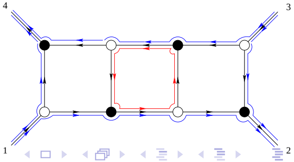
$$= \bigwedge_{i=1}^{4} \frac{d\zeta_i}{\zeta_i} \frac{\left(-\frac{s}{u}\right)^{4-\mathcal{N}} (1-\zeta_1)^{4-\mathcal{N}}(1-\zeta_3)^{4-\mathcal{N}} \left(-\frac{t}{u}\right)^{4-\mathcal{N}} (1-\zeta_2)^{4-\mathcal{N}}(1-\zeta_4)^{4-\mathcal{N}}}{\left[-\frac{s}{u}(1-\zeta_1)(1-\zeta_3) - \frac{t}{u}(1-\zeta_2)(1-\zeta_4)\right]^{4-\mathcal{N}}}$$

Extraction of physical info: Look at the poles

Double cut structure:

$\gamma_0 \equiv \{\zeta_i \in \hat{\mathbb{C}}^4 \,|\, \zeta_i = 0,\ i = 1,3\}$

$$\oint_{\gamma_0} \tilde{\mathcal{M}}_4(\zeta) =$$ 

$+$ 

Conclusion

- On-shell diagrammatics as a starting point for a Lagrangian free definition of field theory
- Locality is lost for individual diagrams
- Symmetry and Structure of the theory more transparent.
- All-loop recursion relation holds for the intergrand of $\mathcal{N} \geq 1$ SYM in the planar sector (diagrammatic proof!)
- Provide a natural definition for the forward limit in $\mathcal{N} = 0$ YM $\leftarrow$ It needs to be checked
- Difference in structure between SYM and YM suggested to be due to the different structure in the tree-level boundary term (deeper understanding needed)
- It is lacking an on-shell interpretation for the all-plus helicity amplitudes