B and D mesons on the lattice

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I review the status of lattice calculations of D and B mesons decay constants and semileptonic decay form factors, as well as $B^0 - \bar{B}^0$ mixing parameters. I will mainly discuss those calculations that fully incorporate vacuum polarization effects by considering three flavours of sea quarks.

1. Introduction

The study of B and D meson decays plays a dominant role in testing the Standard Model (SM) and searching for New Physics (NP) effects, giving information which is complementary to direct searches in colliders. The goal of lattice QCD in these studies is providing the theoretical non-perturbative input needed to compare experimental results for the heavy flavour observables with SM predictions.

Lattice QCD is currently a very exciting field due to the recent claims of possible discrepancies between SM expectations and some flavour observables, like the disagreements in the value of the B_s^0 mixing phase reported by the UT collaboration [1], the value of the CKM angle $\sin(2\beta)$ discussed in [2] or the decay constant f_{D_s}. Improving the lattice calculations from which these analyses are taking inputs is crucial in order to resolve whether the disagreements are indicating the presence of NP effects.

Lattice QCD is a non-perturbative formulation of QCD based only on first principles. It also provides a quantitative calculation methodology, which has become a precise tool capable of providing some of the accurate determinations needed by phenomenology. Accuracy in lattice calculations requires control over all the sources of systematic error. In particular, it is essential to take into account vacuum polarization effects in a realistic way, i.e., including up, down and strange sea quarks on the gauge configurations' generation. The up and down quarks are usually taken to be degenerate, so those simulations are referred to as $N_f = 2+1$. The vacuum polarization effects were almost always neglected in old lattice calculations due to limited computational power. This is known as the quenched approximation and introduces an uncontrolled and irreducible error, which can be as large as 10-30% [3]. Simulations with $N_f = 2$ sea quarks are still missing part of the vacuum polarization effects and the associated systematic error is hard to estimate without repeating the calculation with $N_f = 2+1$ sea quarks.

Another important source of systematic error is associated with the fact that current simulations are unable to simulate up and down quarks as light as the physical ones. The way of connecting lattice results to the physical world in a model independent way is by extrapolating those results using the guide of Chiral Perturbation Theory (ChPT). In order for this extrapolation to have controlled errors and be realistic, simulations must be performed for a range of light sea quark masses smaller than $m_s/2$.

Other systematic errors that must be included in any realistic lattice analysis are discretization, finite volume and renormalization effects. Discretization effects can be estimated by power counting, but this estimate must be explicitly tested by performing the calculation at several values of the lattice spacing. Finite volume effects can be estimated by repeating the calculation at several volumes and/or using ChPT techniques.

Between the quantities relevant for the experimental heavy flavour physics program that can be obtained with lattice QCD techniques in a accurate way, are B and D decay constants and semileptonic decays form factors, and the parameters that describe $B^0 - \bar{B}^0$ mixing and short-distance contributions to $D^0 - \bar{D}^0$ mixing. In the next Sections I will discuss the latest results for these non-perturbative quantities from lattice calculations with all sources of systematic error addressed. Among other things, that means that I will focus on simulations with $N_f = 2 + 1$ sea quarks. A description of the different fermion formulations employed in those calculations, together with the relevant references can be found in [4]. More details about the analyses discussed in this paper and a more complete list of calculations are given in [5].

2. Leptonic decays

2.1. Charmed meson decay constants: the f_{D_s} puzzle

The lattice determination of pseudoscalar decay constants, together with experimental measurements of pseudoscalar leptonic decay widths, can be used to extract the value of the CKM matrix elements involved in the process. For decay constants which are well determined experimentally, those for which the CKM matrix elements involved are known which a good precision and experimental measurements are accurate, the comparison with lattice calculations can be used as a test of the theory. That is true for the decay constants in the charm sector, which were believed to provide a good test of lattice QCD techniques.

For the decays constants f_{D^+} and f_{D_s} there are lattice results available from two groups with $N_f = 2 + 1$ sea quarks, the FNAL/MILC [6] and the HPQCD [7] collaborations, to compare against experiment. Both use configurations generated by the MILC collaboration for three different values of the lattice spacing, $a = 0.15\ fm$, $a = 0.12\ fm$ and $a = 0.9\ fm$. The main difference between the two collaborations is the treatment of the valence quarks. While HPQCD uses an improved staggered action, the HISQ action [8], for all the valence quarks, FNAL/MILC uses another improved staggered action, the Asqtad action [9], for the light quarks (up, down and strange) and the Fermilab action [10] for the charm quark. The HPQCD collaboration has partially conserved currents, so they can extract the value of the decay constant without any renormalization. The FNAL/MILC collaboration needs to renormalize its currents, but they do it in a partially non-perturbative way that generates very small errors, around 1.5%.

The values for f_{D^+}, f_{D_s} and the ratio of both quantities from these two analyses, together with the new CLEO-c experimental results in [11,12] and Belle results for f_{D_s} in [13], are collected in Table 1.

Group	f_{D^+}(MeV)	f_{D_s}(Mev)
experiment	(*CLEO-c*) 205.8(8.9)	(*average*) 269.6(8.3)
HPQCD	207(4)	241(3)
FNAL/MILC	207(11)	249(11)

Table 1
Comparison of f_{D^+} and f_{D_s} as obtained from experiment [11–13] and from unquenched lattice QCD calculations [7,6].

Both lattice collaborations agree very well in the central values obtained for the decay constants. The errors of the HPQCD calculation are smaller than those for the FNAL/MILC due to the fact that the HISQ action is more improved[1] than the Fermilab action and thus discretization errors are sensibly reduced.

Calculations of these decay constants using twisted mass fermions are also making progress, although they are still restricted to $N_f = 2$ simulations. The ETM collaboration has reported preliminary results from this kind of simulations in [14]. They determine the ratios f_{D^+}/f_π and f_{D_s}/f_K that can be calculated more accurately and need smoother chiral extrapolations than the denominators and numerators alone. The numbers presented, $f_D = (197 \pm 14)$MeV and $f_{D_s} = (244 \pm 12)$MeV, are compatible with the $N_f = 2 + 1$ calculations but errors are still preliminary and the extrapolation to the continuum is very large. The effect of the strange sea

[1] Improvement in this context refers to the addition of higher-dimensional operators to the action.

quark must be also incorporated. A value of $f_{D^+} = 201 \pm 22^{+4}_{-9}$ based on a single value of the lattice spacing and using $N_f = 2$ Wilson fermions was obtained in [15].

While the $N_f = 2+1$ and $N_f = 2$ lattice calculations of f_{D^+} agree very well with experiment, there is a clear tendency of the lattice calculations of f_{D_s} to give smaller values than the experimental numbers. In particular, the result from the HPQCD collaboration, which is the most accurate one, disagree by more than 3σ from experiment. The fact that the rest of quantities calculated by the HPQCD calculation with the same actions, configurations, input parameters, etc (f_D, f_K, f_π, m_D, m_{D_s}, $\frac{2m_{D_s}-m_{\eta_c}}{2m_D-m_{\eta_c}}$, ...) agree with experiment at the 2% level, strongly supports the reliability of their results for f_{D_s}, although further confirmation from other collaborations would increase the confidence in the correctness of the lattice techniques used. This discrepancy has been recently suggested to be a hint of beyond the Standard Model effects [16]. More work is needed to resolve this issue. From the experimental side, it is desirable that certain issues like the assumption of three-generation CKM unitarity to set $V_{cs} = V_{ud}$ or the inclusion of radiative corrections from experimental data or Monte Carlo simulations are addressed. From the theory side, the error on the FNAL/MILC result for f_{D_s} is currently larger by roughly a factor of four than the error on the HPQCD result. The FNAL/MILC collaboration plans to reduce the uncertainties in their calculation in the near future by increasing statistics, using a smaller lattice spacing, and improving the determination of the inputs needed. This will constitute an important check of the HPQCD numbers.

2.2. B and B_s decay constants

Lattice results for the decay constants in the B sector are needed more than in the D sector since the corresponding CKM matrix elements to extract the information from experiment are not as well known. The value of the B decay constants are used in the SM predictions for processes very sensitive to beyond SM effects, such as $B_s \to \nu^+\nu^-$. The purely leptonic decays themselves would also be a sensitive probe of effects from charged Higgs bosons with eventual improvements in the experimental measurements.

The way of calculating these decay constants on the lattice is the same as for the charm decay constants. In fact, the FNAL/MILC collaboration has also calculated f_B, f_{B_s} and the ratio of both with the same choice of actions, same ensembles and same procedure as for the D decay constants in [6]. The errors in both the charm and bottom mesons are thus very similar. The results are listed in Table 2. In that table, the results from the other lattice $N_f = 2+1$ calculation, by the HPQCD collaboration [17], are also included. In this case the HPQCD collaboration used a NRQCD action (non-relativistic QCD) [18] to describe the b valence quark. Errors are then significantly larger than in their analysis of D^+ and D_s decay constants where they use the HISQ action for the c quark. A dominant source of uncertainty in their NRQCD calculation is the error associated with the one-loop renormalization applied.

Group	f_B(Mev)	f_{B_s}(MeV)	f_{B_s}/f_B
FNAL/MILC	195(11)	243(11)	1.25(4)
HPQCD	216(22)	260(26)	1.20(3)

Table 2
Values of f_B and f_{B_s} as obtained from the two lattice QCD calculations with $N_f = 2+1$ [6,17].

A suggested way of reducing the uncertainty in the calculation of f_{B_s}/f_B is by extracting it from the double ratio $[f_{B_s}/f_B]/[f_K/f_\pi]$ which can be calculated very accurately since it is very close to one in ChPT.

3. Semileptonic decays

Semileptonic decays can be used to extract CKM matrix elements like $|V_{cb}|$, $|V_{ub}|$, $|V_{cd}|$, $|V_{cs}|$ and $|V_{us}|$. The theory input needed to get those parameters from experimentally measured semileptonic widths are the form factors in terms of which the hadronic matrix elements involved on those decays are parametrized. For example, for the decay $D \to Kl\nu$, the differential decay rate is given by

$$\frac{d\Gamma}{dq^2} = (\text{known factors})|V_{cs}|^2 f_+^2(q^2)\,, \tag{1}$$

where $f_+(q^2)$ is the vector form factor, which can be extracted from the matrix element of the vector current

$$\langle K|V^\mu|D\rangle = f_+(q^2)(p_D+p_K-\Delta)^\mu + f_0(q^2)\Delta^\mu\,, \tag{2}$$

with $\Delta^\mu = (m_D^2 - m_K^2)q^\mu/q^2$.

Lattice QCD can be used to calculate the value of those form factors as a function of the virtual W momentum transfer, q^2, or, equivalently, the recoil momentum of the daughter meson. On the lattice, the smallest discretization errors correspond to the form factor at the largest momentum transfer, where the experimental data are less precise. In addition, the finite volume provides an infrared cutoff and there is a finite minimum value for the momenta that can be simulated. A way of circumventing this limitation is by using twisted boundary conditions that allow for arbitrary small values of the momenta [19]. Another set of important techniques that can be applied to semileptonic decay analysis are double ratio methods [20]. These methods can yield a reduction of both statistical and systematic uncertainties by a partial or total cancellation of those uncertainties between numerator and denominator in the ratios.

The matrix element $|V_{cb}|$ can be extracted from the decay $B \to D^* l\nu$. The experimental results for this process at zero recoil have smaller errors than those for $B \to Dl\nu$. A new lattice calculation of the form factor describing the decay, the axial vector form factor at zero recoil $\mathcal{F}_{\mathcal{B}\to\mathcal{D}^*}(1)$, was presented in the lattice conference last year [21]. This calculation eliminates the quenching errors from previous calculations since it includes $N_f = 2+1$ sea quarks. It also introduces a new double ratio method which gives the form factor at zero recoil directly. The relation between the double ratio and the form factor,

$$|\mathcal{F}_{\mathcal{B}\to\mathcal{D}^*}(1)|^2 = \frac{\langle D^*|\bar{c}\gamma_j\gamma_5 b|\bar{B}\rangle\langle\bar{B}|\bar{b}\gamma_j\gamma_5 c|D^*\rangle}{\langle D^*|\bar{c}\gamma_4 c|D^*\rangle\langle\bar{B}|\bar{b}\gamma_4 b|\bar{B}\rangle}\,, \tag{3}$$

is exact to all orders in the heavy-quark expansion in the continuum. Statistical errors in the numerator and denominator are highly correlated and largely cancel. Most of the renormalization also cancels, yielding a small uncertainty for the perturbative matching.

The final result obtained for the form factor after chiral and continuum extrapolation is $\mathcal{F}_{B\to D^*}(1) = 0.921 \pm 0.013 \pm 0.021$ [22], where the first error is statistical and the second one includes all sources of systematic errors and is dominated by heavy-quark discretization errors. The CKM matrix element $|V_{cb}|$ extracted from this value of the form factor and the experimental averages in [23] is

$$|V_{cb}| = (38.8 \pm 0.6 \pm 1.0) \times 10^{-3}\,. \tag{4}$$

This value differs by 2σ from the one extracted from inclusive decays [23].

An alternative method for the extraction of $|V_{cb}|$ is the analysis of the decay $B \to Dl\nu$. A study of the form factors needed for such determination with quenched Wilson fermions was presented in [24]. An interesting aspect of that analysis is that the authors calculate the scalar form factor as well as the momentum transfer dependence to avoid needing to extrapolate to zero recoil, where the experimental data suffer from phase space suppression compared to the $B \to D^* l\nu$ case. The scalar form factor, which only contributes for $l = \tau$, can be used to constrain BSM physics [25]. The same authors also calculated the form factors needed for the description of $B \to D^* l\nu$ for different values of the momentum transfer in [26]. The result they obtain for $|V_{cb}|$ is consistent with the one in (4), although it is extracted from quenched simulations and then vacuum polarization effects are missing.

The decay $B \to \pi l\nu$ provides a way of extracting $|V_{ub}|$ that is competitive with $b \to u$ inclusive decays. The main problem in studying this exclusive channel with lattice techniques is the poor overlap in q^2 (the momentum transfer) between experimental and lattice data, which inflates the final error. A way of overcoming this problem is using a model independent parametrization of the shape of the form factor, which allows a direct comparison of experimental and lattice data even with a poor q^2 overlap. This is the approach that is being followed by the FNAL/MILC collaboration [27]. The authors in [27] used the so called $z-$expansion, which is a model independent parametrization based only on unitarity and analyticity [28]. This $N_f = 2+1$ calculation uses the Fermilab action for the b quark and an improved staggered formalism, the Asqtad action, to describe the light quark in the numerical simulations for this work. The preliminary results obtained by this collaboration are shown in Figure 1.

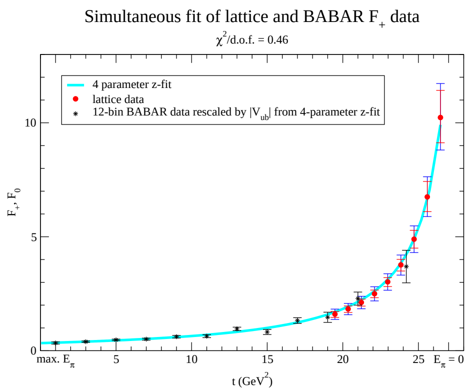

Figure 1. Preliminary results in [27] for the two form factors describing the decay $B \to \pi l \nu$ as obtained from a simultaneous fit to lattice and experimental data.

The value obtained from this simultaneous fit of experimental and lattice data via the z-expansion is [27]

$$|V_{ub}| = (2.94 \pm 0.35) \times 10^{-3}\,, \tag{5}$$

consistent with the global fits of the CKM matrix, but 2σ away from the inclusive determination [23].

The authors in [29] have proposed a new model independent parametrization for form factors, which satisfies unitarity, analyticity, and perturbative scaling. They use existing lattice data from the HPQCD and FNAL/MILC collaborations ($N_f = 2+1$ simulations), experimental results from CLEO, Belle, and BaBar, and cone sum rules results to test their methodology.

The calculation of semileptonic and leptonic decays on the lattice can be used to construct ratios independent of CKM matrix elements, such as $\frac{\Gamma(D\to l\nu)}{\Gamma(D\to\pi l\nu)}$ or $\frac{\Gamma(D_s\to l\nu)}{\Gamma(D\to K l\nu)}$, with which one could test the consistency of lattice calculations against experiment or constrain BSM physics. These are being studied by several lattice collaborations. References to preliminary results from $N_f = 2$ simulations can be found in [5].

4. $B^0 - \bar{B}^0$ mixing

The mixing in the $B^0_q - \bar{B}^0_q$ system is an interesting place to look for NP effects. The BSM effects can appear as new tree level contributions, or through the presence of new particles in the box diagrams. In fact, it has been recently claimed that there is a disagreement between direct experimental measurement of the phase of B^0_s mixing amplitude and the SM prediction [1]. Possible NP effects have also been reported to show up in the comparison between direct experimental measurements of $\sin(2\beta)$ and SM predictions using B^0 mixing parameters [2]. Studies of neutral B meson mixing parameters can also impose important constraints on different NP scenarios [30].

In the SM, the $B^0 - \bar{B}^0$ mixing is due to box diagrams with exchange of two W-bosons. These box diagrams can be rewritten in terms of an effective Hamiltonian with four-fermion operators describing processes with $\Delta B = 2$. The matrix elements of the operators between the neutral meson and antimeson encode the non-perturbative information on the mixing and can be calculated using lattice QCD techniques. Those matrix elements are parametrized by products of B decay constants and bag parameters, and provide the value of quantities experimentally measurable as the mass differences, $\Delta M_{s,d}$, and decay width differences, $\Delta\Gamma_{s,d}$, between the heavy and light B^0_s and B^0_d mass eigenstates. Theoretically, for example, the mass difference is given by

$$\Delta M_{s(d)}|_{theor.} \propto |V^*_{ts(d)} V_{tb}|^2 f^2_{B_{s(d)}} \hat{B}_{B_{s(d)}}\,, \tag{6}$$

with $\langle \bar{B}^0_s | Q_L^{s(d)} | B^0_s \rangle = \frac{8}{3} M^2_{B_{s(d)}} f^2_{B_{s(d)}} B_{B_{s(d)}}(\mu)$ and $O_L^{s(d)} = \left[\bar{b}^i \gamma_\mu (1-\gamma_5) s^i (d^i)\right] \left[\bar{b}^j \gamma^\mu (1-\gamma_5) s^j (d^j)\right]$.

Many of the uncertainties that affect the theoretical calculation of the decay constants and bag parameters cancel totally or partially if one takes the ratio $\xi^2 = f_{B_s}^2 B_{B_s} / f_{B_d}^2 B_{B_d}$. Hence, this ratio and therefore the combination of CKM matrix elements related to it, $|\frac{V_{td}}{V_{ts}}|$, can be determined with a significantly smaller error than the individual matrix elements. The ratio ξ is also an important ingredient in the unitarity triangle analyses.

The first lattice calculation of the B^0 mixing parameters with $N_f = 2+1$ sea quarks, which only studied the B_s^0 sector, was performed by the HPQCD collaboration in [31]. The authors obtained

$$\Delta M_s = 20.3(3.0)(0.8) ps^{-1} \quad \text{and} \qquad (7)$$
$$\Delta\Gamma_s = 0.10(3) ps^{-1} ,$$

which is compatible with experiment.

The FNAL/MILC and HPQCD collaborations are currently working on a more complete study of $B^0 - \bar{B}^0$ mixing, including B_s^0 and B_d^0 parameters. The main goal of both projects is obtaining the ratio ξ fully incorporating vacuum polarization effects. The choice of actions and the setup is the same as for their f_B, f_{B_s} calculations described in Section 2.1 -more details can be found in [32].

Results from the two collaborations are still preliminary. Figures 2 and 3 show some examples of the values of $f_B\sqrt{M_B B_B}$ obtained as function of the light sea quark masses.

The dependency on the light sea quark mass is in both studies very mild, so only the chiral extrapolation in the d quark mass for B_d^0 parameters is expected to be a significant source of error. In Figure 2, it can also be appreciated that the results for the two different lattice spacings are very similar, which indicates small discretization errors.

A comparison of the preliminary results from the two collaborations for the ratio ξ is shown in Figure 4.

Only the full QCD points[2] are included. The results of the two collaborations agree within statistical errors. This is very encouraging since both analyses use completely different descriptions for the heavy quarks.

The final step in those calculations is extrapolating the results to the physical values of the light quark masses and, in the case of the FNAL/MILC collaboration, performing simultaneously the extrapolation to the continuum. Systematic errors on those extrapolations are currently being studied. These two analyses are expected to have final results very soon with total errors ranging $5-7\%$ for $f_{B_q}\sqrt{B_{B_q}}$ and $2-3\%$ for ξ.

The effects of heavy new particles in the box diagrams that describe the B^0 mixing can be seen in the form of effective operators built with SM degrees of freedom. The NP could modify the Wilson coefficients of the four-fermion operators that already contribute to B^0 mixing in the SM and gives rise to new four-fermion operators in the $\Delta B = 2$ effective Hamiltonian -see [33,34] for a list of the possible operators in the SUSY basis. The calculation of those Wilson coefficients for a particular BSM theory, together with the lattice calculation of the matrix elements of all the possible four-fermion operators in the SM and beyond and experimental measurements of B^0 mixing parameters, can constraint BSM parameters and help to understand new physics. To date,

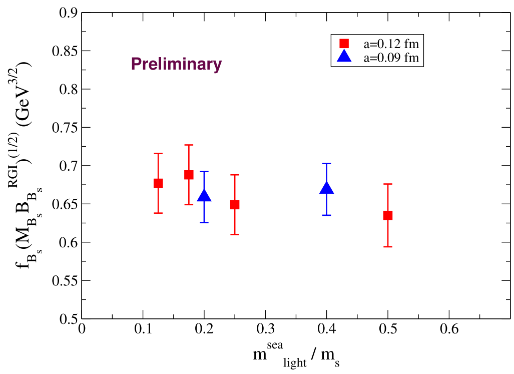

Figure 2. Values of $f_{B_s}\sqrt{M_{B_s}\hat{B}_{B_s}}$ in GeV$^{3/2}$ as a function of the light sea quark mass normalized to the physical strange quark mass from the HPQCD collaboration. The data include statistical, perturbative and scale errors. The bottom valence quark is fixed to its physical value and the strange valence quark is very close to its physical value. The strange sea quark mass is also very close to its physical value.

[2]Where valence and sea quark masses are the same.

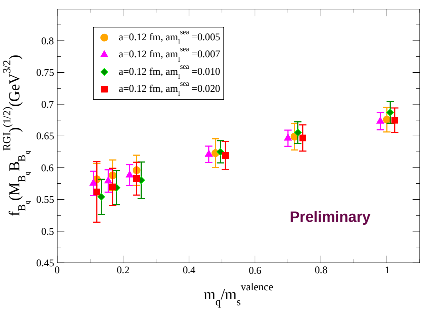

Figure 3. Bare values of $f_{B_q}\sqrt{M_{B_q}B_{B_q}}$ in lattice units as a function of the light valence quark mass m_q normalized to the value of the physical strange quark mass from the FNAL/MILC collaboration. The results correspond to one of the three lattice spacings at which the FNAL/MILC's study is performed. The bottom valence quark is fixed to its physical value and the strange valence quark is very close to its physical value. The strange sea quark mass is also very close to its physical value.

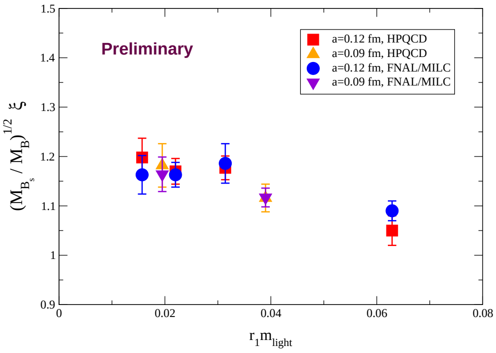

Figure 4. Product of the ratios ξ and M_{B_s}/M_{B_d} as a function of the down quark mass normalized to the strange quark mass. Results for both FNAL/MILC and HPQCD collaborations including only statistical errors are shown for two different values of the lattice spacing a.

there does not exist an unquenched determination of the complete set of matrix elements of four-fermion operators in that general $\Delta B = 2$ effective Hamiltonian. However, the two collaborations currently working on B^0 mixing in the SM are planning to extend their analysis to BSM operators in the near future. Actually, the HPQCD collaboration has already calculated the one-loop matching coefficients needed for such an analysis [35].

5. Conclusions

Hints of discrepancies between SM predictions and experimental measurements have started to show up in some CP violating observables [36]. The precise determination of the CKM matrix elements V_{ub} and $|V_{cb}|$, and the accurate calculation of parameters like ξ, decay constants, ..., involved in those analysis, is crucial in order to resolve the origin of those disagreements and fully exploit the potential of the CP violating observables on constraining NP.

There has recently been important progress in order to achieve realistic lattice calculations, with $N_f = 2 + 1$ sea quarks and a serious study of systematic errors. New results relevant for phenomenology have appeared in the last year in the D meson sector with errors at the few percent level. In the near future, results for B^0 mixing parameters and B decay constants will be also available with errors at the few percent level.

Several lattice collaborations are currently producing accurate $N_f = 2+1$ results as discussed in this paper, and other collaborations are starting to generate $N_f = 2 + 1$ ensembles using different sea quark actions. This will allow an important consistency check of lattice methods.

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