Quantum Corrections to Unimodular Gravity

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Introduction

- Unimodular Gravity is a truncation of General Relativty where the spacetime metric is unimodular,

$$\tilde{g} \equiv \det \tilde{g}_{\mu\nu} = -1$$

- It has the nice property that the vacuum energy does not couple to gravitation.

$$S \equiv \int d^n x \left(-\frac{1}{2\kappa^2} R[\tilde{g}] + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - V(\psi) \right)$$

- Due to the unimodular metric the full diffeomorphism invariance is broken to a subgroup with unit jacobian; *TDiff* (i.e. the generating vectors are transverse $\partial_\mu \xi^\mu = 0$).

The fact that full diffeomoporhism invariance is broken is a technical issue since in order to formulate a path integral one should integrate over constrained variables.

$$\mathscr{D}g_{\mu\nu} \;\; \text{with} \;\; g^{\mu\nu}\delta g_{\mu\nu} = 0$$

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- Introducing a gauge 3-form M. Henneaux and C. Teitelboim, Phys. Lett. B **222**, 195 (1989).
- Formulating the theory in terms of an unconstrained metric $g_{\mu\nu}$ by adding Weyl invariance

$$\tilde{g}_{\mu\nu} = g^{-\frac{1}{n}} g_{\mu\nu}$$

E. Alvarez, D. Blas, J. Garriga and E. Verdaguer,
Nucl. Phys. B **756**, 148 (2006)

Flat Space

The most general action principle for a spin two field

$$\mathscr{L} \equiv \sum_{i=1}^{4} C_i \ \mathscr{O}^{(i)}$$

$$\mathscr{O}^{(1)} \equiv \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial^\mu h^{\rho\sigma}$$
$$\mathscr{O}^{(2)} \equiv -\frac{1}{2}\partial^\rho h_{\rho\sigma}\partial_\mu h^{\mu\sigma}$$
$$\mathscr{O}^{(3)} \equiv \frac{1}{2}\partial_\mu h\partial_\lambda h^{\mu\lambda}$$
$$\mathscr{O}^{(4)} \equiv -\frac{1}{4}\partial_\mu h\partial^\mu h$$

LTDiff forces $C_2 = 1$

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad \text{with} \quad \partial_\mu \xi^\mu = 0$$

Only two theories propagate just spin two

- Fierz-Pauli (*LDiff*).

$$C_3 = C_4 = 1$$

- *WTDiff*.

$$C_3 = \frac{2}{n}$$
$$C_4 = \frac{n+2}{n}$$

WTDiff is obtained by $h_{\mu\nu} \to h_{\mu\nu} - \frac{1}{n} h \eta_{\mu\nu}$. But this is not a field redefinition, because it is not invertible. It is the linear limit of Unimodular Gravity. It has been shown recently that this generalizes also to a curved space C. Barceló, R. Carballo-Rubio and L. J. Garay, Phys. Rev. D **89**, no. 12, 124019 (2014).

We follow this idea of a non-invertible field redefinition to define Unimodular Gravity from General Relativity. The truncation of General Relativity to unimodular metrics is just

$$S_{UG} \equiv -M_P^{n-2} \int d^n x \left(R[\hat{g}] + L_{\text{matt}}[\psi_i, \hat{g}] \right) =$$
$$= -M_P^{n-2} \int d^n x |g|^{\frac{1}{n}} \left(R + \frac{(n-1)(n-2)}{4n^2} \frac{\nabla_\mu g \nabla^\mu g}{g^2} + L_{\text{matt}}[\psi_i, |g|^{-\frac{1}{n}} g_{\mu\nu}] \right)$$

And the equations of motion

$$R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} - \frac{(n-2)(2n-1)}{4n^2} \left(\frac{\nabla_\mu g \nabla_\nu g}{g^2} - \frac{1}{n} \frac{(\nabla g)^2}{g^2} g_{\mu\nu} \right) +$$
$$+ \frac{n-2}{2n} \left(\frac{\nabla_\mu \nabla_\nu g}{g} - \frac{1}{n} \frac{\nabla^2 g}{g} g_{\mu\nu} \right) = M_P^{2-n} \left(T_{\mu\nu} - \frac{1}{n} T g_{\mu\nu} \right)$$

which reminds to the ones posited by Einstein in 1919 when $|g| = 1$.

Working in the gauge $|g| = 1$, the trace can be recovered by using the Bianchi identities

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R \Rightarrow \frac{n-2}{2n}\nabla_\mu R = -\frac{1}{n}\nabla_\mu T$$

$$\frac{n-2}{2n}R + \frac{1}{n}T = -C$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - Cg_{\mu\nu} = T_{\mu\nu}$$

The constant piece of the potential V_0 does not source the cosmological constant.

The aim of the work we have done is to examine whether there are quantum corrections to this situation since, if there were present (which are not) the importance of the classical result would not be great.

Quantum Corrections

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Transverse ghosts are needed for *TDiff*, $\nabla^\mu c^T_\mu = 0$ so we swap transversality with gauge symmetry $c_\mu \to \nabla_\mu f$.

New ghosts are needed in order to close the BRST, we need "ghosts for ghost".

field	s_D	s_W
$g_{\mu\nu}$	0	0
$h_{\mu\nu}$	$\nabla_\mu c^T_\nu + \nabla_\nu c^T_\mu + c^{\rho\,T}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{\rho\,T} h_{\rho\nu} + \nabla_\nu c^{\rho\,T} h_{\rho\mu}$	$2c^{(1,1)}\left(g_{\mu\nu} + h_{\mu\nu}\right)$
$c^{(1,1)\mu}$	$\left(Q^{-1}\right)^\mu_\nu \left(c^{\rho\,T}\nabla_\rho c^{T\,\nu}\right) + \nabla^\mu \phi^{(0,2)}$	0
$\phi^{(0,2)}$	0	0
$b^{(1,-1)}_\mu$	$f^{(0,0)}_\mu$	0
$f^{(0,0)}_\mu$	0	0
$\bar{c}^{(0,-2)}$	$\pi^{(1,-1)}$	0
$\pi^{(1,-1)}$	0	0
$c'^{\,(0,0)}$	$\pi'^{\,(1,1)}$	0
$\pi'^{\,(1,1)}$	0	0
$c^{(1,1)}$	$c^{T\rho}\nabla_\rho c^{(1,1)}$	0
$b^{(1,-1)}$	$c^{T\rho}\nabla_\rho b^{(1,-1)}$	$f^{(0,0)}$
$f^{(0,0)}$	$c^{T\rho}\nabla_\rho f^{(0,0)}$	0

Table: BRST transformations of the fields involved in the path integral.

where $\left(Q^{-1}\right)^\mu_\nu$ denotes the inverse of the operator $Q_{\mu\nu} = g_{\mu\nu}\Box - R_{\mu\nu}$

The operator involving $h_{\mu\nu}$, f and c' is non-minimal. We need to use the Barvinsky & Vilkovisky technique (A. O. Barvinsky and G. A. Vilkovisky, Phys. Rept. **119**, 1 (1985)) to compute it.

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The non minimal piece can be written

$$S = \int d^n x \, \Psi^A F_{AB} \Psi^B$$

$$\Psi^A = \begin{pmatrix} h^{\mu\nu} \\ f \\ c' \end{pmatrix}$$

The main idea is to introduce a parameter λ in the non-minimal part of the operator

$$F_{AB}(\nabla|\lambda) = \gamma_{AB}\Box + \lambda J^{\alpha\beta}_{AB}\nabla_\alpha\nabla_\beta + M_{AB} = D_{AB}(\nabla|\lambda) + M_{AB} \qquad 0 \leq \lambda \leq 1$$

so the effective action can be defined as

$$W(\lambda) = W(0) - \frac{1}{2}\int_0^\lambda d\lambda' \, Tr\left[\frac{d\hat{F}(\lambda)}{d\lambda'}\hat{G}(\lambda')\right]$$

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And if we find the inverse of $\hat{F}$ in the sense

$$\hat{F}(\nabla)\hat{K}(\nabla) = \Box^m + \hat{M}(\nabla)$$

we can expand the Green function as a power series in $\hat{M}$

$$\hat{G} = -\hat{K}\sum_{p=0}^{4}(-1)^p \, \hat{M}_p \frac{\mathbb{I}}{\Box^{m(p+1)}} + \ldots$$

so the trace can be easily computed.

By doing this we find(the divergent part of) the off-shell effective action

$$W_\infty = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\left(\frac{119}{90}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \left(\frac{1}{6\alpha^2} - \frac{359}{90}\right)R_{\mu\nu}R^{\mu\nu} + \frac{1}{72}\left(22 - \frac{3}{\alpha^2}\right)R^2\right)$$

Now we can get the on-shell result using the equations of motion of the background field

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = 0$$
$$R_{\mu\nu}R^{\mu\nu} = R^2$$
$$R = \text{constant}$$

and

$$W_4 = E_4 + 2R_{\mu\nu}R^{\mu\nu} - \frac{2}{3}R^2 = E_4 + constant$$

The one loop (on-shell) quantum effective action is then

$$W_\infty^{\text{on-shell}} = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x \left(\frac{119}{90} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{83}{120} R^2\right) =$$
$$= \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x \left(\frac{119}{90} E_4 - \frac{83}{120} R^2\right)$$

This is not dynamical, in contrast to the GR one (Christensen-Duff)

$$W_\infty^{GR} \equiv \frac{1}{16\pi^2(n-4)}\int \sqrt{|g|} d^4x \ \left(\frac{53}{45}\ W_4 - \frac{1142}{135}\Lambda^2\right)$$

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Indeed, as there is no conformal anomaly one can recover from this the result for an arbitrary background metric.

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Thank you

Some details on calculations: BRST

$$s = s_D + s_W \mid s_D^2 = s_W^2 = 0 \;\&\; \{s_D, s_W\} = 0$$
$$s_D g_{\mu\nu} = s_W g_{\mu\nu} = 0$$
$$s_D h_{\mu\nu} = \nabla_\mu c_\nu^T + \nabla_\nu c_\mu^T + c^{T\rho}\nabla_\rho h_{\mu\nu} + \nabla_\mu c^{T\rho} h_{\rho\nu} + \nabla_\nu c^{T\rho} h_{\rho\mu}$$
$$s_W h_{\mu\nu} = 2c\left(g_{\mu\nu} + h_{\mu\nu}\right)$$

The quadratic piece of the unimodular lagrangian

$$\mathscr{L} = \frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{4n}h\Box h + \frac{1}{2}h^{\alpha\beta}h^\mu_\beta R_{\mu\alpha} + \frac{1}{2}h^{\mu\nu}h^{\alpha\beta}R_{\mu\alpha\nu\beta} - \frac{1}{n}hh^{\mu\nu}R_{\mu\nu} -$$
$$-\frac{1}{2n}h^{\mu\nu}h_{\mu\nu}R + \left(-f\Box f + \frac{\alpha}{2}f\Box h + \frac{\alpha}{2}h\Box f\right) - \frac{1}{2}\left(\nabla_\mu c'^{(0,0)}\nabla^\mu c'^{(0,0)} + \right.$$
$$\left. +2\left(\nabla_\nu h^\nu_\mu - \frac{1}{n}\nabla_\mu h\right)\nabla^\mu c'^{(0,0)}\right) + \frac{1}{2n^2}h^2 R$$

TDiff ghosts are transverse $\nabla^\mu c_\mu^T$. We need several fields to fix the BRST

$$h_{\mu\nu}^{(0,0)},\ c_\mu^{(1,1)},\ b_\mu^{(1,-1)},\ f_\mu^{(0,0)},\ \phi^{(0,2)},$$
$$\pi^{(1,-1)},\ \pi'^{(1,1)},\ \bar{c}^{(0,-2)},\ c'^{(0,0)},$$
$$c^{(1,1)},\ b^{(1,-1)},\ f^{(0,0)}$$

TDiff and *Weyl* are fixed independently (technically convenient)

$$S_{gauge-fixing} = \int d^n x\, s\,(X_{TD} + X_W)$$

$$
\begin{aligned}
S^{TDiff}_{BRST} &= \int d^n x\, b^\mu \Big(\Box^2 c^{(1,1)}_\mu - 2R_{\mu\rho}\nabla^\rho\nabla_\nu c^{\nu(1,1)} - \Box R_{\mu\rho} c^{\rho(1,1)} - \\
&-2\nabla_\sigma R_{\mu\rho}\nabla^\sigma c^{\rho(1,1)} - R_{\mu\rho}R^{\rho\nu}c^{(1,1)}_\nu\Big) + \bar{c}^{(0,-2)}\Box\phi^{(0,2)} + \\
&+ \pi^{(1,-1)}\Box^{-1}\pi'^{(1,1)} - \frac{1}{4\rho_1}\Big(F_\mu F^\mu + \nabla_\mu c'^{(0,0)}\nabla^\mu c'^{(0,0)} + 2F_\mu\nabla^\mu c'^{(0,0)}\Big) = \\
&= S_{bc} + S^{bc}_{gf} + S_{\bar{c}\phi} + S_\pi + S_{hc'} \\
S^{Weyl}_{BRST} &= \int d^n x \left[\nabla_\mu f^{(0,0)}\nabla^\mu\Big(f^{(0,0)} - \alpha\, g(h)\Big) - \alpha\nabla_\mu b^{(1,-1)}\nabla^\mu\left(sg(h)\right)\right]
\end{aligned}
$$

Functional traces

The functional traces

$$Tr\left(\mathscr{O}_{\nu_1\nu_2\ldots\nu_j}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_p}\frac{\mathbb{I}}{\Box^n}\right)$$

can be computed by using the heat kernel expansion

$$\exp(-s\hat{F}(\nabla))\delta(x,x') = \frac{1}{(4\pi)^{n/2}}\frac{\mathscr{D}^{1/2}(x,x')}{s^{n/2}}\exp\left(-\frac{\sigma(x,x')}{2s}\right)\hat{\Omega}(s|x,x')$$

and with

$$\hat{\Omega}(s|x,x') = \sum_{n=0}^{\infty} s^n \hat{a}_n(x,x')$$

$$\frac{\mathbb{I}}{\Box^n} = \frac{1}{(n-1)!}\int_0^{\infty} ds\; s^{n-1}\exp\left(-s\hat{\Box}\right)$$

Now the traces can be computed by acting with derivatives on this representation and using the coincidence limits.

Finally it is needed to integrate over s, where only three types of (logarithmic) divergent integrals arise for dimension $n \to 4$

$$\int_0^\infty \frac{ds}{s^{n/2+k}}, \text{ with } \; k = -1, 0, 1$$

and whose pole part can be obtained by integrating by parts, which gives the Laurent series of the result.