The Quantum Effective Action of Lifshitz theories

with A. O. Barvinsky, D. Blas, G. Perez-Nadal and C. F. Steinwachs

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$$S = \int dt\, d^D x \left(\frac{1}{2} \partial_t \phi \partial_t \phi + \phi (-\Delta)^z \phi \right)$$

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For high energy physicist, the important feature is the improved UV regime

$$G(\omega, p^i) \sim \frac{1}{\omega^2 + p^{2z}}$$

Motivation

Condensed matter

It explains tri-critical phenomena known as "Lifshitz points"

$$S = \int dt\ d^D x \left(\frac{1}{2}\partial_t\phi\partial_t\phi + \phi(-\Delta)^z\phi + \alpha_1\phi(-\Delta)^{z-1}\phi + ... + \alpha_{z-1}\phi\Delta\phi \right)$$

The theory can flow from the anisotropic scaling to the isotropic one depending on the RGE of the coupling constants.

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Quantum Gravity

Following this same idea, it is possible to construct an, a priori, renormalizable theory of Quantum Gravity: Hořava-Lifshitz gravity

P. Hořava, 2009

$$ds^2 = (N^2 - N_i N^i)dt^2 - 2N_i dx^i - \gamma_{ij} dx^i dx^j$$

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$$R^2, \quad R$$

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For example, for $z = 2$

$$R^2, \quad R$$

There is also a non-projectable version

$$a_i = \frac{\nabla_i N}{N} \qquad (a_i)^z \tag{1}$$

D. Blas, O. Pujolas and S. Sibiryakov, 2010

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Lifshitz Theories in curved space

Therefore we are interested in studying Lifshitz field theories in curved space

$$ds^2 = (N^2 - N_i N^i)dt^2 - 2N_i dx^i - \gamma_{ij} dx^i dx^j$$

$$t \to f(t) \qquad x^i \to \tilde{x}^i(x^j, t)$$

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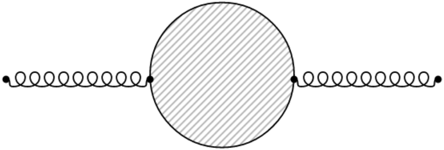

Can produce gravitational counterterms and phenomena analogous to the Weyl anomaly

Motivation

Holography

Lifshitz scalar field theories are conjectured to be holographic duals to Lifshitz space-times

- Massive vector coupled to Einstein-Hilbert gravity in the bulk

  S. Kachru, X. Liu and M. Mulligan, 2008

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Anisotropic Weyl invariance

$$N \to \Omega^{-z} N \qquad \gamma_{ij} \to \Omega^{-2}\gamma_{ij} \qquad \phi \to \Omega^{\frac{D-z}{2}}\phi \qquad zE + T_i^i = 0$$

$$S = \int dt d^D x \sqrt{|g|}\, \left(\mathcal{L}_n\phi\mathcal{L}_n\phi + \phi(-\Delta)^z\phi + ...\right)$$

For $z = D$, this is already invariant

Anisotropic Weyl invariance will be broken at the quantum level by conformal anomalies. Entanglement entropy, general properties of RG flows,...??

The action

$$ds^2 = (N^2 - N_i N^i)dt^2 - 2N_i dx^i - \gamma_{ij} dx^i dx^j$$

$$S = \int dt d^D x N\sqrt{\gamma}\left(\mathcal{L}_n\phi + \mathcal{L}_n\phi + \phi(-\Delta)^z\phi + \ldots\right) =$$
$$= \int dt d^D x N\sqrt{\gamma}\phi\mathcal{D}\phi$$

$$\mathcal{D} = \underbrace{-\mathcal{L}_n^2 + K^2}_{\text{Temporal part}} + \underbrace{\frac{1}{N}\sum_{i=0}^{z}\Omega_i (N\Delta)^i}_{\text{spatial part}}$$

Effective action from the Heat Kernel

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D. V. Vassilevich, 2003

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$$W_{ren} = -\frac{1}{2}\zeta^{'}(0,\mathcal{D}) - \frac{1}{2}\log\left(\mu^2\right)\zeta(0,\mathcal{D})$$

$$\zeta(0,f,\mathcal{D}) = a_{D+z}(f,\mathcal{D})$$

$$K(t,\mathcal{D}) = Tr(f\ e^{-s\mathcal{D}}) = \frac{1}{(4\pi s)^{\frac{z+D}{2z}}}\sum_{n=0}^{\infty} s^{\frac{n}{2z}}a_n(f,\mathcal{D})$$

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Weyl anomalies are going to be given also by the Heat Kernel

$$\delta W = -z\zeta(0,\omega,\mathcal{D}) = -z\ a_{D+z}(\omega,\mathcal{D})$$

The short-time expansion

$$K(t,\mathcal{D}) = Tr(f\ e^{-s\mathcal{D}}) = \frac{1}{(4\pi s)^{\frac{z+D}{2z}}}\sum_{n=0}^{\infty} s^{\frac{n}{2z}} a_n(f,\mathcal{D})$$

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where the coefficients a_n are given by local invariants of dimension n

$$a_n = \int dt\ d^D x \sqrt{\gamma} N \sum C_i \alpha^i$$

$$R \quad R^2 \quad K_{ij}K^{ij} \quad \mathcal{L}_n K$$

$$C_i(D,z)$$

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where the coefficients a_n are constraint

$$\begin{aligned}
&\Delta N + 2\Delta\gamma = d \\
&z(\Delta N + L) + 2\Delta\gamma = n \\
&\Delta N + L \equiv even
\end{aligned}$$

- $\Delta N \equiv$ (# of n^μ) - (# of n_μ)
- $\Delta\gamma \equiv$ (# of $\gamma^{\mu\nu}$) - (# of $\gamma_{\mu\nu}$)
- $d \equiv$ # of spatial derivatives
- $L \equiv$ # of Lie derivatives

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- No Possible term (no possible anomaly) for $D + z \equiv$odd
- The number of Lie derivatives for $z > D$ is constrained to $L = 0, 2$
- For $z > D$ the counterterm (anomaly) is given only by the spatial part of the action

Heat Kernel for $z > D$

The coefficient for $z > D$ is going to be given only by

$$\frac{1}{N}\sum_{i=0}^{z}\Omega_i(N\Delta)^i \sim \sum_{i=0}^{z}\Omega_i(N\Delta)^i$$

This is a standard higher order operator in which we have done a conformal transformation to the metric

$$\gamma_{ij} \rightarrow \frac{1}{N}\gamma_{ij}$$

The Heat Kernel will be then

$$a_{D+z}\left(\frac{1}{N}\sum_{i=0}^{z}\Omega_i(N\Delta)^i\right) = a_{D+z}\left(\sum_{i=0}^{z}\Omega_i\Delta^i\right)\Bigg|_{\gamma'_{ij}=\frac{1}{N}\gamma_{ij}}$$

Heat Kernel for $z = D$

The coefficient for $z = D$ is going to be given by

$$-\mathcal{L}_n^2 + K^2 + \frac{1}{N}\sum_{i=0}^{z}\Omega_i (N\Delta)^i$$

The Heat Kernel will be then

$$a_{2z}\left(\mathcal{D}\right) = \int dtd^D x\sqrt{|g|}\left(C_0 K_{ij}K^{ij} + C_1 K^2 + C_2 \mathcal{L}_n K\right) + a_{2z}\left(\sum_{i=0}^{z}\Omega_i \Delta^i\right)\Big|_{\gamma'_{ij}=\frac{1}{N}\gamma_{ij}}$$

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- Go to a spacetime in which $K_{ij} = 0$, so that the case is the same as with $z > D$.
- Fix C_0, C_1 and C_2 using a brute force algorithm

D. Nesterov and S. Solodukhin, 2010

Heat Kernel for $D \leq z$

$$a_{D+z}\left(\mathcal{D}\right) = \int dtd^Dx\sqrt{|g|}\left(C_0 K_{ij}K^{ij} + C_1 K^2 + C_2 \mathcal{L}_n K\right) + a_{D+z}\left(\sum_{i=0}^{z}\Omega_i \Delta^i\right)\Big|_{\gamma'_{ij}=\frac{1}{N}\gamma_{ij}}$$

$$C_0 = -\frac{\Gamma\left(\frac{D}{2z}\right)}{z\Gamma\left(\frac{D}{2}\right)}\frac{D+2z}{D+2}$$

$$C_1 = -\frac{\Gamma\left(\frac{D}{2z}\right)}{z\Gamma\left(\frac{D}{2}\right)}\frac{3+z-D}{D+2} - 2$$

$$C_2 = -2$$

Examples

Examples

$z = D = 2$

$$\mathcal{D} = -\mathcal{L}_n^2 + K^2 + \Delta(N\Delta) + \Omega_1^{ij}\nabla_i\nabla_j + \Omega_0$$

$$a_4(f,\mathcal{D}_i) = \int dt\ d^2xN\sqrt{\gamma}\ f\ \sqrt{\pi}\left[\frac{1}{32}\Omega_{1\ ij}\Omega_1^{ij} + \frac{1}{64}(\Omega_{1\ i}^{i})^2 - \frac{1}{8}\tilde{\nabla}_i\tilde{\nabla}_j\Omega_1^{ij} + \right.$$
$$\left. + \frac{1}{16}\tilde{\nabla}^j\tilde{\nabla}_j\Omega_{1\ i}^{i} - \frac{1}{2}\Omega_0 - \frac{1}{8}\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right) - \frac{2}{\sqrt{\pi}}\left(\mathcal{L}_nK + K^2\right)\right]$$

The anomaly agrees with previous computations.

We find the cocycles computed by I. arav, S. Chapman and Y. Oz, 2014

Examples

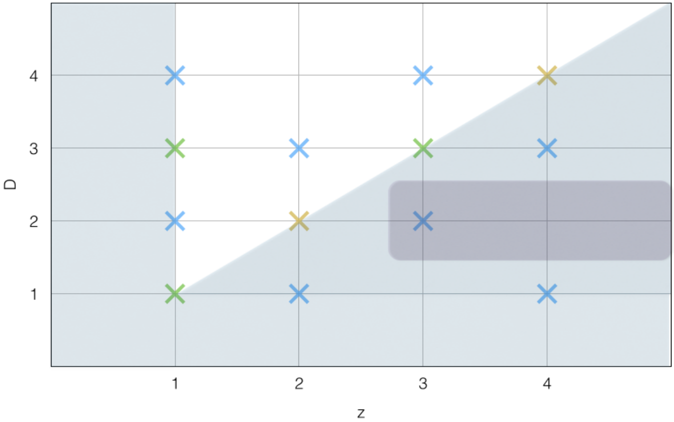
$z = D = 3$

$$\mathcal{D} = \mathcal{L}_n^2 + K^2 + \Delta(N\Delta(N\Delta)) + \Omega_2\Delta(N\Delta) + \Omega_1\Delta + \Omega_0$$

$$a_6(\mathcal{D}) = \int d^t d^3x \sqrt{|g|}\, f\left\{ -\frac{1}{2}\frac{1}{7!}\left\{ \frac{784}{9}\mathcal{R}_i^k\mathcal{R}^{ij}\mathcal{R}_{jk} - \frac{398}{3}\mathcal{R}_{ij}\mathcal{R}^{ij}\mathcal{R} + \frac{289}{9}\mathcal{R}^3 + \right.\right.$$
$$\left. + 16\mathcal{R}\tilde{\Delta}\mathcal{R} + 18\tilde{\Delta}^2\mathcal{R} + 24\mathcal{R}^{ij}\tilde{\nabla}_k\tilde{\nabla}_j\mathcal{R}_i^k + 40\mathcal{R}^{ij}\tilde{\Delta}\mathcal{R}_{ij} - \tilde{\nabla}_j\mathcal{R}_{ik}\tilde{\nabla}^k\mathcal{R}^{ij} + 34\tilde{\nabla}_k\mathcal{R}_{ij}\tilde{\nabla}^k\mathcal{R}^{ij} \right\} -$$
$$\left. - \frac{1}{5}\left[K_{ij}K^{ij} - \frac{1}{3}K^2\right] - 2\left(\mathcal{L}_nK + K^2\right) + \frac{2}{3}\Omega_0 \right\} - \frac{1}{2160}\Omega_2\left(12\tilde{\Delta}\mathcal{R} + 3\mathcal{R}^2 + 6\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\right) +$$
$$\left. + \frac{1}{6}\Omega_1\Omega_2 - \frac{1}{24}\Omega_2^3 - \frac{1}{18}\Omega_1\mathcal{R} + \frac{1}{72}\Omega_2^2\mathcal{R} \right\}$$

We see that the structure is more complicated and that we find terms with intrinsic objects.

Examples



Conclusions and open questions

- Theories with $z + D$ odd are one-loop finite
- There are many situations in which the theory is also one-loop finite if there are not deformations
- We give a general recipe to compute the Quantum Effective Action and the Weyl anomalies

Anomalies

- Is there an analogous to the A-theorem for Lifshitz theories?

Conjecture

- Einstein gravity with a massive vector field is dual only to the minimal action

$$S = \int dt d^D x \sqrt{|g|} \left(\mathcal{L}_n \phi \mathcal{L}_n \phi + \phi \left(-\Delta \right)^z \phi \right)$$

In order to take account of deformations, we need to use Hořava-Lifshitz gravity.

Arbitrary spin

- Generalization to fields with higher spin is almost straightforward. First step towards quantizing HL gravity.

Backup

H-L Gravity

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$$a_{D+z}\left(\sum_{i=0}^{z}\Omega_i\Delta^i\right)\Big|_{\gamma'_{ij}=\frac{1}{N}\gamma_{ij}}$$

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$$Tr\left(e^{-s\Delta^z}\right)=Tr\left(\int_0^\infty dt\ \delta(t-\Delta)e^{-st^z}\right)=Tr\left(\int_0^\infty dt\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\ e^{i\lambda t}e^{-st^z}e^{-i\lambda\Delta}\right)=$$
$$=\int_0^\infty dt\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{-st^z}e^{i\lambda t}\,Tr\left(e^{-i\lambda\Delta}\right)=\frac{1}{(4\pi)^{\frac{D}{2}}}\sum_k C_k a_k$$

$$C_k=\begin{cases}\frac{1}{z}\frac{\Gamma\left(-\frac{u}{z}\right)}{\Gamma(-u)}s^{\frac{u}{z}} & u=\frac{k-D}{2}<0\\ \left(-\partial_t\right)^u e^{-st^z}\Big|_{t=0} & u=\frac{k-D}{2}>0\quad and\quad D\ even\\ \frac{(-1)^u}{\Gamma\left(\frac{1}{2}\right)}\frac{d^{[u]+1}}{dt^{[u]+1}}\int_{-\infty}^{t}dx\ e^{-sx^z}(t-x)^{-\frac{1}{2}}\Big|_{t=0} & u=\frac{k-D}{2}>0\quad and\quad D\ odd\end{cases}$$

Heat Kernel for $D \leq z$

$$a_0 = 1$$
$$a_2 = \frac{1}{6} R$$
$$\ldots$$

$$C_k = \begin{cases} \frac{1}{z}\frac{\Gamma\left(-\frac{u}{z}\right)}{\Gamma(-u)} s^{\frac{u}{z}} & u = \frac{k-D}{2} < 0 \\ \left(-\partial_t\right)^u e^{-st^z}\Big|_{t=0} & u = \frac{k-D}{2} > 0 \quad and \quad D\ even \\ \frac{(-1)^u}{\Gamma\left(\frac{1}{2}\right)} \frac{d^{[u]+1}}{dt^{[u]+1}} \int_{-\infty}^t dx\ e^{-sx^z} (t-x)^{-\frac{1}{2}}\Big|_{t=0} & u = \frac{k-D}{2} > 0 \quad and \quad D\ odd \end{cases}$$