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Symmetrical approach to three-body phase space

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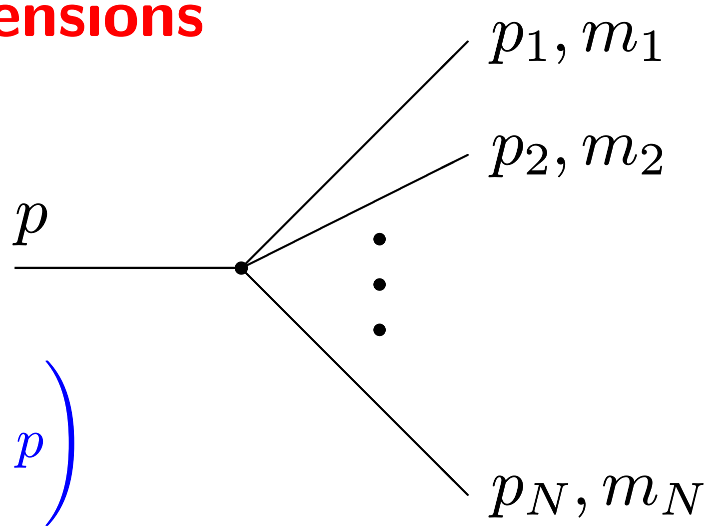
based on work with

R. Delbourgo

N-particle phase space in D dimensions



Definition:

$$I_N^{(D)}(p, m_1, \ldots, m_N)$$
$$= \int \cdots \int \left\{ \prod_{i=1}^{N} \mathrm{d}^D p_i \; \delta(p_i^2 - m_i^2) \; \theta(p_i^0) \right\} \delta\left(\sum_{i=1}^{N} p_i - p \right)$$

where p is the total momentum (we will assume that $p^2 > 0$).

For kinematical reasons, it is clear that the results for the integrals $I_N^{(D)}$ have no physical meaning if the absolute value of the momentum p is less than the sum of the masses. Therefore, in what follows we will imply that all results for $I_N^{(D)}$ are accompanied by

$$\theta\{p^2 - (m_1 + \ldots + m_N)^2\}$$

without writing this theta function explicitly.

[B. Almgren, Arkiv för Physik **38** (1968) 161]

Recurrence relation

Recurrence relation:

$$I_N^{(D)}(p, m_1, \ldots, m_N) = \int \mathrm{d}s \; I_{R+1}^{(D)}(p, \sqrt{s}, m_{N-R+1}, \ldots, m_N) \; I_{N-R}^{(D)}(\sqrt{s}, m_1, \ldots, m_{N-R})$$

Taking into account the theta functions associated with $I_{N-R}^{(D)}$ and $I_{R+1}^{(D)}$, one can see that the actual limits of the integration variable s in this recurrence relation extend from $\left(\sum_{i=1}^{N-R} m_i\right)^2$ to $\left(p - \sum_{i=N-R+1}^{N} m_i\right)^2$.

For example,

$$I_3^{(D)}(p, m_1, m_2, m_3) = \int\limits_{(m_1+m_2)^2}^{(p-m_3)^2} \mathrm{d}s \; I_2^{(D)}(p, \sqrt{s}, m_3) \; I_2^{(D)}(\sqrt{s}, m_1, m_2)$$

[B. Almgren, Arkiv för Physik **38** (1968) 161]

[R. Hagedorn, Nuovo Cim. **25** (1962) 1017]

[E. Byckling, K. Kajantie, Nucl. Phys. **B9** (1969) 568]

Two-particle phase space ($N = 2$)

For $N = 2$ and $D = 4$, the phase-space integral can be easily evaluated as

$$I_2 = \frac{\pi}{2p^2}\sqrt{\lambda(p^2, m_1^2, m_2^2)}\ ,$$

where

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

is nothing but the well-known Källen function.

In D dimensions,

$$I_2^{(D)} = \frac{\pi^{(D-1)/2}}{(2p)^{D-2}\,\Gamma\left(\frac{D-1}{2}\right)}\left[\lambda(p^2, m_1^2, m_2^2)\right]^{(D-3)/2}$$

[R. Delbourgo and M.L. Roberts, J. Phys. **A36** (2003) 1719]

Integral representation for three-particle phase space ($N = 3$)

Using recurrence relation and the result for two-body phase space, one can obtain the following non-symmetric integral representation (for $D = 4$):

$$I_3 = \frac{\pi^2}{4p^2} \int\limits_{s_2}^{s_3} \frac{\mathrm{d}s}{s} \sqrt{(s-s_1)(s-s_2)(s_3-s)(s_4-s)} \; ,$$

with

$$s_1 = (m_1 - m_2)^2, \quad s_2 = (m_1 + m_2)^2, \quad s_3 = (p - m_3)^2, \quad s_4 = (p + m_3)^2,$$

so that $s_1 \le s_2 \le s_3 \le s_4$.

In D dimensions,

$$I_3^{(D)} = \frac{\pi^{D-1}}{(4p)^{D-2}\Gamma^2\left(\frac{D-1}{2}\right)} \int\limits_{s_2}^{s_3} \frac{\mathrm{d}s}{s^{D/2-1}} \; \left[(s-s_1)(s-s_2)(s_3-s)(s_4-s)\right]^{(D-3)/2}$$

[A. Bashir, R. Delbourgo, M.L. Roberts, J. Math. Phys. **42** (2001) 5553]

Explicit non-symmetric result for three-particle phase space

The result (for $D=4$) can be expressed in terms of the elliptic integrals

$$
\begin{aligned}
I_3 \;=\;& \frac{\pi^2}{4p^2\sqrt{Q_+}}\Biggl\{\frac{1}{2}Q_+(m_1^2+m_2^2+m_3^2+p^2)E(k) \\
&+4m_1m_2\left[(p-m_3)^2-(m_1-m_2)^2\right]\left[(p+m_3)^2-m_3p+m_1m_2\right]K(k) \\
&+8m_1m_2\left[(m_1^2+m_2^2)(p^2+m_3^2)-2m_1^2m_2^2-2m_3^2p^2\right]\Pi\left(\alpha_1^2,k\right) \\
&-8m_1m_2(p^2-m_3^2)^2\,\Pi\left(\alpha_2^2,k\right)\Biggr\},
\end{aligned}
$$

with

$$
\begin{aligned}
Q_+ &\equiv (p+m_1+m_2+m_3)(p+m_1-m_2-m_3)(p-m_1+m_2-m_3)(p-m_1-m_2+m_3)\,, \\
Q_- &\equiv (p-m_1-m_2-m_3)(p-m_1+m_2+m_3)(p+m_1-m_2+m_3)(p+m_1+m_2-m_3)\,, \\
k &\equiv \sqrt{\frac{Q_-}{Q_+}}\,, \quad \alpha_1^2=\frac{(p-m_3)^2-(m_1+m_2)^2}{(p-m_3)^2-(m_1-m_2)^2}\,, \quad \alpha_2^2=\frac{(m_1-m_2)^2}{(m_1+m_2)^2}\alpha_1^2\,.
\end{aligned}
$$

[B. Almgren, Arkiv för Physik **38** (1968) 161]

[S. Bauberger, F. Berends, M. Böhm, M. Buza, Nucl. Phys. **B434** (1995) 383]

Special case of equal masses and other comments

For equal masses, $m_1 = m_2 = m_3 \equiv m$ we get

$$I_{3,\text{eq}} = \frac{\pi^2}{4p^2}\sqrt{(p-m)(p+3m)}\left\{\frac{1}{2}(p-m)(p^2+3m^2)E(k_{\text{eq}}) - 4m^2 pK(k_{\text{eq}})\right\},$$

with

$$k_{\text{eq}} = \sqrt{\frac{(p+m)^3(p-3m)}{(p-m)^3(p+3m)}}\,.$$

Coming back to the general unequal masses, we note that the product of Q_+ and Q_- produces the quantity

$$\begin{aligned} D_{123} \quad \equiv \quad Q_+Q_- = & \left[p^2-(m_1+m_2+m_3)^2\right]\left[p^2-(-m_1+m_2+m_3)^2\right] \\ & \times\left[p^2-(m_1-m_2+m_3)^2\right]\left[p^2-(m_1+m_2-m_3)^2\right] \end{aligned}$$

that occurs in recurrence relations for the sunset diagram.

[O.V. Tarasov, Nucl. Phys. **B502** (1997) 455]

[A.I. Davydychev, V.A. Smirnov, Nucl. Phys. **B554** (1999) 391]

Elliptic integrals

The normal elliptic integrals of the first and second kind are defined as

$$F(\varphi,k) \;=\; \int\limits_0^{\sin\varphi} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int\limits_0^{\varphi} \frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}}$$

$$E(\varphi,k) \;=\; \int\limits_0^{\sin\varphi} \mathrm{d}t\, \sqrt{\frac{1-k^2t^2}{1-t^2}} = \int\limits_0^{\varphi} \mathrm{d}\psi\, \sqrt{1-k^2\sin^2\psi}\;.$$

At $\varphi=\pi/2$ we get the complete elliptic integrals of the first and second kind,

$$K(k) \;=\; F\left(\tfrac{\pi}{2},k\right) = \int\limits_0^{1} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2}\; {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \right| k^2 \right)\;,$$

$$E(k) \;=\; E\left(\tfrac{\pi}{2},k\right) = \int\limits_0^{1} \mathrm{d}t\, \sqrt{\frac{1-k^2t^2}{1-t^2}} = \frac{\pi}{2}\; {}_2F_1\left(\left. \begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \right| k^2 \right)$$

Elliptic integrals (continued)

Complete elliptic integral of the third kind:

$$\Pi(c,k) \;=\; \int\limits_0^1 \frac{\mathrm{d}t}{(1-ct^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2}\, F_1\left(\tfrac{1}{2};\; 1,\; \tfrac{1}{2};\; 1 \middle|\; c,\; k^2\right),$$

where F_1 is the Appell hypergeometric function of two arguments.

The Jacobian zeta function, $Z(\beta,k)$, is defined through

$$K(k)\; Z(\beta,k) = K(k)\; E(\beta,k) - E(k)\; F(\beta,k).$$

To represent the elliptic functions $\Pi(\alpha_i^2,k)$ in terms of Z functions, we can use

$$\Pi(\alpha_i^2,k) = K(k) + \frac{\alpha_i\; K(k)\; Z(\beta_i,k)}{\sqrt{(1-\alpha_i^2)(k^2-\alpha_i^2)}},$$

with $\beta_i = \arcsin(\alpha_i/k)$.

N-particle phase space in D dimensions (continued)

$$I_N^{(D)}(p, m_1, \ldots, m_N) = \int \cdots \int \left\{ \prod_{i=1}^{N} \mathrm{d}^D p_i \; \delta(p_i^2 - m_i^2) \; \theta(p_i^0) \right\} \; \delta\left(\sum_{i=1}^{N} p_i - p \right) \; ,$$

The D-dimensional vector p can be presented as $(p^0, \mathbf{p})$, where $\mathbf{p}$ is the $(D-1)$-dimensional Euclidean vector of space components.
Without loss of generality, we can work in the center-of-mass frame, $p = (p^0, \mathbf{0})$.

Trick with the δ-function (in the center-of-mass frame, $(px) = p^0 x^0$):

$$\delta\left(\sum_{i=1}^{N} p_i - p \right) = \frac{1}{(2\pi)^D} \int \mathrm{d}^D x \; \exp\left\{ \mathrm{i} \sum_{i=1}^{N} (p_i x) - \mathrm{i}(px) \right\} \; ,$$

[B.A. Arbuzov, E.E. Boos, S.S. Kurennoy, K.Sh. Turashvili, Yad. Fiz. **44** (1986) 1565]

In this way, we get

$$I_N^{(D)} = \frac{1}{(2\pi)^D} \int \mathrm{d}^D x \; e^{-\mathrm{i} p^0 x^0} \left\{ \prod_{i=1}^{N} \int \mathrm{d}^D p_i \; \delta(p_i^2 - m_i^2) \; \theta(p_i^0) \; e^{\mathrm{i}(p_i x)} \right\} \; .$$

N-particle phase space in D dimensions (continued)

Integrating over $(D-1)$-dimensional angles of $\mathbf{p}_i$ we get

$$\int \mathrm{d}^D p_i \; \delta(p_i^2 - m_i^2)\; \theta(p_i^0)\; e^{\mathrm{i}(p_i x)} = \frac{(2\pi)^{(D-1)/2}}{2\xi^{(D-3)/2}} \int\limits_0^\infty \frac{\rho_i^{(D-1)/2}\mathrm{d}\rho_i}{\sqrt{\rho_i^2+m_i^2}}\; J_{(D-3)/2}(\rho_i \xi)\; e^{\mathrm{i}x^0\sqrt{\rho_i^2+m_i^2}}$$

with $\rho_i \equiv |\mathbf{p}_i|$ and $\xi \equiv |\mathbf{x}|$.

At $D=4$ the Bessel function reduces to an elementary function,

$$J_{1/2}(\rho_i \xi) = \sqrt{\frac{2}{\pi \rho_i \xi}}\; \sin(\rho_i \xi)$$

Note an analogy with the calculation of Feynman integrals in the coordinate space, when each massive propagator yields a (modified) Bessel function.

[E. Mendels, Nuovo Cim. **A45** (1978) 87]

[S. Groote, J.G. Körner and A.A. Pivovarov, Nucl. Phys. **B542** (1999) 515]

Two-particle phase space in D dimensions

For $N=2$ the integration over ξ gives (we denote $\nu=(D-3)/2$)

$$\int\limits_0^\infty \xi \mathrm{d}\xi \; J_\nu(\rho_1\xi)\; J_\nu(\rho_2\xi) = 2\delta(\rho_1^2-\rho_2^2) \; ,$$

so that we can put $\rho_1=\rho_2\equiv\rho$, whereas the integration over x^0 yields

$$\delta\left(p-\sqrt{\rho^2+m_1^2}-\sqrt{\rho^2+m_2^2}\right)$$

in the center-of-mass frame. The resulting integral

$$I_2^{(D)} = \frac{\pi^{(D-1)/2}}{2\Gamma\left(\frac{D-1}{2}\right)} \int\limits_0^\infty \frac{\rho^{D-2}\;\mathrm{d}\rho}{\sqrt{\rho^2+m_1^2}\sqrt{\rho^2+m_2^2}}\;\delta\left(p-\sqrt{\rho^2+m_1^2}-\sqrt{\rho^2+m_2^2}\right)$$

can be easily evaluated, yielding the known result

$$I_2^{(D)} = \frac{\pi^{(D-1)/2}}{(2p)^{D-2}\;\Gamma\left(\frac{D-1}{2}\right)} \left[\lambda(p^2,m_1^2,m_2^2)\right]^{(D-3)/2}$$

Three-particle phase space in D dimensions

Here and below we follow
[A. I. Davydychev, R. Delbourgo, J. Phys. **A37** (2004) 4871]

For the three-particle phase-space integral we get

$$I_3^{(D)} = \frac{2^{(D-7)/2}\pi^{D-2}}{\Gamma\left(\frac{D-1}{2}\right)} \int\limits_0^\infty \frac{\mathrm{d}\xi}{\xi^{(D-5)/2}} \int\limits_{-\infty}^\infty \mathrm{d}x^0 \; e^{-\mathrm{i}p^0x^0} \prod_{i=1}^3 \int\limits_0^\infty \frac{\rho_i^{(D-1)/2}\mathrm{d}\rho_i}{\sqrt{\rho_i^2+m_i^2}} \; J_{(D-3)/2}(\rho_i\xi)\; e^{\mathrm{i}x^0\sqrt{\rho_i^2+m_i^2}}$$

Here we can integrate over ξ, using

$$\int\limits_0^\infty \frac{\mathrm{d}\xi}{\xi^{\nu-1}}\; J_\nu(\rho_1\xi)\; J_\nu(\rho_2\xi)\; J_\nu(\rho_3\xi) = \frac{2\theta\{-\lambda(\rho_1^2,\rho_2^2,\rho_3^2)\}[-\lambda(\rho_1^2,\rho_2^2,\rho_3^2)]^{\nu-1/2}}{\pi^{1/2}\Gamma\left(\nu+\frac{1}{2}\right)(8\rho_1\rho_2\rho_3)^\nu}$$

(with $\nu=(D-3)/2$), where λ is the Källen function. In our case, when all $\rho_i \geq 0$,

$$\theta\{-\lambda(\rho_1^2,\rho_2^2,\rho_3^2)\} = \theta(\rho_1+\rho_2-\rho_3)\;\theta(\rho_2+\rho_3-\rho_1)\;\theta(\rho_3+\rho_1-\rho_2)\;,$$

equals 1 when one can compose a triangle with sides ρ_1, ρ_2, ρ_3, and 0 otherwise.

Three-particle phase space in D dimensions (continued)

Denoting $\sigma_i = \sqrt{\rho_i^2 + m_i^2}$ (so that $\sigma_1 + \sigma_2 + \sigma_3 = p$, due to a δ function), integrating over x^0 and introducing Mandelstam-type variables

$$s = p^2 + m_3^2 - 2p\sigma_3, \quad t = p^2 + m_1^2 - 2p\sigma_1, \quad u = p^2 + m_2^2 - 2p\sigma_2 \ ,$$

satisfying

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + p^2 \equiv w_0 \qquad (*)$$

we get another integral representation (the integration limits are discussed below),

$$I_3^{(D)} = \frac{\pi^{D-2}}{4p^{D-2}\Gamma(D-2)} \iiint \mathrm{d}s\ \mathrm{d}t\ \mathrm{d}u\ \delta(s+t+u-w_0)\ \left[\Phi(s,t,u)\right]^{(D-4)/2} \theta\left\{\Phi(s,t,u)\right\}$$

where

$$\Phi(s,t,u) = -\frac{1}{16p^2}\, \lambda\left\{\lambda(s,m_3^2,p^2), \lambda(t,m_1^2,p^2), \lambda(u,m_2^2,p^2)\right\}$$

can also be written in a more familiar Kibble cubic form (provided that $(*)$ holds)

$$\begin{aligned}\Phi(s,t,u) \quad &= \quad stu - s(m_1^2 m_2^2 + p^2 m_3^2) - t(m_2^2 m_3^2 + p^2 m_1^2) - u(m_3^2 m_1^2 + p^2 m_2^2) \\ &\quad +2(m_1^2 m_2^2 m_3^2 + p^2 m_1^2 m_2^2 + p^2 m_2^2 m_3^2 + p^2 m_3^2 m_1^2)\end{aligned}$$

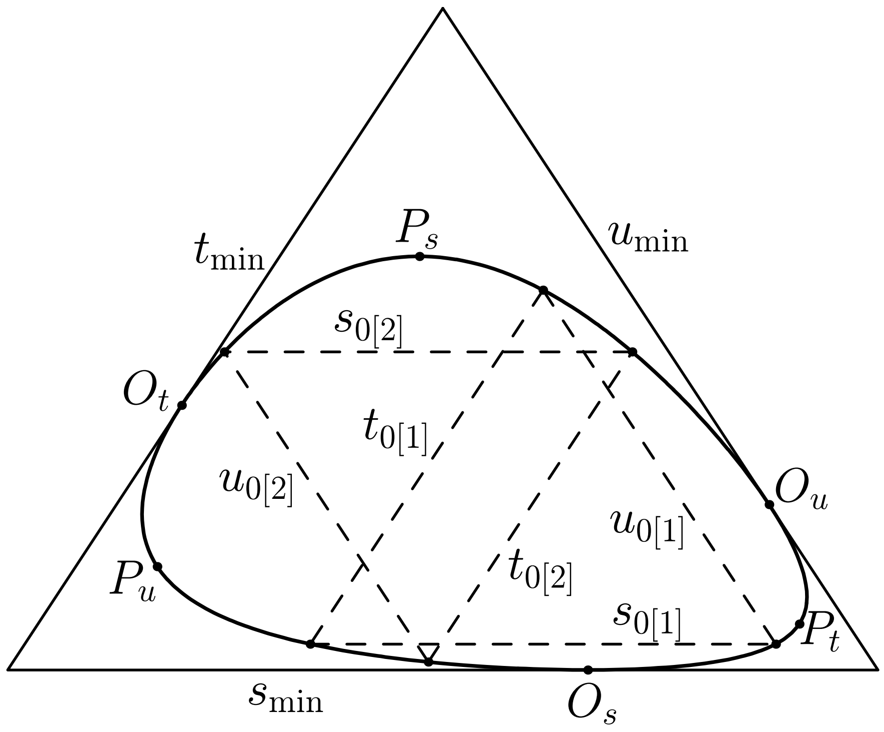
The Dalitz–Kibble integration area

The maximal values $(P_s,\ P_t,\ P_u)$:

$s_{\rm max} = (p - m_3)^2$,
$t_{\rm max} = (p - m_1)^2$,
$u_{\rm max} = (p - m_2)^2$.

The minimal values $(O_s,\ O_t,\ O_u)$:

$s_{\rm min} = (m_1 + m_2)^2$,
$t_{\rm min} = (m_2 + m_3)^2$,
$u_{\rm min} = (m_1 + m_3)^2$.



Moreover, due to the theta function $\theta\left\{\Phi(s,t,u)\right\}$ the region of integration is in fact restricted by the interior of the cubic curve $\Phi(s,t,u) = 0$.

The function $\Phi(s,t,u)$ has a maximum within the region of integration.

For equal masses, $\Phi_{\rm max} = \frac{1}{27}\, p^2 (p^2 - 9m^2)^2$ occurs at $s = t = u = \frac{1}{3}\,(p^2 + 3m^2)$.

For the general unequal masses, one needs to solve a fourth-order algebraic equation to find the position of the maximum.

Geometrical interpretation

[A. I. Davydychev, R. Delbourgo, J. Math. Phys. **39** (1998) 4299]

Let us introduce

$$c_{12} = \frac{s - m_1^2 - m_2^2}{2m_1 m_2}, \quad c_{23} = \frac{t - m_2^2 - m_3^2}{2m_2 m_3}, \quad c_{13} = \frac{u - m_1^2 - m_3^2}{2m_1 m_3} .$$

Then, the function $\Phi(s,t,u)$ can be presented as a Gram determinant,

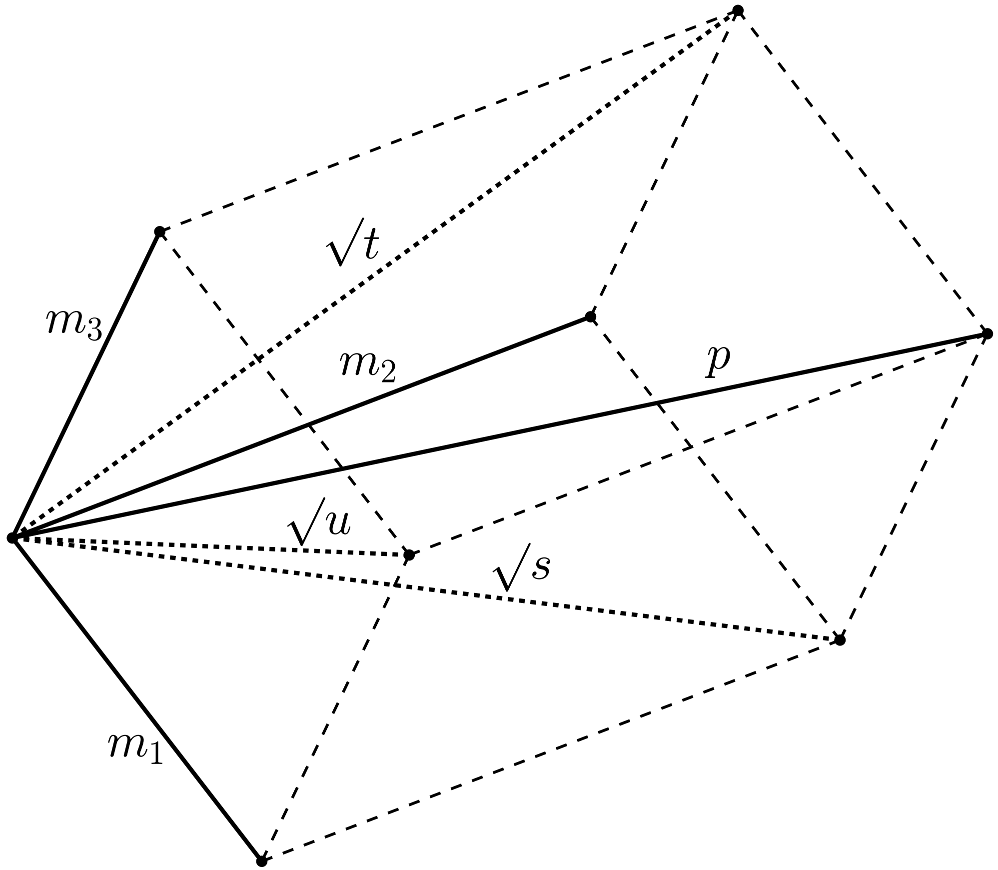
$$\Phi(s,t,u) = 4m_1^2 m_2^2 m_3^2 \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix}$$

For $D=4$, we get (the integration extends over $c_{jl} \geq 1$)

$$I_3 = \frac{2\pi^2}{p^2} m_1^2 m_2^2 m_3^2 \int\int\int \mathrm{d}c_{12}\, \mathrm{d}c_{13}\, \mathrm{d}c_{23}\, \theta \left(\begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix} \right)$$
$$\times \delta \left(m_1^2 + m_2^2 + m_3^2 + 2m_1 m_2 c_{12} + 2m_2 m_3 c_{23} + 2m_1 m_3 c_{13} - p^2 \right)$$

Geometrical interpretation (continued)

If we were to interpret c_{jl} as the cosines of the angles between the m_j and m_l sides of a vertex of a parallelepiped then all these quantities would have a straightforward geometrical interpretation:



$$\Phi(s,t,u) \leftrightarrow 4\{\text{volume of parallelepiped}\}^2,$$

the δ function would tell us that the "principal" diagonal should be equal to p,

the quantities $\sqrt{s}$, $\sqrt{t}$ and $\sqrt{u}$ could be identified as the diagonals of the faces,

the quantities

$$\frac{p^2+m_1^2-t}{2pm_1}, \quad \frac{p^2+m_2^2-u}{2pm_2} \quad \text{and} \quad \frac{p^2+m_3^2-s}{2pm_3}$$

could be understood as cosines of the angles between the diagonal p and the m_i sides of the parallelepiped.

Kibble cubic characteristics

To remind, $s + t + u = m_1^2 + m_2^2 + m_3^2 + p^2 \equiv w_0$.
Suppose

$$(s_0, t_0, w_0 - s_0 - t_0), \quad (s_0, w_0 - s_0 - u_0, u_0), \quad (w_0 - t_0 - u_0, t_0, u_0)$$

all are the roots of the equation $\Phi(s,t,u) = 0$. Then, we can present $\Phi(s,t,u)$ as

$$\Phi(s,t,u) = stu - st_0u_0 - s_0tu_0 - s_0t_0u + 2s_0t_0u_0 \; .$$

Defining

$$c_{tu} \equiv \sqrt{\frac{t_0u_0}{tu}} \, , \qquad c_{st} \equiv \sqrt{\frac{s_0t_0}{st}} \, , \qquad c_{su} \equiv \sqrt{\frac{s_0u_0}{su}} \, ,$$

we arrive at another Gram determinant representation for $\Phi(s,t,u)$,

$$\Phi(s,t,u) = stu \begin{vmatrix} 1 & c_{tu} & c_{st} \\ c_{tu} & 1 & c_{su} \\ c_{st} & c_{su} & 1 \end{vmatrix} \; .$$

Kibble cubic characteristics (continued)

There are (at least) two sets of solutions that can be described as

$$s_0 = \frac{A_1 A_2}{A_3} \, , \qquad t_0 = \frac{A_2 A_3}{A_1} \, , \qquad u_0 = \frac{A_1 A_3}{A_2} \, ,$$

so that

$$\Phi(s,t,u) = stu - A_1^2 t - A_2^2 u - A_3^2 s + 2 A_1 A_2 A_3 \, .$$

The first set of solutions corresponds to

$$A_1 \equiv p m_1 + m_2 m_3 \, , \quad A_2 \equiv p m_2 + m_3 m_1 \, , \quad A_3 \equiv p m_3 + m_1 m_2 \, .$$

For this set, we have

$$c_{tu} = \frac{p m_3 + m_1 m_2}{\sqrt{tu}}, \quad c_{st} = \frac{p m_2 + m_1 m_3}{\sqrt{st}}, \quad c_{su} = \frac{p m_1 + m_2 m_3}{\sqrt{su}} \, .$$

Note that if we change $p \to -p$, this would also be a solution, which would correspond to a "non-physical" branch of the Kibble cubic.

Kibble cubic characteristics (continued)

There are (at least) two sets of solutions that can be described as

$$s_0 = \frac{A_1 A_2}{A_3}\,, \qquad t_0 = \frac{A_2 A_3}{A_1}\,, \qquad u_0 = \frac{A_1 A_3}{A_2}\,,$$

so that

$$\Phi(s,t,u) = stu - A_1^2 t - A_2^2 u - A_3^2 s + 2A_1 A_2 A_3\ .$$

The second set of solutions corresponds to

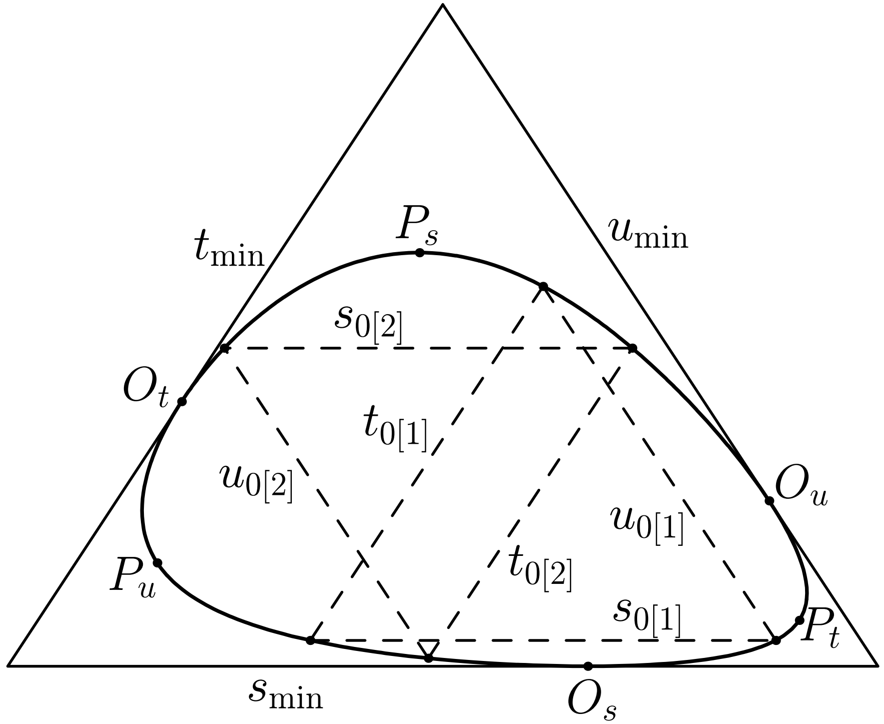
$$A_1 \equiv \tfrac{1}{2}(p^2{+}m_1^2{-}m_2^2{-}m_3^2)\ , \quad A_2 \equiv \tfrac{1}{2}(p^2{-}m_1^2{+}m_2^2{-}m_3^2)\ , \quad A_3 \equiv \tfrac{1}{2}(p^2{-}m_1^2{-}m_2^2{+}m_3^2)\ .$$

For this set, we get

$$c_{tu} = \frac{p^2{-}m_1^2{-}m_2^2{+}m_3^2}{2\sqrt{tu}}, \quad c_{st} = \frac{p^2{-}m_1^2{+}m_2^2{-}m_3^2}{2\sqrt{st}}, \quad c_{su} = \frac{p^2{+}m_1^2{-}m_2^2{-}m_3^2}{2\sqrt{su}}.$$

Kibble cubic characteristics (continued)

In the Dalitz–Kibble plot
we connect the points
for each of the two sets
by dashed lines, introducing
subscripts $[1]$ and $[2]$.
The two "dashed" triangles
indicate that the two sets
are complementary to each other:
the boundary of the Dalitz plot
confines tu, st and su as follows:



$$(t_0 u_0)_{[1]} \le tu \le (t_0 u_0)_{[2]} \; ,$$

$$(s_0 t_0)_{[1]} \le st \le (s_0 t_0)_{[2]} \; ,$$

$$(s_0 u_0)_{[1]} \le su \le (s_0 u_0)_{[2]} \; .$$

Kibble cubic characteristics (continued)

Consider the values of the "cosines" c_{su}, c_{st} and c_{tu}.

For the first set, c_{su}, c_{st} and c_{tu} would vary between 1 and $\cos\varphi_i$ $(i = 1, 2, 3)$, respectively, where

$$\cos\varphi_1 = \frac{2(pm_1 - m_2m_3)}{p^2+m_1^2-m_2^2-m_3^2}, \quad \cos\varphi_2 = \frac{2(pm_2 - m_3m_1)}{p^2-m_1^2+m_2^2-m_3^2}, \quad \cos\varphi_3 = \frac{2(pm_3 - m_1m_2)}{p^2-m_1^2-m_2^2+m_3^2}.$$

Their sines can be presented as

$$\sin\varphi_1 = \frac{\sqrt{Q_+}}{p^2+m_1^2-m_2^2-m_3^2}, \quad \sin\varphi_2 = \frac{\sqrt{Q_+}}{p^2-m_1^2+m_2^2-m_3^2}, \quad \sin\varphi_3 = \frac{\sqrt{Q_+}}{p^2-m_1^2-m_2^2+m_3^2}.$$

For the second set, c_{su}, c_{st} and c_{tu} would vary between 1 and $1/\cos\varphi_i$. This means that we need to understand them in the sense of analytic continuation.

Bridge between integral representations

For $D = 4$, using the representation for $\Phi(s,t,u)$ in terms of s_0, t_0 and u_0, we get

$$I_3 = \frac{\pi^2}{4p^2} \iiint \mathrm{d}s\ \mathrm{d}t\ \mathrm{d}u\ \delta(s+t+u-w_0)\ \theta(stu - st_0u_0 - s_0tu_0 - s_0t_0u + 2s_0t_0u_0)\ ,$$

with $w_0 = p^2 + m_1^2 + m_2^2 + m_3^2$. Integrating over u yields

$$I_3 = \frac{\pi^2}{4p^2} \iint \mathrm{d}s\ \mathrm{d}t\ \theta\{(st - s_0t_0)(w_0 - s - t) - st_0u_0 - s_0tu_0 + 2s_0t_0u_0\}\ .$$

Integrating over t, we get difference between the roots of the quadratic argument,

$$\frac{1}{s}\sqrt{s^4 - 2w_0s^3 + (w_0^2 + 2s_0t_0 + 2s_0u_0 - 4t_0u_0)s^2 - 2(w_0t_0 + w_0u_0 - 4u_0t_0)s_0s + s_0^2(t_0 - u_0)^2}.$$

For both sets of (s_0, t_0, u_0) the square root takes the familiar form,

$$I_3 = \frac{\pi^2}{4p^2} \int\limits_{s_2}^{s_3} \frac{\mathrm{d}s}{s}\ \sqrt{(s-s_1)(s-s_2)(s_3-s)(s_4-s)}\ ,$$

Three-particle phase space in odd dimensions ($D = 3, 5, \ldots$)

Starting from the (non-symmetric) D-dimensional representation,

$$I_3^{(D)} = \frac{\pi^{D-1}}{(4p)^{D-2}\Gamma^2\left(\frac{D-1}{2}\right)} \int\limits_{s_2}^{s_3} \frac{\mathrm{d}s}{s^{D/2-1}} \left[(s-s_1)(s-s_2)(s_3-s)(s_4-s)\right]^{(D-3)/2} ,$$

we can easily see (just substituting $s = x^2$) that all *odd*-dimensional phase-space integrals can be expressed in terms of polynomial functions

$$\begin{aligned} I_3^{(3)} &= \frac{\pi^2}{2p}(p - m_1 - m_2 - m_3) , \\ I_3^{(5)} &= \frac{\pi^4}{60p^3}(p - m_1 - m_2 - m_3)^3 \Big[\tfrac{1}{7}(p - m_1 - m_2 - m_3)^4 + (m_1 + m_2 + m_3)p^3 \\ &\quad -2(m_1^2 + m_2^2 + m_3^2)p^2 + (m_1^3 + m_2^3 + m_3^3)p + 12m_1m_2m_3p \\ &\quad -(m_1+m_2+m_3)(m_1+m_2)(m_2+m_3)(m_3+m_1) + 4m_1m_2m_3(m_1+m_2+m_3)\Big] , \end{aligned}$$

etc., which are explicitly symmetric in the masses m_i.

Three-particle phase space in two dimensions

Consider the case $D = 2$. Then, we get just the elliptic integral $K(k)$,

$$I_3^{(2)} = \int\limits_{s_2}^{s_3} \frac{\mathrm{d}s}{\sqrt{(s-s_1)(s-s_2)(s_3-s)(s_4-s)}} = \frac{2}{\sqrt{Q_+}} K(k) .$$

This is explicitly symmetric in the masses, because Q_+, Q_- and k are symmetric,

$$\begin{aligned} Q_+ &\equiv (p+m_1+m_2+m_3)(p+m_1-m_2-m_3)(p-m_1+m_2-m_3)(p-m_1-m_2+m_3) , \\ Q_- &\equiv (p-m_1-m_2-m_3)(p-m_1+m_2+m_3)(p+m_1-m_2+m_3)(p+m_1+m_2-m_3) , \\ k &\equiv \sqrt{\frac{Q_-}{Q_+}}. \end{aligned}$$

We can also obtain a very useful relation between the three $Z(\varphi_i, k)$ functions,

$$Z(\varphi_1, k) + Z(\varphi_2, k) + Z(\varphi_3, k) = k^2 \ \sin\varphi_1 \ \sin\varphi_2 \ \sin\varphi_3 .$$

Three-particle phase space in four dimensions

Use a trick of inserting the unity (look at the argument of the δ-function), to get rid of $1/s$, $1/t$ and $1/u$ (t_i and u_i are obtained from s_i by permutation of masses):

$$\frac{\pi^2}{4p^2} \int\!\!\int\!\!\int \mathrm{d}s\ \mathrm{d}t\ \mathrm{d}u\ \delta(s+t+u-w_0)\ \theta(stu-st_0u_0-s_0tu_0-s_0t_0u+2s_0t_0u_0)$$

$$= \frac{\pi^2}{4p^2} \int\!\!\int\!\!\int \mathrm{d}s\ \mathrm{d}t\ \mathrm{d}u\ \frac{s+t+u}{w_0}\ \delta(s+t+u-w_0)\ \theta(stu-st_0u_0-s_0tu_0-s_0t_0u+2s_0t_0u_0)$$

$$= \frac{\pi^2}{4p^2 w_0} \left\{ \int\limits_{s_2}^{s_3} \mathrm{d}s\ \sqrt{(s-s_1)(s-s_2)(s_3-s)(s_4-s)} + \int\limits_{t_2}^{t_3} \mathrm{d}t\ \sqrt{(t-t_1)(t-t_2)(t_3-t)(t_4-t)} + \int\limits_{u_2}^{u_3} \mathrm{d}u\ \sqrt{(u-u_1)(u-u_2)(u_3-u)(u_4-u)} \right\}$$

Three-particle phase space in four dimensions

Collecting the results for all three integrals and using the relation for Z-functions, we arrive at the symmetric result

$$I_3 = \frac{\pi^2}{8p^2}\left\{ \sqrt{Q_+}(p^2+m_1^2+m_2^2+m_3^2)\,[E(k)-K(k)] \right.$$
$$\left. +Q_+ K(k)\left[\frac{Z(\varphi_1,k)}{\sin^2\varphi_1}+\frac{Z(\varphi_2,k)}{\sin^2\varphi_2}+\frac{Z(\varphi_3,k)}{\sin^2\varphi_3}\right] \right\},$$

where

$$\sin\varphi_1 = \frac{\sqrt{Q_+}}{p^2+m_1^2-m_2^2-m_3^2}, \quad \sin\varphi_2 = \frac{\sqrt{Q_+}}{p^2-m_1^2+m_2^2-m_3^2}, \quad \sin\varphi_3 = \frac{\sqrt{Q_+}}{p^2-m_1^2-m_2^2+m_3^2}.$$

This result can also be presented in terms of the elliptic integrals Π, using

$$K(k)\;Z(\varphi_i,k) = \cot\varphi_i\,\sqrt{1-k^2\sin^2\varphi_i}\,\left[\Pi(k^2\sin^2\varphi_i,k)-K(k)\right] .$$

Three-particle phase space in six dimensions

Using the same approach we can also obtain symmetric results $D = 6$ (and higher dimensions):

$$
\begin{aligned}
I_3^{(6)} = \frac{\pi^4}{144p^4}\Bigg\{ & \frac{Q_+^{1/2}}{20}\left[E(k)-K(k)\right]\Big[192(p^8+m_1^8+m_2^8+m_3^8) - 112(p^4+m_1^4+m_2^4+m_3^4)^2 \\
& -6(p^2+m_1^2+m_2^2+m_3^2)^4 - 156(p^6+m_1^6+m_2^6+m_3^6)(p^2+m_1^2+m_2^2+m_3^2) \\
& +83(p^4+m_1^4+m_2^4+m_3^4)(p^2+m_1^2+m_2^2+m_3^2)^2\Big] \\
& +\frac{1}{40}Q_-Q_+^{1/2}K(k)\Big[3(p^2+m_1^2+m_2^2+m_3^2)^2 - 16(p^4+m_1^4+m_2^4+m_3^4)\Big] \\
& +\frac{3}{4}\,\frac{Q_+^{5/2}\,K(k)}{\sin\varphi_1\,\sin\varphi_2\,\sin\varphi_3}\left[\frac{Z(\varphi_1,k)}{\sin^2\varphi_1}+\frac{Z(\varphi_2,k)}{\sin^2\varphi_2}+\frac{Z(\varphi_3,k)}{\sin^2\varphi_3}\right] \\
& -\frac{3}{8}Q_+^2(p^2+m_1^2+m_2^2+m_3^2)K(k)\left[\frac{Z(\varphi_1,k)}{\sin^4\varphi_1}+\frac{Z(\varphi_2,k)}{\sin^4\varphi_2}+\frac{Z(\varphi_3,k)}{\sin^4\varphi_3}\right]\Bigg\}.
\end{aligned}
$$

Summary and conclusions

- We have considered several representations for the three-particle phase space (in terms of the Kibble cubic $\Phi(s,t,u)$, etc.), exploring their symmetry properties and geometrical meaning.

- A number of representations are given for N-particle phase space, for an arbitrary dimension D.

- It was shown that the angles φ_i are convenient to describe the results for the three-particle phase-space integral I_3.

- The result for I_3 in four dimensions (given in terms of the Jacobian Z function) is very compact and explicitly symmetric with respect to all masses m_i.

- In similar way, explicitly symmetric results for higher dimensions ($D = 6$, etc.) can be also obtained.