Classical Spinning Strings in AdS_3/CFT_2

The study of an integrable deformation

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Based on [R. Hernandez and JMN, 2014]
and [R. Hernandez and JMN, 2015]

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Outline

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AdS/CFT generalities

The prime example is the duality between $\mathcal{N} = 4$ SYM in four dimensions and type IIB string theory on $AdS_5 \times S^5$ with RR flux. There we can use integrability to find the spectrum of anomalous dimensions in the SYM.

Integrability methods are potentially applicable to other AdS_d backgrounds with RR flux. Examples of that are the $AdS_4/ABJM$ duality and the AdS_3/CFT_2 (in particular integrability have been proven for $AdS_3 \times S^3 \times T_4$ and $AdS_3 \times S^3 \times S^3 \times S^1$) we are going to treat here.

AdS_3/CFT_2 correspondence and integrability

However we are going to consider the case where we have an arbitrary combination of RR and NS-NS fluxes controlled by a parameter $q \in [0, 1]$. The case of pure RR flux (corresponding to $q = 0$) can be studied using usual integrability methods, while the case of pure NS-NS flux (corresponding to $q = 1$) can be rewritten as a WZW model.

Nonetheless, using supercoset formulation in GS action with a WZ term, the Lax connection for general q for both T_4 and $S^3 \times S^1$ cases (and for any semi-symmetric permutation coset) was found in [Cagnazzo, Zarembo, 2012], therefore we can apply the integrability toolbox to this case.

Outline

Spinning string ansatz

Let's start with the simpler case of spinning strings with most of the AdS_3 dynamics frozen

$$Y_1 + iY_2 = 0 \, , \qquad Y_3 + iY_0 = e^{iw_0 \tau} \, ,$$
$$X_1 + iX_2 = r_1(\sigma)\, e^{i[\omega_1 \tau + \alpha_1(\sigma)]} \, , \quad X_3 + iX_4 = r_2(\sigma)\, e^{i[\omega_2 \tau + \alpha_2(\sigma)]} \, ,$$

where $r_1^2(\sigma) + r_2^2(\sigma) = 1$. Later we will see that the generalization to full $AdS_3 \times S^3$ is straightforward from this results.

The $S^3 \subset AdS_3 \times S^3 \times T_4$ Lagrangian

If we freeze the dynamics on the AdS_3 space, the Polyakov action can be written as

$$L_{S^3} = \frac{\sqrt{\lambda}}{2\pi}\left[\sum_{i=1}^{2}\frac{1}{2}\left[(r_i')^2 + r_i^2(\alpha_i')^2 - r_i^2\omega_i^2\right] - \frac{\Lambda}{2}(r_1^2 + r_2^2 - 1)\right] + \frac{\sqrt{\lambda}}{2\pi}\left[q r_2^2\left(\omega_1\alpha_2' - \omega_2\alpha_1'\right)\right] \ ,$$

supported with the Virasoro constrains

$$\sum_{i=1}^{2}\left(r_i'^2 + r_i^2(\alpha_i'^2 + \omega_i^2)\right) = w_0^2 \ , \qquad \sum_{i=1}^{2} r_i^2\omega_i\alpha_i' = 0 \ .$$

The Neumann-Rosochatius Lagrangian

The lagrangian we have obtained is a deformation of the Neumann-Rosochatius lagrangian presented in [Arutyunov, Russo, Tseytlin, 2003] for $AdS_5 \times S^5$

$$L = \frac{1}{2}\sum_{i=1}^{3}(r_i'^2 + r_i^2\alpha_i'^2 - \omega_i^2 r_i^2) - \frac{1}{2}\Lambda(\sum_{i=1}^{3} r_i^2 - 1) ,$$

this lagrangian is integrable (because is an extension of a set of oscillators on a sphere), and the flux deformation we are looking at do not break the integrability.

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$$L = \frac{1}{2}\sum_{i=1}^{3}(r_i'^2 - \frac{v_i^2}{r_i^2} - \omega_i^2 r_i^2) - \frac{1}{2}\Lambda(\sum_{i=1}^{3} r_i^2 - 1) \ ,$$

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Equations of motion and First integrals

There are two ways to obtain the dynamics from this lagrangian

- To solve directly the equations of motion. This method is useful for calculating constant radii solutions.
- To use the (deformed) Uhlenbeck constants. This method is useful for calculating elliptic solutions.

Constant radii solutions: E. O. M.

The equations of motion for our lagrangian are

$$\alpha_i' = \frac{v_i + q r_2^2 \epsilon_{ij}\omega_j}{r_i^2} \ , \quad i = 1, 2 \ ,$$
$$r_1'' = -r_1\omega_1^2 + r_1\alpha_1'^2 - \Lambda r_1 \ ,$$
$$r_2'' = -r_2\omega_2^2 + r_2\alpha_2'^2 - \Lambda r_2 + 2q r_2(\omega_1\alpha_2' - \omega_2\alpha_1') \ ,$$

and we are going to take $r_1 =$const. and $\alpha_i' = \bar{m}_i =$const. (which have the interpretation of winding numbers) therefore we can join them as

$$(\omega_1^2 - \omega_2^2) - (\bar{m}_1^2 - \bar{m}_2^2) + 2q(\omega_1\bar{m}_2 - \omega_2\bar{m}_1) = 0$$

Constant radii solutions: finding the ω's and the dispersion relation (as a power series)

Using the result from the previous slide and the definition of total angular momentum $J=\sqrt{\lambda}\sum_i \left(r_i^2\omega_i - \sum_j \epsilon_{ij} q r_2^2 \bar{m}_j \right)$ we can find the ω_i as a power series on J:

$$\omega_1 = \frac{J}{\sqrt{\lambda}} + \frac{\sqrt{\lambda}}{2J}\bar{m}_1(\bar{m}_1+\bar{m}_2)(1-q^2)\left[1-\frac{\sqrt{\lambda}}{J}q\bar{m}_2+\dots\right],$$

$$\omega_2 = \frac{J}{\sqrt{\lambda}} - q(\bar{m}_1-\bar{m}_2) + \frac{\sqrt{\lambda}}{2J}\bar{m}_2(\bar{m}_1+\bar{m}_2)(1-q^2) \cdot \left[1-\frac{\sqrt{\lambda}}{J}q(\bar{m}_1+\bar{m}_2)+\dots\right].$$

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$$\omega_2=\frac{J}{\sqrt{\lambda}}-q(\bar{m}_1-\bar{m}_2)+\frac{\sqrt{\lambda}}{2J}\bar{m}_2(\bar{m}_1+\bar{m}_2)(1-q^2)\cdot\left[1-\frac{\sqrt{\lambda}}{J}q(\bar{m}_1+\bar{m}_2)+\dots\right].$$

$$E^2=J^2-2\sqrt{\lambda}q\bar{m}_1J+\frac{\lambda}{J}\left[(\bar{m}_1^2J_1+\bar{m}_2^2J_2)(1-q^2)+q^2\bar{m}_1^2J\right]+\cdots$$

Constant radii solutions: interesting simplifications

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- The limit $\bar{m}_1 = -\bar{m}_2 = \bar{m}$ ($\Longrightarrow J_1 = J_2$, already considered in [Hoare, Stepanchuk, Tseytlin, 2014]). In this case

$$\omega_1 = \frac{J}{\sqrt{\lambda}} \ , \quad \omega_2 = \frac{J}{\sqrt{\lambda}} - 2q\bar{m} \ , \quad E^2 = J^2 - 2q\bar{m}J + \bar{m}^2 \ .$$

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- The limit $q = 1$, which corresponds to pure NS-NS flux and to WZW model. In this case:

$$\omega_1 = \frac{J}{\sqrt{\lambda}} \ , \quad \omega_2 = \frac{J}{\sqrt{\lambda}} - (\bar{m}_1 - \bar{m}_2) \ , \quad E^2 = (J - \bar{m}_1)^2 \ .$$

Snoidal solutions: Uhlenbeck constants

To find non-constant radii solutions we can use the (deformed) Uhlenbeck constant

$$\bar{I}_1 = r_1^2(1-q^2)+\frac{1}{\omega_1^2-\omega_2^2}\left[(r_1 r_2' - r_1' r_2)^2 + \frac{(v_1+q\omega_2)^2}{r_1^2} r_2^2 + \frac{v_2^2}{r_2^2} r_1^2\right] .$$

If we change to coordinates $\frac{r_1^2}{\zeta-\omega_1^2} + \frac{r_2^2}{\zeta-\omega_2^2} = 0$ we get

$$\zeta'^2 = -4(1-q^2)\prod_{i=1}^{3}(\zeta-\zeta_i) ,$$

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which can be easily solved

$$r_1^2(\sigma) = \frac{\zeta_3 - \omega_1^2}{\omega_2^2 - \omega_1^2} + \frac{\zeta_2 - \zeta_3}{\omega_2^2 - \omega_1^2}\,\text{sn}^2\left(\sigma\sqrt{(1-q^2)(\zeta_3 - \zeta_1)}, \frac{\zeta_3 - \zeta_2}{\zeta_3 - \zeta_1}\right) .$$

Snoidal solutions: $q = 1$ limit

The dispersion relation is not easy to find for general q because, to do that, we have to invert the periodicity condition

$$\pi\sqrt{(1-q^2)(\zeta_3-\zeta_1)} = n\mathsf{K}\left(\frac{\zeta_3-\zeta_2}{\zeta_3-\zeta_1}\right) \ .$$

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However when $q = 1$ the elliptic functions and periodicities become trigonometric ($\zeta'^2 = -4\omega^2 \prod_{i=1}^{2}(\zeta - \tilde{\zeta}_i)$) and we find

$$r_1^2(\sigma) = \frac{\tilde{\zeta}_2 - \omega_1^2}{\omega_2^2 - \omega_1^2} + \frac{\tilde{\zeta}_1 - \tilde{\zeta}_2}{\omega_2^2 - \omega_1^2}\sin^2(\omega\sigma) \ ,$$

where $2\omega \in \mathbb{Z}$.

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where $2\omega \in \mathbb{Z}$. And the dispersion relation can be found analytically

$$E^2 = \lambda\left(\bar{m}_1^2-\bar{m}_2^2+4\omega\bar{m}_2-3\omega^2\right) - 2\sqrt{\lambda}J(\bar{m}_1+\bar{m}_2-2\omega) \; .$$

Outline

The $AdS_3 \times S^3 \subset AdS_3 \times S^3 \times T_4$ Lagrangian

We can repeat the same steps with a dynamic AdS_3 space. The lagrangian now can be written as

$$L = L_{S^3} + L_{AdS_3} = L_{S^3} + \frac{\sqrt{\lambda}}{4\pi} \Big[g^{ab} \left(z'_a z'_b + z_a z_a \beta'^2_b - z_a z_a w_b^2 \right) - \frac{\tilde{\Lambda}}{2} \left(g^{ab} z_a z_b + 1 \right) - 2 q z_1^2 (w_0 \beta'_1 - w_1 \beta'_0) \Big] \ ,$$

supported with the Virasoro constrains

$$z_0'^2 + z_0^2 (\beta_0'^2 + w_0^2) = z_1'^2 + z_1^2 (\beta_1'^2 + w_1^2) + \sum_{i=1}^{2} \left(r_i'^2 + r_i^2 (\alpha_i'^2 + \omega_i^2) \right) \ ,$$

$$z_1^2 w_1 \beta'_1 + \sum_{i=1}^{2} r_i^2 \omega_i \alpha'_i = z_0^2 w_0 \beta'_0 \ .$$

Constant radii solutions: E. O. M.

The equations of motion for the part of the sphere does not change and those for the AdS are nearly the same but with factors of g^{aa}. In the constant radii case (with $\beta_1' = \bar{k}$), can be written as

$$(\omega_1^2 - \omega_2^2) - (\bar{m}_1^2 - \bar{m}_2^2) + 2q(\omega_1 \bar{m}_2 - \omega_2 \bar{m}_1) = 0 \ ,$$
$$w_1^2 - \bar{k}^2 - w_0^2 + 2q\kappa\bar{k} = 0 \ .$$

Constant radii solutions: Dispersion relation

After using the Virasoro constrains and some algebra we can find the (two possible) dispersion relation

$$E_+ = J + S - \sqrt{\lambda} q \left(\bar{m}_1 + 2\frac{\bar{k} S}{J} \right) + \frac{\lambda}{2J^2} (\bar{m}_1^2 J_1 + \bar{m}_2^2 J_2 + \bar{k}^2 S) \cdot (1 - q^2) - \frac{\lambda}{J^3} 2q^2 \bar{k} S (\bar{m} J + \bar{k} S) + \cdots$$

$$E_- = J - S - \sqrt{\lambda} q \bar{m}_1 + \frac{\lambda}{2J^2} (\bar{m}_1^2 J_1 + \bar{m}_2^2 J_2 - \bar{k}^2 S)(1 - q^2) + \cdots$$

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$$E_- = J - S - \sqrt{\lambda} q \bar{m}_1 + \frac{\lambda}{2J^2} (\bar{m}_1^2 J_1 + \bar{m}_2^2 J_2 - \bar{k}^2 S)(1 - q^2) + \cdots$$

The dynamics again simplifies in $q = 1$ where

$$E_+ = S + \sqrt{(J - \bar{m}_1)^2 - 4\bar{k}S} \, , \qquad E_- = J - S - \bar{m}_1 \, .$$

Snoidal solutions

Again, we can repeat step by step the calculations done for the sphere (with some changes of signs) and we get

$$z_0^2(\sigma) = \frac{\mu_3 - w_0^2}{w_1^2 - w_0^2} + \frac{\mu_2 - \mu_3}{w_1^2 - w_0^2} \,\text{sn}^2 \left(\sigma \sqrt{(1-q^2)(\mu_3 - \mu_1)}, \frac{\mu_3 - \mu_2}{\mu_3 - \mu_1} \right) \ .$$

Again, the dispersion relation is not easy to find unless $q = 1$ because, to do that, we have to invert the periodicity condition

$$\pi \sqrt{(1-q^2)(\mu_3 - \mu_1)} = n' \mathrm{K} \left(\frac{\mu_3 - \zeta_2}{\mu_3 - \mu_1} \right) \ .$$

Snoidal solutions: $q = 1$ limit

In that limit elliptic functions and periodicities become trigonometric and we find

$$z_0^2(\sigma) = \frac{\tilde{\mu}_2 - w_0^2}{w_1^2 - w_0^2} + \frac{\tilde{\mu}_1 - \tilde{\mu}_2}{w_1^2 - w_0^2} \sin^2(\omega' \sigma) \ ,$$

where $2\omega' \in \mathbb{Z}$.

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where $2\omega' \in \mathbb{Z}$.

$$E_+ = \big[\sqrt{\lambda}\big(\bar{k}_1^2 + \bar{m}_1^2 - (\bar{m}_2 - 3\omega)(\bar{m}_2 - \omega) - \omega'^2\big) - 2\bar{k}_1 S \\ - 2J(\bar{m}_1 + \bar{m}_2 - 2\omega)\big] / \big(2\bar{k}_1\big) \ ,$$

and something similar for E_-

Outline

A comment on the pulsating strings

Another interesting anstaz I want to talk about is the pulsating string anstaz, which have the same functional form as the spinning string ansatz but changing $\tau \leftrightarrow \sigma$. That is

$$Y_1 + iY_2 = z_1(\tau)\, e^{i[w_1\sigma + \beta_1(\tau)]} \,, \quad Y_3 + iY_0 = z_0(\tau)\, e^{i[w_0\sigma + \beta_0(\tau)]} \,,$$
$$X_1 + iX_2 = r_1(\tau)\, e^{i[\omega_1\sigma + \alpha_1(\tau)]} \,, \quad X_3 + iX_4 = r_2(\tau)\, e^{i[\omega_2\sigma + \alpha_2(\tau)]} \,,$$

where $r_1^2(\tau) + r_2^2(\tau) = z_0^2(\tau) - z_1^2(\tau) = 1$.

A comment on the pulsating strings

The same procedure can be applied to pulsating strings (with some subtleties related with the periodicity conditions). For the case $S^1 \times AdS_3$ we have found that

$$\frac{E}{\sqrt{\lambda}} = qk_1 + \frac{\omega^2 - k_1^2}{2k_1} \left[q \mp \sqrt{q^2 - 1 + \frac{\alpha^4}{(k_1^2 - \omega^2)^2}} \right] ,$$

which agrees with the result obtained in [Maldacena, Ooguri, 2000] by means of the WZW model limit ($q = 1$).

Outline

Conclusions

- We have found a general class of solutions (with and without constant radii) of the flux-deformed Neumann-Rosochatius system.
- The ellipsoidal solutions contains more general solutions I have not covered here, like giant magnon solutions.
- We have seen a huge simplification of the expressions (in particular the dispersion relation) when we take the limit $q = 1$. This is a consequence of the description as a WZW model in this limit, which have more symmetries.
- We already had a good description for the $q = 0$ case (using integrability) and $q = 1$ case (using WZW models). This results provides a description in the interpolating regime between those two well known cases.