Deformations of special geometry and the holomorphic anomaly equation

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with Bernard de Wit and Swapna Mahapatra, arXiv:1406.5478

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Introduction

In supersymmetric field theories and string theories:

class of terms in effective action that play a special role:

F-terms.

Example: Type II superstring theory on $M_4 \times CY_3$

N=2 supersymmetry in four dimensions.

For every integer $g \geq 0$: $\exists$ F-term at precisely g-loop order in string perturbation theory, schematically

$$\int d^4x \, d^4\theta \sum_{g\geq 0} F^{(g)}(Y^I)\,(W^2)^g$$

Y^I: conical affine special Kähler manifold $\mathcal{M}$, $Y^I \to \lambda\, Y^I$, $\lambda \in \mathbb{C}^*$, $I = 0, \ldots, n = h^{1,1}$ (or $h^{2,1}$)

Special Kähler geometry. de Wit, van Proeyen

$F^{(0)}(Y)$ determines Kähler metric $N_{IJ} = 2\,\mathrm{Im}\left(\partial^2 F^{(0)}/\partial Y^I \partial Y^J\right)$.

Projective special Kähler manifold $\bar{\mathcal{M}} = \mathcal{M}/\mathbb{C}^* \quad (t^I = Y^I/Y^0)$.

Assemble: $g \geq 1$

$$w(Y) = \sum_{g=1}^{\infty} F^{(g)}(Y)$$

Special significance of F-terms:

- determine entropy of half-BPS black holes OSV, 2004
- $F^{(g)}$ topological string amplitudes.

Actually, the $F^{(g)}$ are not quite holomorphic ($g \geq 1$):

holomorphic anomaly equation (recursive relation, $g \geq 2$)

Bershadsky, Cecotti, Ooguri, Vafa, 1993

Deformed special geometry

Q1: what is the precise relation between TST and the LEEA?

Q2: Consistent extension of special geometry for incorporating non-holomorphic corrections encoded in $F^{(g)}, g \geq 1$?

- Q2: deformed special geometry, based on

$$F(Y, \bar{Y}) = F^{(0)}(Y) + 2i\,\Omega(Y, \bar{Y}) \;, \quad \Omega \quad \text{real}$$

Symplectic $Sp(2(n+1), \mathbb{R})$-transformations act on

$$(Y^I, F_I) \quad , \quad F_I = \frac{\partial F(Y, \bar{Y})}{\partial Y^I} \quad , \quad I = 0, \dots, n$$

LEEA. When Ω harmonic: Wilsonian limit

$$\Omega(Y, \bar{Y}) = w(Y) + \bar{w}(\bar{Y})$$

Natural deformation leads to perturbative topological string.

Deformed special geometry

- Q1: Generalized Hesse potential $H(\phi, \chi)$ Freed 1997

$$\phi^I = Y^I + \bar{Y}^I \qquad , \qquad \chi_I = F_I + \bar{F}_I$$

Obtained by Legendre transform with respect to $Y^I - \bar{Y}^I$:

$$H(\phi, \chi) = 4\left[\mathrm{Im} F^{(0)}(Y) + \Omega(Y, \bar{Y})\right] + i\,\chi_I\,(Y - \bar{Y})^I$$

Significance:

  - H 'Hamiltonian' (Legendre transform of LEEA), symplectic function $\longrightarrow$ Ω transforms in prescribed, non-trivially way.
  - Want to understand the structure of the Hesse potential:

  a unique subsector captures perturbative topological string.

Evaluating the Hesse potential

- Q1: New variables:

$$\begin{pmatrix} \phi^I \\ \chi_I \end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix} Y^I \\ F_I(Y,\bar{Y}) \end{pmatrix} = 2\,\mathrm{Re}\begin{pmatrix} \mathcal{Y}^I \\ F_I^{(0)}(\mathcal{Y}) \end{pmatrix}$$

Y^I: sugra variables $\qquad$ $\mathcal{Y}^I$: TST variables

Evaluate H in terms of $\mathcal{Y}^I \Rightarrow$ expansion in powers of $\Omega(\mathcal{Y},\bar{\mathcal{Y}})$.

  - H as series of symplectic functions, $H = \sum_{k=0}^{\infty} H^{(k)}(\mathcal{Y},\bar{\mathcal{Y}})$
  -

$$H^{(1)} = 4\Omega - 4N^{IJ}\left(\Omega_I\Omega_J + \Omega_{\bar{I}}\Omega_{\bar{J}}\right) + \mathcal{O}(\Omega^3)$$

$$N_{IJ} = -i\left(F_{IJ}^{(0)} - \bar{F}_{\bar{I}\bar{J}}^{(0)}\right)$$

- Pick Ω: non-holomorphic term whose variation is harmonic:

$$\Omega(\mathcal{Y},\bar{\mathcal{Y}}) = w(\mathcal{Y}) + \bar{w}(\bar{\mathcal{Y}}) + \alpha\ln\det N_{IJ} + \mathcal{O}(\alpha^2) \quad , \quad \alpha \in \mathbb{R}$$

Expanding $H^{(1)}$ in powers of (w, α):

$$H^{(1)} = 4\Big[F^{(1)}(\mathcal{Y},\bar{\mathcal{Y}}) + \Big(F^{(2)}(\mathcal{Y},\bar{\mathcal{Y}}) + \text{h.c.}\Big) + \Big(F^{(3)}(\mathcal{Y},\bar{\mathcal{Y}}) + \text{h.c.}\Big) + 4\,\alpha\, D_I F_J^{(1)}\, N^{IK}\, N^{JL}\, \bar{D}_{\bar{K}} \bar{F}_{\bar{L}}^{(1)} + \mathcal{O}(\alpha^4)\Big]$$

$F^{(g)}$ are symplectic functions. $F^{(1)} \propto \alpha \ln\det N_{IJ}$.
For $g \geq 2$: polynomials in N^{IJ}. Satisfy the holomorphic anomaly equation of topological string theory,

$$\partial_{\bar{I}} F^{(g)} = i\, \bar{F}^{(0)}_{\bar{I}\bar{P}\bar{Q}} N^{PJ} N^{QK} \left(2\alpha\, D_J \partial_K F^{(g-1)} + \sum_{r=1}^{g-1} \partial_J F^{(r)} \partial_K F^{(g-r)} \right)$$

where $\partial_{\bar{I}} F^{(g)} = \frac{\partial F^{(g)}(\mathcal{Y},\bar{\mathcal{Y}})}{\partial \bar{\mathcal{Y}}^I}$, etc.

Outlook: Perturbative TST captured by $H^{(1)}$. Formal series in inverse powers of $(\mathcal{Y}^0)^2$. Requires non-perturbative completion. RCS's talk.