Geometry, Entanglement & Scrambling

Joan Simón

University of Edinburgh and Maxwell Institute of Mathematical Sciences

Iberian Strings 2015
Salamanca, May 27, 2015

Based on arXiv:1503.08161 with P. Caputa, A. Štikonas, T. Takayanagi & K. Watanabe
See also Roberts & Stanford, arXiv:1412.5123

Outline

1. From QM to geometry through entanglement
   - Foundations of statistical mechanics $\Rightarrow$ black holes
   - Rindler physics + entanglement properties $\Rightarrow$ Einstein's equations
   - Entanglement renormalisation, tensor networks & MERA (many body strongly correlated systems in 1+1 at criticality) $\Rightarrow$ space of geodesics in AdS_3
2. AdS/CFT : from geometry to QM
   - Spacetime connectivity vs entanglement
   - EPR=ER
3. Scrambling

The whole or its parts

Entanglement is the feature that *may* allow you to

$$\boxed{\text{know about the entire quantum state knowing nothing about its parts}}$$

- Consider an entangled pure state in $\mathbb{C} \times \mathbb{C}$

$$|\Psi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|\right)$$

$$\rho_1 = \mathrm{tr}_2\rho = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{1}{2}\mathbb{I}$$

- Reduced density matrix for subsystem 1 has maximal ignorance

Two features of entanglement

1. Entangled states in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with no interactions have non-trivial correlations
2. Concentration of measure : Typical reduced matrices are close to maximally entangled ones (Hayden, Leung,Winter)

Theorem

Let φ be a random pure state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $d_2 \geq d_1 \geq 3$, then

$$Prob\left\{ S(\varphi_1) < \log d_1 - \alpha - \frac{d_1}{d_2 \log 2} \right\} \leq \exp\left(-\frac{(d_1 d_2 - 1) C_3 \alpha^2}{(\log d_1)^2} \right)$$

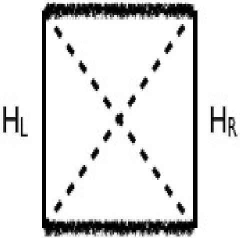
where $C_3 = (8\pi^2 \log 2)^{-1}$, $\forall \alpha > 0$.

the larger the ratio $\frac{d_2}{d_1}$ is, the more probable it is !!

Thermal/BH analogy

1. Typical pure states look thermal for low energy/complexity operators
   - Foundations of quantum statistical mechanics (Popescu, Short, Winter)
   - Most BH microstates look thermal ($\exists$ horizon)
2. Eternal AdS black holes (Maldacena)

$$\rho_{\text{BH}} = \frac{1}{Z(\beta)} \sum_i e^{-\beta E_i} |E_i\rangle\langle E_i| , \qquad |E_i\rangle \in \mathcal{H}_R$$



$$\rho_{\text{BH}} = \text{tr}_{\mathcal{H}_L} |\Psi\rangle\langle\Psi| \ \text{ with } \ |\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_i e^{-\beta E_i/2} |E_i\rangle \otimes |E_i\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R$$

Quantum entanglement is responsible for the existence of correlations

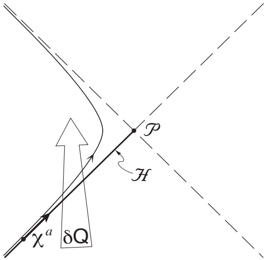
Rindler physics vs Einstein's equations

Einstein's equations as an equation of state of the thermal vacuum (Jacobson)

- Assume Clausius relation

$$\delta S = \frac{\delta Q}{T}$$

  holds for all local Rindler horizons
- T is local Unruh's temperature
- heat δQ is local boost energy crossing local Rindler horizon
- Assume holography : $\delta S = \eta\, \delta A$



Horizon entropy as vacuum entanglement entropy

If entanglement underlies spacetime geometry,
it should govern its deformations $\Rightarrow$

Einstein's equations should follow from entanglement properties

This is true in AdS/CFT (van Raamsdonck, Takayanagi, Myers, ...)

Beyond the AdS/CFT

Jacobson claims that Einstein's equations hold if the entropy of small causal diamonds is stationary at constant volume

- Energy enters through the causal diamond modular Hamiltonian

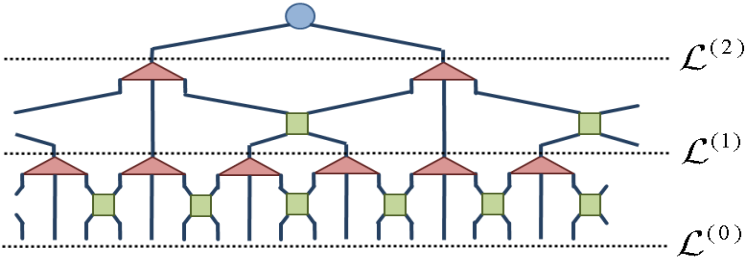
Message : under some UV assumptions on the behaviour of quantum entanglement, local equlibrium physics selects Einstein's equatios

Surely the last word has not been said on this front ...

Entanglement renormalisation

Definition

Real space coarse-graining transformation that removes UV degrees of freedom consistently with the quantum entanglement properties of the quantum state (Vidal)



It combines Wilson's RG picture, with White's rule as part of the density matrix renormalisation group (DMRG) while dealing with quantum entanglement properties between different blocks

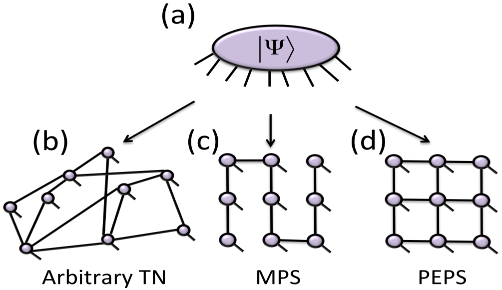
Tensor networks

Given a quantum many body state

$$|\psi\rangle = \sum_{i_1,\dots i_N} C_{i_1\dots i_N} |i_1 \dots i_N\rangle$$

To reduce the complexity in the representation of the state

- replace tensor C $\Rightarrow$ network of smaller interconnected tensors

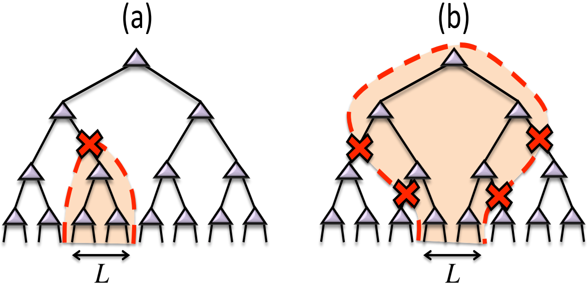


MERA

Definition

The multi-scale entanglement renormalisation ansatz (MERA) is a type of tensor network

- *Tensors must be unitaries or isometries*
- *Entanglement is built locally at all length scales of the system*
- *Scale invariance : translation invariance + same tensor at each layer*

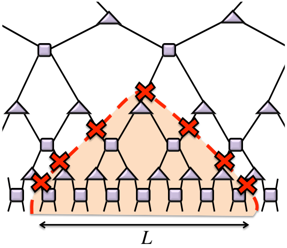


MERA and AdS/CFT

- Entanglement entropy satisfies an area law in holographic space

$$S(L) \sim \log L$$

- MERA has a causal structure : expectations values of local observables only depend on the tensors inside that causal cone



MERA & holography

∃ a relation between the holographic space in MERA and the space of geodesics in AdS_3 (Evenbly, Vidal, Swingle, Czech, Sully, ...)

AdS/CFT : Entanglement vs Geometry

Many recent developments relating geometry with entanglement
Main subset relevant for this talk

1. Entanglement vs spacetime connectivity (van Raamsdonk)
    - Correlations vs Mutual Information vs causal Geodesics
    - Entanglement vs Minimal surfaces (Ryu & Takayanagi)

2. Eternal BH $\simeq$ thermo field double (Maldacena)
    - EPR=ER reinterpretation (Maldacena & Susskind)

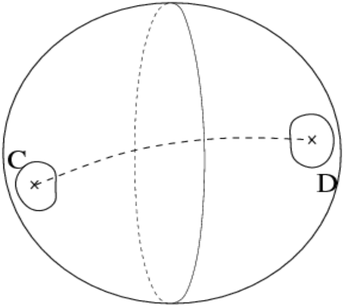
Entanglement vs Correlations

Question : quantum entanglement vs correlation lengths ?
Consider as measure of entanglement, mutual information

$$I(C:D) = S(\rho_C) + S(\rho_D) - S(\rho_{CD})$$

Using Pinksler's inequality, one can show (Wolf, Verstraete, Hastings, Cirac)

$$I(C:D) \geq \frac{(\langle \mathcal{O}_C \mathcal{O}_D \rangle - \langle \mathcal{O}_C \rangle \langle \mathcal{O}_D \rangle)^2}{2\|\mathcal{O}_C\|^2 \|\mathcal{O}_D\|^2}$$



Connected correlators as geodesics in AdS/CFT

The connected 2-pt correlation function of a heavy operator behaves like (Balasubramanian & Ross)

$$\langle \mathcal{O}_C(x_c)\mathcal{O}_D(x_D)\rangle - \langle \mathcal{O}_C(x_c)\rangle\langle \mathcal{O}_D(x_D)\rangle \sim e^{-mL_{\text{bulk}}(x_C,\,x_D)}$$

- $L_{\text{bulk}}(x_C, x_D)$ is the bulk geodesic distance between the boundary points x_C and x_D
- $\Delta_{\mathcal{O}} = m\ell \gg 1$ (not scaling with N or c)
- Holographic dual correlation only depends mildly on the dual operator (through $\Delta_{\mathcal{O}}$)

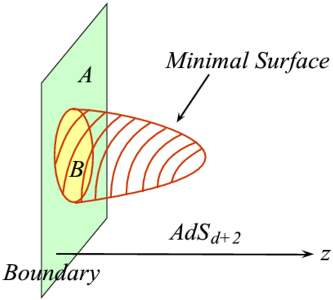
Entanglement entropy in AdS/CFT

Entanglement entropy in an strongly coupled CFT vs bulk geometry.
Ryu & Takayanagi

$$S(\rho_B) \propto \mathrm{Area}(\partial B) \propto \mathrm{Area}(\Sigma_{\mathrm{bulk}})$$

where Σ_{bulk} is a bulk minimal surface anchored to ∂B

- Non-local diffeomorphism invariant observables
- Deep relation between the set of minimal surfaces and Einstein's equations



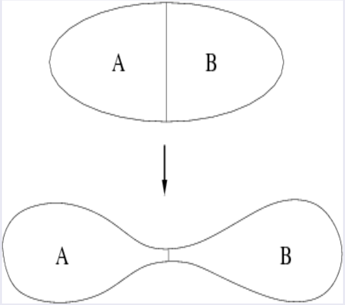
Entanglement vs spacetime connectedness

Consider a full quantum system described by $A \cup B$
Study the limit of vanishing entanglement holographically

Spacetime connectedness

Sending entanglement to zero, requires :

1. Proper bulk distance to infinity
2. Area of the common boundary to zero $\Rightarrow$ pinching (van Raamsdonk)



Consequences

Quantum mechanics

Consider a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ with no interactions

1. Product states have vanishing connected correlators
2. Entangled states have non-vanishing correlators !!

AdS/CFT

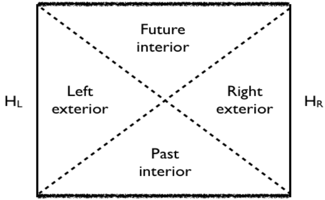
Consider 2 decoupled CFTs

1. Product states having holographic duals correspond to disconnected asymptotically AdS spacetimes
   Example : $|\text{vac}\rangle \otimes |\text{vac}\rangle \Rightarrow$ 2 disconnected AdS spacetimes
2. Entangled states $\Rightarrow \exists$ correlations $\Rightarrow$ they may allow some connected geometric description !!
   Example : eternal AdS black hole

Eternal AdS BH revisited

1. *Single* BH in thermal equilibrium : evolution by a boost ($H_R - H_L$)

$$H_{\text{tf}} = \mathbb{I}_L \otimes H_R - H_L \otimes \mathbb{I}_R \,.$$



- Time propagates upwards in $\mathcal{H}_R$ and downwards in $\mathcal{H}_L$.
- Thermofield double is (boost) invariant

2. Approximate description of the state at $t = 0$ of *two* AdS black holes ($H_R + H_L$)

$$H = \mathbb{I}_L \otimes H_R + H_L \otimes \mathbb{I}_R \equiv H_R + H_L \,.$$

Time propagates upwards in both boundaries

The wormhole interpretation

Consider the 4d Schwarzschild black hole metric

$$ds^2 = -e^{2\Phi}dt^2 + \frac{dr^2}{1-B/r} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Study a fixed t slice at $\theta = \pi/2$:

$$ds^2\big|_\Sigma = \frac{dr^2}{1-B/r} + r^2 d\phi^2$$

View this section as a surface $z(r)$ in one higher euclidean dimension

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 = \left(1 + (z')^2\right) dr^2 + r^2 d\phi^2$$
$$z(r) = \pm 2B\,(r/B - 1)^{1/2}$$

This is a non-traversable wormhole, but it illustrates that black holes can be reinterpreted in terms of Einstein-Rosen (ER) bridges (wormholes)

EPR = ER (Maldacena & Susskind)

Eternal black hole re-interpreted

1. Non-vanishing correlators between $\mathcal{H}_L$ and $\mathcal{H}_R$ are due to quantum entanglement (EPR)
2. These correlations are holographically captured by the bulk geodesic distance between opposite boundaries $\Rightarrow$ length of the ER bridge
3. Entanglement entropy = black hole entropy $\Rightarrow$ maximal cross-section of the ER bridge

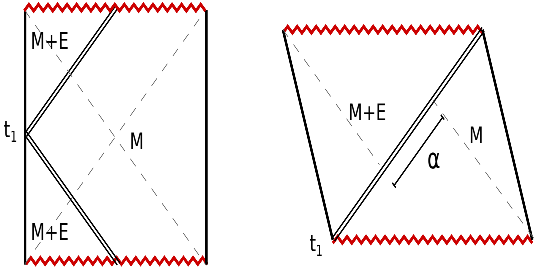
EPR=ER conjecture

In short, it takes the above picture and states it is always correct

Question : perturbations of this scenario : time evolution of the throat ?

Perturbing eternal BH (Shenker & Stanford)

- Perturbation turned on at time t_1 on the left boundary
- Backreaction can be non-trivial, no matter how light the perturbation is, depending on the t_1 scale
- Shock-wave description



Small perturbations get blue shifted near horizon (Shenker-Stanford)

$$t^\star \sim \beta \log m_p \beta$$

Connection to scrambling

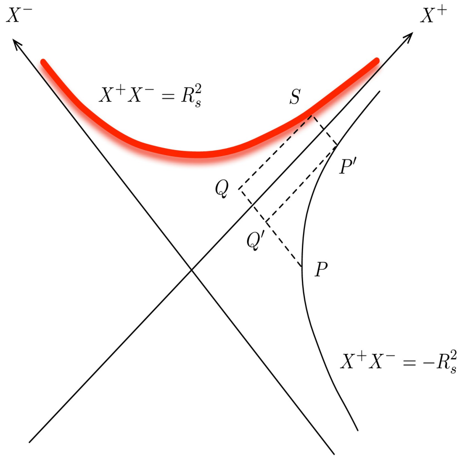
BH physics suggest speed at which thermality is regained is faster than in diffusive systems (scrambling) (Susskind-Sekino)

1. scrambling time

   $$\tau_{\text{scrambling}} \sim \beta \log S$$

2. Faster than diffusion

   $$\tau_{\text{diff}} \sim S^{2/d} \gg \log S$$



Question : Any CFT evidence for this behaviour ?

Motivation

Consider a critical physical system in (1+1)-d in some thermal state

$$\rho_\beta$$

Perturb the state by a local primary operator

$$\mathcal{O}_w(x_0,0)\,\rho_\beta\,\mathcal{O}_w^\dagger(x_0,0)$$

Evolve the system unitarily

$$e^{-iHt}\,\Big(\mathcal{O}_w(x_0,0)\,\rho_\beta\,\mathcal{O}_w^\dagger(x_0,0)\Big)\,e^{iHt}$$

Questions

1. Time scale $t_\omega^\star$ at which subsystems behave thermally

$$\Delta I_{A:B}(t_\omega^\star)=0$$

2. Time scale t_ω at which correlations die off

$$I_{A:B}(t_\omega)=0$$

Remaining's Talk Outline

1. 2d CFT discussion
   - Technical set-up
   - Large c 2d CFTs at finite T & thermo field double
2. Holographic discussion
   - Brief discussion on strategy & calculation
3. Final remarks

CFT : overall strategy

Our 2d CFT will be non-compact : no Poincaré recurrences

Our discussion will get quite technical, but its logic should be clear

1. Describe the quantum state and its regularisation (on top of the UV cut-off)
2. Use replica trick to compute entanglement entropy $\Rightarrow$ correlators in 2d CFT
3. Use large c limit to compute these correlators analytically
4. Solve for the time scales $t_\omega^\star$ and $t^\star$

Remark : this main set-up was already considered in Roberts & Stanford

Single CFT set-up

Consider a perturbation, generated at time $t = 0$ at $x = -\ell$, created by a primary operator $\mathcal{O}$ acting on the vacuum of the 2d CFT :

$$|\Psi_{\mathcal{O}}(t)\rangle = \sqrt{\mathcal{N}}\, e^{-iHt}\, e^{-\epsilon H}\mathcal{O}(0, -\ell)|0\rangle$$

- $\mathcal{O}$ is inserted at $t = 0$ and $x = -\ell$ and dynamically evolved afterwards
- ϵ is a parameter smearing the local operator (time splitting)

Density matrix :

$$\begin{aligned}\rho(t) &= \mathcal{N}\, e^{-iHt}\, e^{-\epsilon H}\mathcal{O}(0, -\ell)|0\rangle\langle 0|\mathcal{O}^\dagger(0, -\ell)\, e^{iHt}\, e^{-\epsilon H} \\ &= \mathcal{N}\, \mathcal{O}(\omega_2, \bar{\omega}_2)|0\rangle\langle 0|\mathcal{O}^\dagger(\omega_1, \bar{\omega}_1)\end{aligned}$$

where $\omega_1 = -\ell + i(\epsilon - it)$, $\omega_2 = -\ell - i(\epsilon + it)$ $(\bar{\omega}_1 = -\ell - i(\epsilon - it))$

Extension to finite temperature

Same set-up as before, but now

1. we perturb a thermal state at $t = -t_\omega$:

$$\rho(t) \equiv \mathcal{N}\mathcal{O}(\omega_2, \bar{\omega}_2)\, e^{-\beta H}\, \mathcal{O}^\dagger(\omega_1, \bar{\omega}_1)$$

   with

$$\begin{aligned} \omega_1 &= x_0 + t + t_\omega + i\epsilon & \bar{\omega}_1 &= x_0 - t - t_\omega - i\epsilon \\ \omega_2 &= x_0 + t + t_\omega - i\epsilon & \bar{\omega}_2 &= x_0 - t - t_\omega + i\epsilon\,. \end{aligned}$$

2. A pair of operators will be inserted on a cylinder, separated $2i\epsilon$

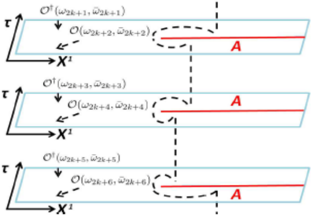
Replica trick - I

Following Cardy & Calabrese

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left(\frac{\mathrm{Tr}\rho_A^n}{\mathrm{Tr}\left(\rho_A^{(0)}\right)^n}\right)$$
$$= \frac{1}{1-n}\log\left[\frac{\langle\mathcal{O}(\omega_1,\bar\omega_1)\mathcal{O}^\dagger(\omega_2,\bar\omega_2)...\mathcal{O}^\dagger(\omega_{2n},\bar\omega_{2n})\rangle_{\Sigma_n}}{\left(\langle\mathcal{O}(\omega_1,\bar\omega_1)\mathcal{O}^\dagger(\omega_2,\bar\omega_2)\rangle_{\Sigma_1}\right)^n}\right]$$

Notice no twisted operators but CFT defined on a Riemann surface

- $\omega_{2k+1} = e^{2\pi i k}\omega_1$
- $\omega_{2k+2} = e^{2\pi i k}\omega_2$



Replica trick - II

Following Cardy & Calabrese

$$\text{Tr}\rho_A^n \sim \langle\psi|\sigma(\omega_1,\bar{\omega}_1)\tilde{\sigma}(\omega_2,\bar{\omega}_2)|\psi\rangle$$

$|\psi\rangle$ stands for whatever CFT state you want to consider (vacuum or excited state)

- Non-trivial topology replaced by twist operators
- Calculation done in n-copies of the original CFT
- Twist operators emerge because of the existence of some internal symmetry when swapping these copies

Our calculation & notion of "scrambling"

- Consider a thermofield double set-up.
- Perturbed the system at $-t_\omega$ by a primary localised operator $\mathcal{O}$
- Evolve unitarily

Measure the amount of entanglement at $t = 0$ using the mutual information

$$I(A : B; t_\omega) = S_A + S_B - S_{A\cup B}$$

We can ask what the time scale t_ω has to be so that the perturbation can not be distinguished from the original thermal state (scrambling time)

$$\Delta I(A : B; t_\omega) = \Delta S_A + \Delta S_B - \Delta S_{A\cup B} = 0$$

What our condition boils down to

Hartman & Maldacena showed that in the absence of perturbation :

- at early times, mutual information decreases linearly
- at late times, i.e. $t > \frac{L}{2}$, $S_{A\cup B} = S_A + S_B$ saturates and the mutual information vanishes.

Thus, if we assume $t^\star_\omega > \frac{L}{2}$, our condition reduces to

$$I(A:B;t^\star_\omega) = 0$$

This is what was analysed by Shenker & Stanford and what we will end up discussing today.

Thermofield double set-up

Consider two non-interacting 2d CFTs, say CFT_L and CFT_R, with isomorphic Hilbert spaces $\mathcal{H}_{L,R}$

Thermofield double state :

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |n\rangle_L |n\rangle_R$$

- $|n\rangle_L$ is an eigenstate of the hamiltonian H_L acting on $\mathcal{H}_L$ with eigenvalue E_n (and similarly for $|n\rangle_R$).
- $|n\rangle_L$ is the CPT conjugate of the state $|n\rangle_R$
- Notation : $|n\rangle_L \otimes |n\rangle_R$ as $|n\rangle_L |n\rangle_R$.
- Thermal reduced density

$$\rho_R(\beta) = \text{tr}_{\mathcal{H}_L}\left(|\Psi_\beta\rangle\langle\Psi_\beta|\right) = \frac{1}{Z(\beta)} \sum_{n\in\mathcal{H}_R} e^{-\beta E_n} |n\rangle_R \langle n|_R \ ,$$

Thermofield double : observables

- Single sided correlators are thermal

$$\langle \Psi_\beta |\, \mathcal{O}_R(x_1, t_1) \ldots \mathcal{O}_R(x_n, t_n) \,| \Psi_\beta \rangle = \operatorname{tr}_{\mathcal{H}_R} \left(\rho_R(\beta) \mathcal{O}_R(x_1, t_1) \ldots \mathcal{O}_R(x_n, t_n) \right) .$$

- Two sided correlators : by analytic continuation

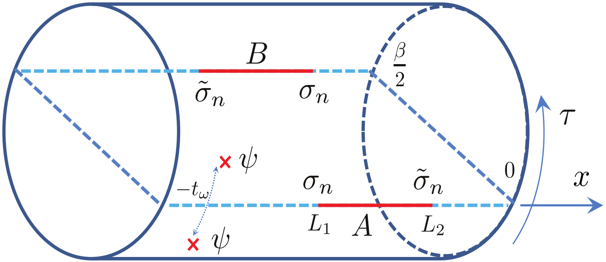
$$\langle \Psi_\beta |\, \mathcal{O}_L(x_1, -t) \ldots \mathcal{O}_R(x_n', t_n') \,| \Psi_\beta \rangle = \operatorname{tr}_{\mathcal{H}_R} \left(\rho_R(\beta) \mathcal{O}_R(x_1, t - i\beta/2) \ldots \mathcal{O}_R(x_n', t_n') \right) .$$

Will use this observation when computing Renyi entropies

CFT considerations

As discussed by Morrison & Roberts (see also Hartman & Maldacena) :

- *single* sided thermal correlation functions are computed on a *single* cylinder with periodicity $\tau \sim \tau + \beta$
- *two-sided* correlators involve a path integral over a cylinder with the same periodicity $\tau \sim \tau + \beta$, where *all* operators $\mathcal{O}_R$ are inserted at $\tau = i\beta/2$, whereas $\mathcal{O}_L$ are inserted at $\tau = 0$



Calculation of S_A

$$S_A = -\lim_{n\to 1} \frac{1}{n-1} \log\left(\mathrm{Tr}\, \rho_A^n(t)\right)$$

where

$$\mathrm{Tr}\, \rho_A^n(t) = \frac{\langle \psi(x_1, \bar{x}_1)\sigma(x_2, \bar{x}_2)\tilde{\sigma}(x_3, \bar{x}_3)\psi^\dagger(x_4, \bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x, \bar{x}_1)\psi^\dagger(x_4, \bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$\begin{array}{llll} x_1 = -i\epsilon\,, & x_2 = y - t_\omega - t_-, & x_3 = y + L - t_\omega - t_-\,, & x_4 = +i\epsilon \\ \bar{x}_1 = +i\epsilon\,, & \bar{x}_2 = y + t_\omega + t_-, & \bar{x}_3 = y + L + t_\omega + t_-\,, & \bar{x}_4 = -i\epsilon \end{array}$$

with conformal dimensions

$$H_\psi = nh_\psi\,, \quad H_\sigma = \frac{c}{24}\left(n - \frac{1}{n}\right)$$

Conformal maps

1. From the cylinder to the plane

$$\omega(x) = e^{2\pi x/\beta}$$

2. Standard map : $\omega_1 \to 0$, $\omega_2 \to z$, $\omega_3 \to 1$ and $\omega_4 \to \infty$

$$z(\omega) = \frac{(\omega_1 - \omega)\omega_{34}}{\omega_{13}(\omega - \omega_4)}$$

where the cross-ratio satisfies

$$z = \frac{\omega_{12}\omega_{34}}{\omega_{13}\omega_{24}}$$

Result

$$S_A^{(n)} = \frac{c(n+1)}{6} \log\left(\frac{\beta}{\pi\epsilon_{UV}} \sinh\frac{\pi L}{\beta}\right) + \frac{1}{n-1} \log\left(|1-z|^{4H_\sigma} G(z,\bar{z})\right)$$

where

$$G(z,\bar{z}) = \langle\psi|\,\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\psi\rangle$$

Using the large c results derived by Fitzpatrick, Kaplan & Walters in the limit $n \to 1$

$$\Delta S_A = \frac{c}{6} \log\left(\frac{z^{\frac{1}{2}(1-\alpha_\psi)}\bar{z}^{\frac{1}{2}(1-\bar{\alpha}_\psi)}(1-z^{\alpha_\psi})(1-\bar{z}^{\bar{\alpha}_\psi})}{\alpha_\psi\bar{\alpha}_\psi(1-z)(1-\bar{z})}\right)$$

where $\alpha_\psi = \sqrt{1 - \frac{24h_\psi}{c}}$.

Cross-ratios

The cross-ratios are

$$z = \frac{\sinh\left(\frac{\pi x_{12}}{\beta}\right)\sinh\left(\frac{\pi x_{34}}{\beta}\right)}{\sinh\left(\frac{\pi x_{13}}{\beta}\right)\sinh\left(\frac{\pi x_{24}}{\beta}\right)}$$

$$\simeq 1 + \frac{2\pi i\epsilon}{\beta}\frac{\sinh\frac{\pi L}{\beta}}{\sinh\frac{\pi(y+L-t_- -t_\omega)}{\beta}\sinh\frac{\pi(y-t_- -t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

$$\bar{z} = \frac{\sinh\left(\frac{\pi \bar{x}_{12}}{\beta}\right)\sinh\left(\frac{\pi \bar{x}_{34}}{\beta}\right)}{\sinh\left(\frac{\pi \bar{x}_{13}}{\beta}\right)\sinh\left(\frac{\pi \bar{x}_{24}}{\beta}\right)}$$

$$\simeq 1 - \frac{2\pi i\epsilon}{\beta}\frac{\sinh\frac{\pi L}{\beta}}{\sinh\frac{\pi(y+L+t_- +t_\omega)}{\beta}\sinh\frac{\pi(y+t_- +t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2)$$

Final result

Analysing the imaginary parts, we reach the conclusions :

- $(z, \bar{z}) \to (1, 1)$ for $t + t_\omega < y$ and $t + t_\omega > y + L$
- $(z, \bar{z}) \to (e^{2\pi i}, 1)$ for $y < t + t_\omega < y + L$

The importance of this monodromy has been emphasized by several groups including Asplund, Bernamonti, Galli & Hartman and Roberts & Stanford

$$\boxed{\Delta S_A = 0\,, \quad t_- + t_\omega < y \text{ and } t_- + t_\omega > y + L}$$

$$\boxed{\begin{array}{l} \Delta S_A = \dfrac{c}{6}\log\left[\dfrac{\beta}{\pi\epsilon}\dfrac{\sin\pi\alpha_\psi}{\alpha_\psi}\dfrac{\sinh\left(\frac{\pi(y+L-t_--t_\omega)}{\beta}\right)\sinh\left(\frac{\pi(t_-+t_\omega-y)}{\beta}\right)}{\sinh\left(\frac{\pi L}{\beta}\right)}\right] \\ \qquad y < t_- + t_\omega < y + L \end{array}}$$

Calculation of S_B

Very similar, but with different insertion points :

$$\mathrm{Tr}\,\rho_A^n(t) = \frac{\langle\psi(x_1,\bar{x}_1)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle\psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$x_5 = y + L + i\frac{\beta}{2} - t_+ - t_\omega, \qquad x_6 = y + i\frac{\beta}{2} - t_+ - t_\omega$$

$$\bar{x}_5 = y + L - i\frac{\beta}{2} + t_+ + t_\omega, \qquad \bar{x}_6 = y - i\frac{\beta}{2} + t_+ + t_\omega$$

We always obtain the expected thermal answer at all times

$$\boxed{S_B = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon_{\mathrm{UV}}}\sinh\frac{\pi L}{\beta}\right)}$$

Calculation of $S_{A\cup B}$

Very similar, but with different insertion points :

$$\mathrm{Tr}\,\rho^n_{A\cup B}(t) = \frac{\langle \psi(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\tilde{\sigma}(x_2\bar{x}_3)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle \psi(x,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

with the insertion points

$$\begin{aligned}
x_1 &= -i\epsilon, \quad x_2 = y - t_- - t_\omega, \quad x_3 = y + L - t_- - t_\omega, \quad x_4 = +i\epsilon \\
\bar{x}_1 &= +i\epsilon, \quad \bar{x}_2 = y + t_- + t_\omega, \quad \bar{x}_3 = y + L + t_- + t_\omega, \quad \bar{x}_4 = -i\epsilon \\
x_5 &= y + L + i\frac{\beta}{2} - t_+ - t_\omega, \quad x_6 = y + i\frac{\beta}{2} - t_+ - t_\omega, \\
\bar{x}_5 &= y + L - i\frac{\beta}{2} + t_+ + t_\omega, \quad \bar{x}_6 = y - i\frac{\beta}{2} + t_+ + t_\omega\,.
\end{aligned}$$

Strategy

Using conformal maps

$$\mathrm{Tr}\,\rho^n_{A\cup B} = \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma} |1-z|^{4H_\sigma}\,|z_{56}|^{4H_\sigma}$$
$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

where all cross-ratios z, z_i are analytically known.

- $\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$ expected 6-pt function

S-channel (I)

Let us introduce a resolution of the identity

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle = \sum_\alpha \langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\alpha\rangle\,\langle\alpha|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

- $(z,\bar{z}) \to (1,1)$ for $\frac{\epsilon}{\beta} \ll 1 \Rightarrow$ use OPE !!
- $\sigma(z,\bar{z})\tilde{\sigma}(1,1) \sim \mathbb{I} +$ corrections in $(z-1)^r\ \mathcal{O}_r$
- Orthogonality of 2-pt functions $\Rightarrow$ $|\alpha\rangle = |\psi\rangle$ dominant

Thus,

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle \simeq \langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\,|\psi\rangle\;\langle\psi|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$

S-channel (II)

Using conformal maps

$$\langle\psi|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle = |1-\tilde{z}_5|^{4H_\sigma}\,|z_{56}|^{-4H_\sigma}\,\langle\psi|\,\sigma(\tilde{z}_5,\bar{\tilde{z}}_5)\tilde{\sigma}(1,1)|\psi\rangle\,,$$

we obtain

$$\mathrm{Tr}\,\rho^n_{A\cup B} \simeq \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma} |1-z|^{4H_\sigma}\,|1-\tilde{z}_5|^{4H_\sigma}\,G(z,\bar{z})G(\tilde{z}_5,\bar{\tilde{z}}_5)+\ldots$$

Since $\tilde{z}_5 = z_5$, the cross-ratio determining S_B, we derive

$$S_{A\cup B} = S_A + S_B, \qquad \text{and} \qquad I_{A:B} = 0$$

This resembles the bulk calculation from two geodesics joining pairs of points in the same boundary !!

T-channel (I)

We could introduce the resolution of the identity as follows

$$\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(1,1)\sigma(z_5,\bar{z}_5)\tilde{\sigma}(z_6,\bar{z}_6)|\psi\rangle$$
$$= \sum_{\alpha}\langle\psi|\sigma(z,\bar{z})\tilde{\sigma}(z_6,\bar{z}_6)\,|\alpha\rangle\,\langle\alpha|\,\sigma(z_5,\bar{z}_5)\tilde{\sigma}(1,1)|\psi\rangle\,.$$

- $(z_5,\bar{z}_5)\to(1,1)$ for $\frac{\epsilon}{\beta}\ll 1 \Rightarrow$ use OPE !!
- As before, $|\alpha\rangle = |\psi\rangle$ dominant contribution !!

T-channel (II)

In this case,

$$\mathrm{Tr}\,\rho^n_{A\cup B} \simeq \left|\frac{\beta}{\pi\epsilon_{UV}}\sinh\left(\frac{\pi L}{\beta}\right)\right|^{-8H_\sigma}\left|\frac{x}{1-x}\right|^{4H_\sigma}|1-z_5|^{4H_\sigma}|1-\tilde{z}_2|^{4H_\sigma}$$
$$G(\tilde{z}_2,\bar{\tilde{z}}_2)G(z_5,\bar{z}_5)+\ldots$$

where $(x,\bar{x})$ are the cross-ratios computed out of the insertion points of the four twist operators

$$x = \frac{z_{23}z_{56}}{z_{25}z_{36}} = \frac{w_{23}w_{56}}{w_{25}w_{36}} = \frac{2\sinh^2\frac{\pi L}{\beta}}{\cosh\frac{2\pi L}{\beta}+\cosh\frac{2\pi(t_- - t_+)}{\beta}} = \bar{x}\,,$$

T-channel (III)

For $t_- + t_\omega > y + L$, we derive

$$\boxed{\begin{aligned} S_{A\cup B} \simeq \frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_-+t_\omega-y)}{\beta}\cosh\frac{\pi(t_++t_\omega-y)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right) \\ +\frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_-+t_\omega-y-L)}{\beta}\cosh\frac{\pi(t_++t_\omega-y-L)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right) \\ +\frac{2c}{3}\log\left|\frac{\beta}{\pi\varepsilon_{UV}}\cosh\left(\frac{\pi\Delta t}{\beta}\right)\right| + \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\right)\end{aligned}}$$

where $\Delta t = t_- - t_+$

- To derive this result we used Fitzpatrick, Kaplan & Walters

Mutual information & "Scrambling" time (I)

In the regime $t_{\mp} + t_{\omega} > y + L > y$,

$$\boxed{\begin{aligned} I_{A:B} \simeq \; & \frac{2c}{3}\log\left(\frac{\beta}{\pi\varepsilon_{UV}}\sinh\frac{\pi L}{\beta}\right) - \frac{2c}{3}\log\left|\frac{\beta}{\pi\varepsilon_{UV}}\cosh\left(\frac{\pi\Delta t}{\beta}\right)\right| \\ & - \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_{\psi}}{\alpha_{\psi}}\right) \\ & - \frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_- + t_\omega - y)}{\beta}\cosh\frac{\pi(t_+ + t_\omega - y)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right) \\ & - \frac{c}{6}\log\left(\frac{\sinh\frac{\pi(t_- + t_\omega - y - L)}{\beta}\cosh\frac{\pi(t_+ + t_\omega - y - L)}{\beta}}{\cosh\frac{\pi\Delta t}{\beta}}\right) \end{aligned}}$$

- take $t_- = t_+ = 0$ and look for $t_\omega^\star$ satisfying

$$I_{A:B}(t_\omega^\star) = 0$$

Mutual information & "Scrambling" time (II)

- $t^\star_\omega > y + L$ requires small perturbation
- Assuming $t^\star_\omega/\beta \gg 1$

$$\boxed{t^\star_\omega = y + \frac{L}{2} - \frac{\beta}{2\pi}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\right) + \frac{\beta}{\pi}\log\left(2\sinh\frac{\pi L}{\beta}\right)}$$

- Expanding in $h_\psi \ll c$ (small perturbation)

$$t^\star_\omega = y + \frac{L}{2} + \frac{\beta}{2\pi}\log\left(\frac{\pi S_{density}}{4E_\psi}\right) + \frac{\beta}{\pi}\log\left(2\sinh\frac{\pi L}{\beta}\right)$$

where we used

$$\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi} \sim \frac{\pi E_\psi}{S_{\text{density}}}$$

with $S_{\text{density}} = \frac{\pi c}{3\beta}$ and $E_\psi = \frac{\pi h_\psi}{\epsilon}$

Holographic considerations

Main idea & strategy :

- Static point particle at $r = 0$ in global AdS_3

$$ds^2 = -\left(r^2 + R^2 - \mu\right) d\tau^2 + \frac{R^2 dr^2}{r^2 + R^2 - \mu} + r^2 d\varphi^2 \,,$$

- Holographic entanglement entropy known

$$S_A = \frac{c}{6}\left[\log\left(\frac{r_\infty^{(1)} \cdot r_\infty^{(2)}}{R^2}\right) + \log\frac{2\cos\left(|\Delta\tau_\infty|\alpha_\mu\right) - 2\cos\left(|\Delta\varphi_\infty|\alpha_\mu\right)}{\alpha_\mu^2}\right]$$

- Map metric to Kruskal coordinates, while boosting the particle, to describe a free falling particle in eternal BTZ
  - Use an initial condition ensuring the particle carries the right energy, from CFT and stress tensor perspective
- Map endpoints & compute entanglement entropy

Holographic comments

Calculations involve many explicit technical details, leading to

1. Exact matching of dominant CFT contributions with the holographic model geodesic calculations
   - S-channel and T-channel contributions precisely match the two dominant geodesics computing $S_{A\cup B}$
2. In the limit of large t_ω :
   - free falling particle becomes almost null with energy localised at the horizon
   - matches the schock-wave descriptions proposed/used by Shenker, Stanford, Roberts, Susskind, ...

Final remarks

- $\exists$ deep relations between quantum entanglement and geometry
  - extended in the AdS/CFT context leading to EPR=ER
- First principle calculation of scrambling time in 2d CFTs
  - Stringy corrections (Shenker & Stanford)
- Our results in the CFT follow from properties of 2d correlators in the large c limit
  - they may exist in other slicings of AdS, i.e. hyperbolic slicing responsible for AdS-Rindler physics
  - this may be related to the bulk expectation that scrambling occurs more generally than for event horizons (Susskind, Fischler et al)
- Question : role of quantum complexity ? (Susskind)