Double Soft Limits from Asymptotic Symmetries

Eduardo Conde Pena (ongoing work
with Pujian Mao @ Université Libre de Bruxelles)

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MOTIVATION

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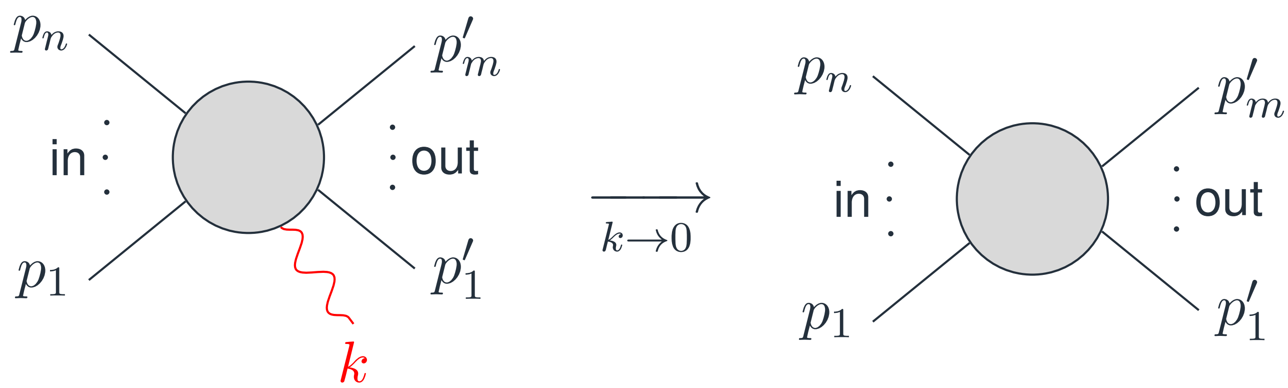

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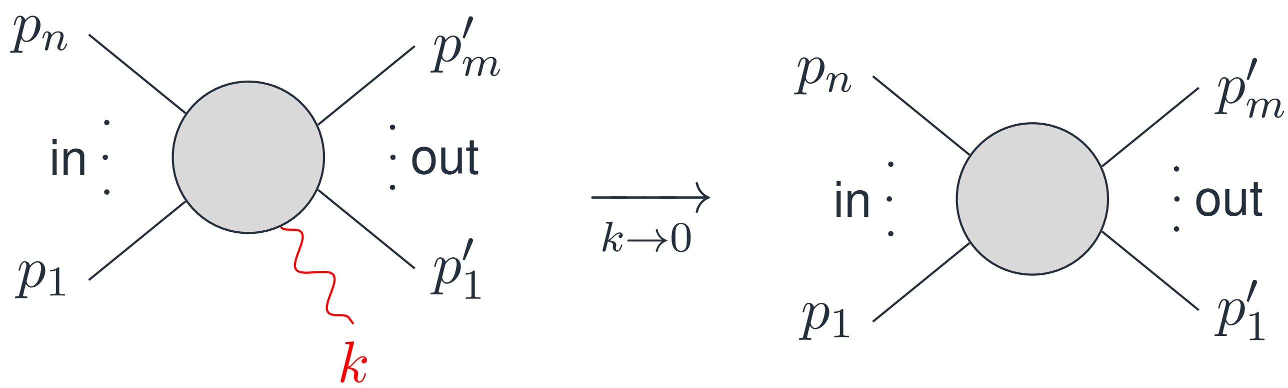

The precise relation is

$$\lim_{k\to 0} M_{n\to m+1} = \lim_{k\to 0} S_1^{(0)} M_{n\to m} \,,$$

$S_1^{(0)}$ is called a *soft factor*. It was originally determined by Weinberg in 1964, using just general arguments of unitarity plus Lorentz invariance. This formula is referred to as *soft photon theorem*.

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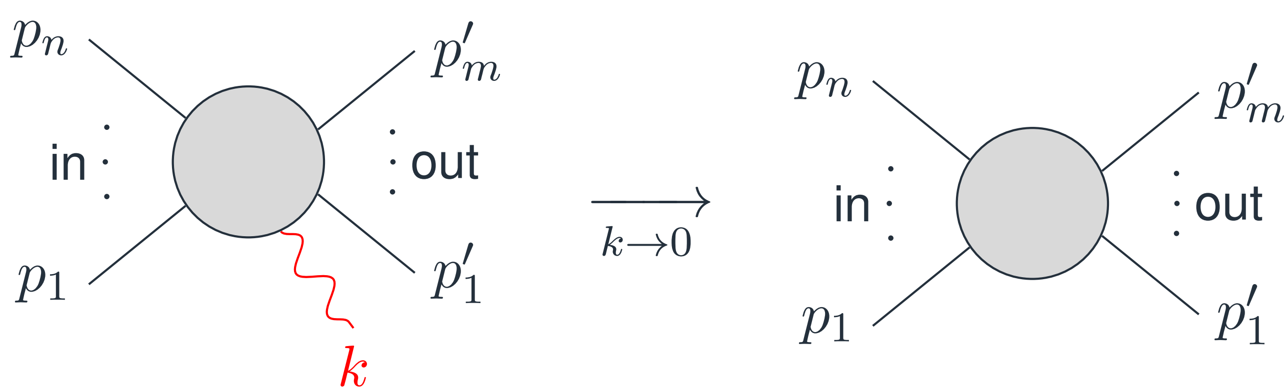

To give an example: soft factor for QED:

$$S_1^{(0)} = e \sum_{\text{out}} q'_j \frac{\epsilon_\pm \cdot p'_j}{k \cdot p'_j} - e \sum_{\text{in}} q_i \frac{\epsilon_\pm \cdot p_i}{k \cdot p_i}$$

where $\epsilon_\pm$ is the polarization tensor of of the soft photon, and q_i (q'_j) are the electric charges of the incoming (outgoing) particles.

We can think of the formula $\lim_{k\to 0} M_{n\to m+1} = \lim_{k\to 0} S_{1}^{(0)} M_{n\to m}$ as the first term in a series expansion:

$$M_{n\to m+1} = S_{1}\, M_{n\to m}\,,$$

$$\begin{array}{c} q = \omega\tilde{q} \\ \omega \to 0 \end{array}\,, \qquad S_{1} = \frac{\tilde{S}_{1}^{(0)}}{\omega} + \tilde{S}_{1}^{(1)} + \tilde{S}_{1}^{(2)}\omega + \tilde{S}_{1}^{(3)}\omega^2 + \cdots$$

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- Determining S_1 amounts to determine the whole S-matrix.
- There has been recent interest in (re)computing the first subleading soft factors $\tilde{S}_1^{(i)}$, $i = 1, 2$.
- [Strominger et al.'13-'15] argue that some of these factors follow from a particular type of symmetries: asymptotic symmetries.
- For instance, for gravity, $\tilde{S}_1^{(0)}$, $\tilde{S}_1^{(1)}$, $\tilde{S}_1^{(2)}$ are determined by symmetries (BMS group and extensions). For QED or Yang-Mills, only $\tilde{S}_1^{(0)}$, $\tilde{S}_1^{(1)}$ can be fixed.

Even more recently, there is interest in *double soft limits*:

$$M_{n\to m+2} = S_2 \, M_{n\to m} \, ,$$

So far, the leading and subleading piece of S_2 have been determined for some theories at tree level: gravity, Yang-Mills and several theories with scalar massless particles.[Cachazo et al., Klose et al., Volovich et al.'15]
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Question	Are double soft limits constrained by asymptotic symmetries?

For the rest of the talk, just two points:

- General discussion on how one discusses symmetries in the language of scattering amplitudes
- Connection between asymptotic symmetries and soft theorems (in a concrete example: QED).

HOW SYMMETRIES CONSTRAIN SCATTERING AMPLITUDES

Unbroken symmetries

Definition of a symmetry: Unitary operator U that acts on the same way on the “in” and “out” states:

$$\left.\begin{array}{rcl} |\mathrm{in}> & \to & U|\mathrm{in}> \\ <\mathrm{out}| & \to & <\mathrm{out}|\mathrm{U}^\dagger \end{array}\right\} \qquad <\mathrm{out}|\mathrm{in}>=<\mathrm{out}|\mathrm{U}^\dagger\mathrm{U}|\mathrm{in}>$$

Ward identity: Condition above rewritten in terms of charges

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- Rotational invariance: $Q = J$, dictates how scattering amplitudes scale with the spin of the particles.
- Supersymmetry: identities between scattering amplitudes with particles on the same supermultiplet.
- Etc.

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When the broken charges do not commute, the operator S_2 in $M_{n\to m+2} = S_2\, M_{n\to m}$ is not analytic, and the double soft limit is ambiguous.

$$\text{e.g.} \qquad \lim_{\mathrm{k}_\alpha\to 0}\lim_{\mathrm{k}_\beta\to 0} \mathrm{M}_{\mathrm{n}\to\mathrm{m}+2} \neq \lim_{\mathrm{k}_\beta\to 0}\lim_{\mathrm{k}_\alpha\to 0} \mathrm{M}_{\mathrm{n}\to\mathrm{m}+2}$$

This ambiguity encodes the commutation relations $[Q_\alpha, Q_\beta] \neq 0$.
[Arkani-Hamed et al.'08]

SOFT PHOTONS AS GOLDSTONE BOSONS

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Asymptotic symmetries of QED

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There is a precise (and technical) way to ask, and answer, this question.
The transformations are the so-called *large gauge transformations*

$$\delta \lim_{r\to\infty} A_\mu = \partial_\mu \epsilon(z,\bar{z})$$

where

$$\mathrm{ds}^2 = -\mathrm{du}^2 - 2\mathrm{dudr} + 2\mathrm{r}^2\gamma_{\mathrm{z\bar{z}}}\mathrm{dzd\bar{z}}\,, \qquad \mathrm{r} = \infty \text{ is the null boundary}$$

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- Classical charge : $\int_{r=\infty} \mathrm{dud^2z}\,\epsilon\,\gamma_{\mathrm{z}\bar{\mathrm{z}}}\mathrm{j_u} + \frac{1}{\mathrm{e}^2}\int_{\mathrm{r}=\infty} \mathrm{dud^2z}\,\epsilon\,\partial_{\mathrm{u}}\left(\partial_{\mathrm{z}}\mathrm{A}_{\bar{\mathrm{z}}} + \partial_{\bar{\mathrm{z}}}\mathrm{A_z}\right)$
- Quantum charge : $\epsilon\, Q_e \quad - \frac{1}{e}\int_{r=\infty} \frac{\mathrm{d^2z}}{1+z\bar{z}}\left(\partial_{\bar{z}}\epsilon\right) \lim_{k^0\to 0} k^0 a^\dagger(k)$

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- Quantum charge : $Q_\mathrm{L} \quad + \quad Q_\mathrm{NL}$

Assume that the asymptotic symmetries discussed before remain valid at the quantum level. We can then impose the condition

$$< \mathrm{out}|Q^\dagger - Q|\mathrm{in} >= 0 \quad \text{with} \quad Q = Q_L + Q_{NL}$$

$$\Downarrow$$

$$< \mathrm{out}| \int_{r=\infty} \frac{d^2 z}{1 + z\bar{z}} (\partial_{\bar{z}}\epsilon) \lim_{k^0 \to 0} k^0 a^\dagger(k)|\mathrm{in} >= \left(\sum_{\mathrm{out}} q'_j \epsilon(z_j, \bar{z}_j) - \sum_{\mathrm{in}} q_i \epsilon(z_i, \bar{z}_i) \right) < \mathrm{out}|\mathrm{in} >$$

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Emission of a soft photon in the direction w is characterized by $\epsilon = \frac{1}{z-w}$. (Notice that $\partial_{\bar{z}}\epsilon = \delta(z-w)$).
Translating into momentum space, the resulting formula becomes

Soft photon theorem

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$$\lim_{k^0 \to 0} k^0 < \mathrm{out}|\mathrm{a}^\dagger(\mathrm{k})|\mathrm{in} >=$$

$$e\, k^0 \left(\sum_{\mathrm{out}} q_j' \frac{\epsilon_\pm \cdot p_j}{k \cdot p_j} - e \sum_{\mathrm{in}} q_i \frac{\epsilon_\pm \cdot p_i}{k \cdot p_i} \right) < \mathrm{out}|\mathrm{in} >$$

This is the soft photon formula by Weinberg, that determines $S_1^{(0)}$

$$\lim_{k \to 0} M_{n \to m+1} = \lim_{k \to 0} S_1^{(0)} M_{n \to m} \, ,$$

QED possesses another extra set of asymptotic symmetries, different from the large gauge transformations. They sort of "rotate" the fields.

The same the large gauge transformations provide in some sense a generalization of electric charge (the linear part was precisely the electric charge), this new set of symmetries provide a generalization of dipole charge.

One can play the same game for these new charges. The associated Ward identities are equivalent to the subleading soft photon theorem, i.e. they determine $S_1^{(1)}$.

Double soft limits

As we said, QED possesses two types of asymptotic symmetries. We can denote their respective charges as Q_{e} and Q_{d}. The former creates a “leading soft photon” and the latter a “subleading soft photon”.

These charges verify an algebra reminiscent of that among translations and rotations:

$$[Q_{\mathrm{e}}, Q_{\mathrm{e}}] = 0\,, \quad [Q_{\mathrm{e}}, Q_{\mathrm{d}}] \sim Q_{\mathrm{e}}\,, \quad [Q_{\mathrm{d}}, Q_{\mathrm{d}}] \sim Q_{\mathrm{d}}$$

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According to the picture we gave for broken symmetries, we should expect that at leading order there is no ambiguity in the double soft limit. The ambiguity must arise though at subleading order, and should reflect the commutation relations given above.

These results match exactly what has been found recently in the scattering amplitudes literature!

The main message is that one can identify asymptotic symmetries of a QFT (this step must be done theory by theory) with spontaneously broken symmetries of its S-matrix. Not only this yields soft theorems, but it seems it also gives “double-soft” theorems.

Keep working...

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Thanks!