Functional equations for multiloop Feynman integrals

O.V. Tarasov

July 24, 2015

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Method for deriving FE

Functional equations (FE) for Feynman integrals were proposed in O.V.T. Phys.Lett. B670 (2008) 67.

Feynman integrals satisfy recurrence relations which can be written as

$$\sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}$$

where Q_j, R_k are polynomials in masses, scalar products of external momenta, d, and powers of propagators. $I_{k,r}$ - are integrals with r external lines. In recurrence relations some integrals are more complicated than the others: they have more arguments than the others.

General method for deriving functional equations:

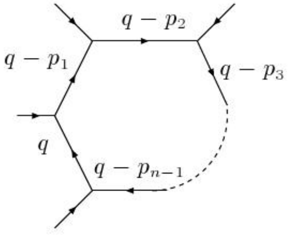
By choosing kinematic variables, masses, indices of propagators remove most complicated integrals, i.e. impose conditions :

$$Q_j = 0$$

keeping at least some other coefficients $R_k \neq 0$.

One-loop n-point integrals

Let's consider one-loop n-point integrals



$$I_n^{(d)} = \int \frac{d^d q}{i\pi^{d/2}} \prod_{j=1}^{n} \frac{1}{c_j^{\nu_j}}, \qquad c_j = (q - p_j)^2 - m_j^2 + i\epsilon.$$

p_j external momentum flowing through j-th propagator with mass m_j.

Introduction

Integrals $I_n^{(d)}$ satisfy generalized recurrence relations (O.T. in Phys.Rev.D54 (1996) p.6479):

$$G_{n-1}\nu_j \mathbf{j}^+ I_n^{(d+2)} - (\partial_j \Delta_n) I_n^{(d)} = \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n) \mathbf{k}^- I_n^{(d)},$$

where $\mathbf{j}^{\pm}$ shifts indices $\nu_j \to \nu_j \pm 1$, $\partial_j \equiv \frac{\partial}{\partial m_j^2}$,

$$G_{n-1} = -2^n \begin{vmatrix} p_1 p_1 & p_1 p_2 & \ldots & p_1 p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 p_{n-1} & p_2 p_{n-1} & \ldots & p_{n-1} p_{n-1} \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}, \quad Y_{ij} = m_i^2 + m_j^2 - p_{ij}, \qquad p_{ij} = (p_i - p_j)^2,$$

One-loop propagator type integral

At $n=3$, $j=1$, $m_3^2=0$, imposing conditions $G_2=0$, $\Delta_3=0$ we get

$$I_2^{(d)}(m_1^2,m_2^2,p_{12}) = \frac{p_{12}+m_1^2-m_2^2-\alpha_{12}}{2p_{12}} I_2^{(d)}(m_1^2,0,s_{13}) + \frac{p_{12}-m_1^2+m_2^2+\alpha_{12}}{2p_{12}} I_2^{(d)}(0,m_2^2,s_{23})$$

where

$$s_{13} = \frac{\Delta_{12}+2p_{12}m_1^2-(p_{12}+m_1^2-m_2^2)\alpha_{12}}{2p_{12}},$$

$$s_{23} = \frac{\Delta_{12}+2p_{12}m_2^2+(p_{12}-m_1^2+m_2^2)\alpha_{12}}{2p_{12}},$$

$$\alpha_{12} = \pm\sqrt{\Delta_{12}}.$$

$$\Delta_{ij} = p_{ij}^2+m_i^4+m_j^4-2p_{ij}m_i^2-2p_{ij}m_j^2-2m_i^2m_j^2.$$

Integral with arbitrary masses and momentum can be expressed in terms of integrals with one propagator massless !!!

Analytic continuation of $I_2^{(d)}$

Setting $m_2 = 0$ in the previous functional equation we have :

$$I_2^{(d)}(m_1^2, 0, p_{12}) = \frac{m_1^2}{p_{12}} I_2^{(d)}\left(m_1^2, 0, \frac{m_1^4}{p_{12}}\right) + \frac{(p_{12}-m_1^2)}{p_{12}} I_2^{(d)}\left(0, 0, \frac{(p_{12}-m_1^2)^2}{p_{12}}\right).$$

where

$$I_2^{(d)}(0,0,p^2) = \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma(d-2)}(-p^2)^{\frac{d}{2}-2}.$$

Integral $I_2^{(d)}$ on the r.h.s. has inverse argument . In fact this equation corresponds to the well known formula for analytic continuation:

$${}_2F_1\left[\begin{array}{c} 1, 2-\frac{d}{2}; \\ \frac{d}{2}; \end{array} z\right] = \frac{1}{z}\; {}_2F_1\left[\begin{array}{c} 1, 2-\frac{d}{2}; \\ \frac{d}{2}; \end{array} \frac{1}{z}\right] + \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(d-2)}(-z)^{\frac{d}{2}-2}\left(1-\frac{1}{z}\right)^{d-3}.$$

One-loop vertex type integral

At $n=4,\ j=1,\ m_4=0$, imposing conditions $G_3=0,\ \Delta_4=0$ we get

$$
\begin{aligned}
&I_3^{(d)}(m_1^2,m_2^2,m_3^2,s_{23},s_{13},s_{12}) = \\
&\frac{s_{13}+m_3^2-m_1^2+\alpha_{13}}{2s_{13}} \\
&\times I_3^{(d)}(m_2^2,m_3^2,0,s_{34}^{(13)},s_{24}(m_1^2,m_3^2,s_{23},s_{13},s_{12}),s_{23}) \\
&+\frac{s_{13}-m_3^2+m_1^2-\alpha_{13}}{2s_{13}} \\
&\times I_3^{(d)}(m_1^2,m_2^2,0,s_{24}(m_1^2,m_3^2,s_{23},s_{13},s_{12}),s_{14}^{(13)},s_{12})
\end{aligned}
$$

Again as it was for integral $I_2^{(d)}$ integral $I_3^{(d)}$ with arbitrary arguments can be expressed in terms of integrals with at least one propagator massless!!!

one-loop vertex integral

Applying functional equations several times integrals $I_3^{(d)}(m_1^2, m_2^2, m_3^2, p_{23}, p_{13}, p_{12})$ can be expressed in terms of simpler integrals. The most complicated will be integral with two massless propagators and one external momentum squared equal to zero. Analytic expression for such an integral is:

$$I_3^{(d)}(0, m^2, 0, 0, p_{13}, p_{12}) =$$
$$= I_2^{(d)}(0, m^2, 0) F_1\left(1, 1, 2 - \frac{d}{2}, \frac{d}{2}; \frac{p_{12} - p_{13}}{m^2}, \frac{p_{12}}{m^2}\right)$$
$$+ \frac{I_2^{(d)}(0, 0, p_{13})}{m^2} {}_2F_1\left[\begin{matrix} 1, \frac{d-2}{2}; \\ d-2; \end{matrix} \frac{p_{12} - p_{13}}{m^2}\right]$$

where F_1 is well known Appell's function.

Introduction

Profit from FE

- FE reduce integrals with complicated kinematics to simpler integrals
- FE can be used for analytic continuation of Feynman integrals without knowing explicit analytic result.

Problems

The method of derivation of FE for technical reasons is not easy to apply for multiloop integrals.

All recurrence relations follow from the equation:

$$\int d^d k \frac{\partial}{\partial k_\mu} f(k, \{s_{ij}\}, \{m_r^2\}) = 0.$$

One can raise the question:

Functional equations hold for integrals or they can be obtained as a consequence of a relation between integrands?

Algebraic relations for propagators

Analyzing one-loop FE one can see that integrands are rather similar and differ only by one propagator:

Integrands for the one-loop propagator type integrals

$$\frac{1}{D_1 D_2}, \quad \frac{1}{D_0 D_2}, \frac{1}{D_1 D_0},$$

Integrands for the one-loop vertex type integrals

$$\frac{1}{D_1 D_2 D_3}, \quad \frac{1}{D_0 D_2 D_3}, \frac{1}{D_1 D_0 D_3}, \frac{1}{D_1 D_2 D_0}$$

where

$$D_0 = (k_1 - p_0)^2 - m_0^2 + i0, \quad D_1 = (k_1 - p_1)^2 - m_1^2 + i0,$$
$$D_2 = (k_1 - p_2)^2 - m_2^2 + i0, \quad D_3 = (k_1 - p_3)^2 - m_3^2 + i0,$$

Observation: since $G_n = 0$ vectors $p_1, p_2, \ldots$ are linearly dependent

Algebraic relations for propagators

Question: Would it be possible to find algebraic relations of the form:

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}$$

where

$$p_0 = y_{01} p_1 + y_{02} p_2$$

and x_1, x_2, y_{01}, y_{02} being independent of k_1.

The answer is - YES! Putting all terms over the common denominator and equating coefficients in front of different products of $(k_1^2)^a (k_1 p_1)^b (k_1 p_2)^c$ to zero we obtain system of equations:

$$y_{02} - x_2 = 0, \quad y_{01} - x_1 = 0, \quad x_1 + x_2 = 1,$$
$$p_1^2(x_1 - y_{01}^2) + p_2^2(x_2 - y_{02}^2) + y_{01} y_{02}(s_{12} - p_1^2 - p_2^2) - m_1^2 x_1 - m_2^2 x_2 + m_0^2 = 0.$$

where $s_{12} = (p_1 - p_2)^2$

Algebraic relations for propagators

Solution of this system of equations is:

$$x_1 = y_{01} = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} - \frac{\sqrt{\Lambda_2 + 4s_{12}m_0^2}}{2s_{12}},$$

$$x_2 = y_{02} = \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}} + \frac{\sqrt{\Lambda_2 + 4s_{12}m_0^2}}{2s_{12}}.$$

and

$$\Lambda_2 = s_{12}^2 + m_1^4 + m_2^4 - 2s_{12}(m_1^2 + m_2^2) - 2m_1^2 m_2^2.$$

Solutions for y_{01}, y_{02} will be substituted into scalar invariants like

$$(p_1 - p_0)^2 = (y_{01} - 1)^2 p_1^2 + 2y_{02}(y_{01} - 1)\ p_1 p_2 + y_{02}^2\ p_2^2,$$
$$(p_2 - p_0)^2 = y_{01}^2 p_1^2 + 2y_{01}(y_{02} - 1)\ p_1 p_2 + (y_{02} - 1)^2\ p_2^2.$$

Algebraic relations for propagators

Integrating obtained algebraic relation w.r.t. k_1 gives the following FE:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = \frac{s_{12} + m_1^2 - m_2^2 - \lambda}{2s_{12}} \, I_2^{(d)}(m_1^2, m_0^2, s_{13}(m_1^2, m_2^2, m_0^2, s_{12})) + \frac{s_{12} - m_1^2 + m_2^2 + \lambda}{2s_{12}} \, I_2^{(d)}(m_2^2, m_0^2, s_{23}(m_1^2, m_2^2, m_0^2, s_{12})).$$

where

$$s_{13} = \frac{\Lambda_2 + 2s_{12}(m_1^2 + m_0^2)}{2s_{12}} + \frac{m_1^2 - m_2^2 + s_{12}}{2s_{12}}\lambda$$

$$s_{23} = \frac{\Lambda_2 + 2s_{12}(m_2^2 + m_0^2)}{2s_{12}} + \frac{m_1^2 - m_2^2 - s_{12}}{2s_{12}}\lambda.$$

$$\lambda = \sqrt{\Lambda_2 + 4s_{12}m_0^2}$$

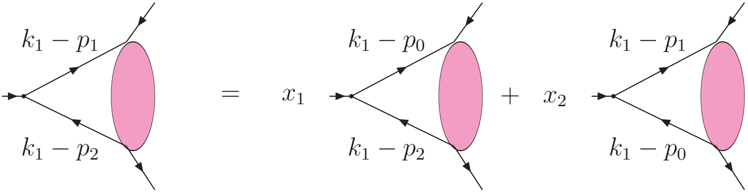
Parameter m_0 is arbitrary and can be taken at will. The same equation was obtained from recurrence relations by imposing conditions on Gram determinants.

Algebraic relations for propagators

Multiplying relation

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}$$

by products of propagators or multi-loop integrals and integrating over k_1 we obtain FE for multi-leg and multi-loop integrals.



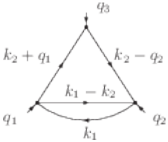
Two-loop vertex integral

Derivation of the FE for the integral

$$R(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2) =$$
$$\int\int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} P(k_2+q_1, m_1) P(k_2-q_2, m_2) P(k_1, m_3) P(k_1-k_2, m_4).$$

where

$$P(k_j, m_r) = \frac{1}{k_j^2 - m_r^2 + i0}$$



Two-loop vertex integral

Multiplying two-propagator algebraic relation by the integral

$$\int \frac{d^d k_2}{[k_2^2 - m_4^2][(k_1 - k_2)^2 - m_5^2]}$$

and integrating over momentum k_1 the following relation follows:

$$\begin{aligned} & R(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2) \\ & = \alpha R(0, m_2^2, m_3^2, m_4^2;\ Q^2, q_2^2, (m_2^2 - m_1^2 + q_3^2)\alpha - m_2^2) \\ & + (1-\alpha) R(m_1^2, 0, m_3^2, m_4^2;\ q_1^2, Q^2, (m_2^2 - m_1^2 - q_3^2)\alpha + q_3^2 - m_2^2), \end{aligned}$$

where

$$\begin{aligned} Q^2 &= (q_1^2 - q_2^2 - m_1^2 + m_2^2)\alpha + q_2^2 - m_2^2, \\ \alpha &= \frac{q_3^2 - m_1^2 + m_2^2 \pm \sqrt{\Delta}}{2q_3^2}, \\ \Delta &= q_3^4 + m_1^4 + m_2^4 - 2q_3^2 m_1^2 - 2q_3^2 m_2^2 - 2m_1^2 m_2^2. \end{aligned}$$

Algebraic relations for propagators

Similar to the relation with two propagators one can find relation for three propagartors:

$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_2 D_3 D_0} + \frac{x_2}{D_1 D_3 D_0} + \frac{x_3}{D_1 D_2 D_0}.$$

Here p_1, p_2 and p_3 are independent external momenta, k_1 will be integration momentum and

$$p_0 = y_{01} p_1 + y_{02} p_2 + y_{03} p_3.$$

Multiplying both sides of equation by the product $D_1 D_2 D_3 D_0$ and equating coefficients in front of $k_1^2, k_1 p_1, k_1 p_2, k_1 p_3$ and term independent of k_1 we obtain system of equations

$$y_{01} - x_1 = 0, \quad y_{02} - x_2 = 0, \quad y_{03} - x_3 = 0, \quad x_3 + x_2 + x_1 - 1 = 0,$$
$$[x_1 - y_{01}(y_{03} + y_{02} + y_{01})] p_1^2 + [x_2 - y_{02}(y_{03} + y_{02} + y_{01})] p_2^2$$
$$+[x_3 - y_{03}(y_{03} + y_{01} + y_{02})] p_3^2$$
$$+y_{02} y_{03} p_{23} + y_{01} y_{03} p_{13} + y_{01} y_{02} p_{12} - m_1^2 x_1 - m_2^2 x_2 - m_3^2 x_3 + m_0^2 = 0.$$

Algebraic relations for propagators

This system has the following solution

$$x_1 = y_{01} = 1 - \alpha - y_{02}, \qquad x_2 = y_{02}, \qquad x_3 = y_{03} = \alpha,$$

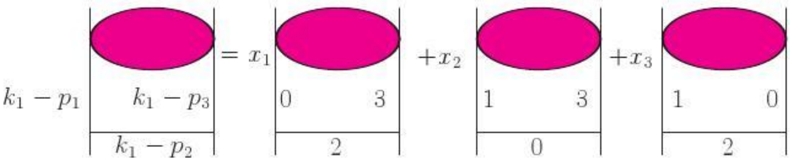
where α is solution of the quadratic equation

$$\alpha^2 p_{13} + [m_3^2 - m_1^2 - p_{13} + y_{02}(p_{13} + p_{12} - p_{23})]\alpha$$
$$+m_1^2 - m_0^2 + (m_2^2 - m_1^2 - p_{12} + p_{12} y_{02}) y_{02} = 0.$$

Solution depends on 2 arbitrary parameters- m_0, y_{02}.

Algebraic relations for propagators

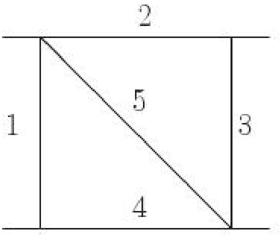
Multiplying algebraic relation for 3 propagators by other propagators or multiloop integrals we obtain the functional equation



In fact algebraic relations do exist not only for products of 2 and 3 propagators but also for products of any number of propagators. In previous examples we considered only one integration variable. In fact more general type of reparametrization of products of propagators is possible.

Relation for two-loop box integrals

Example:



$$D_1 = (k_1 - p_1)^2 - m_1^2, \quad D_2 = (k_2 - p_2)^2 - m_2^2,$$
$$D_3 = (k_2 - p_3)^2 - m_3^2,$$
$$D_4 = k_1^2 - m_4^2, \quad D_5 = (k_1 - k_2)^2 - m_5^2,$$

Relation for the two-loop box integrals

For this integral the following reparametrization exists:

$$
\begin{aligned}
\frac{1}{D_1 D_2 D_3 D_4 D_5} &= \frac{x_1}{D_{01} D_2 D_3 D_4 D_5} + \frac{x_2}{D_1 D_{02} D_3 D_4 D_5} + \frac{x_3}{D_1 D_2 D_{03} D_4 D_5} \\
&+ \frac{x_4}{D_1 D_2 D_3 D_{04} D_5} + \frac{x_5}{D_{01} D_{02} D_3 D_4 D_5} + \frac{x_6}{D_{01} D_2 D_{03} D_4 D_5} \\
&+ \frac{x_7}{D_{01} D_2 D_3 D_{04} D_5} + \frac{x_8}{D_1 D_{02} D_{03} D_4 D_5} + \frac{x_9}{D_1 D_{02} D_3 D_{04} D_5} \\
&+ \frac{x_{10}}{D_1 D_2 D_{03} D_{04} D_5} + \frac{x_{11}}{D_1 D_{02} D_{03} D_{04} D_5} + \frac{x_{12}}{D_{01} D_2 D_{03} D_{04} D_5} \\
&+ \frac{x_{13}}{D_{01} D_{02} D_3 D_{04} D_5} + \frac{x_{14}}{D_{01} D_{02} D_{03} D_4 D_5} + \frac{x_{15}}{D_{01} D_{02} D_{03} D_{04} D_5}
\end{aligned}
$$

where

$$
\begin{aligned}
D_{01} &= (k_1 - p_{01})^2 - M_1^2, \quad D_{02} = (k_1 - p_{02})^2 - M_2^2, \\
D_{03} &= (k_2 - p_{03})^2 - M_3^2, \quad D_{04} = (k_2 - p_{04})^2 - M_4^2,
\end{aligned}
$$

Relation for the two-loop box integrals

$$p_{01} = z_{11}p_1 + z_{12}p_2 + z_{13}p_3,$$
$$p_{02} = z_{21}p_1 + z_{22}p_2 + z_{23}p_3,$$
$$p_{03} = z_{31}p_1 + z_{32}p_2 + z_{33}p_3,$$
$$p_{04} = z_{41}p_1 + z_{42}p_2 + z_{43}p_3,$$

Bringing all terms to the common denominator and equating coefficients in front of different products of $k_1^2, k_1k_2, k_2^2, k_1p_1,\ k_1p_2, k_1p_3, k_2p_1, k_2p_2, k_2p_3,$ leads to a system of 225 equations for unknowns $x_1, ..., x_{15},\ z_{ij},$ $M_1^2, ..., M_4^2.$

There are tens of solutions of this system of equations!!!.
Many arbitrary parameters and therefore very reach structure of FE's.

Summary

- A new simple systematic method for deriving FE was proposed.
- Large number of parameters in FE can be used for transforming initial integrals to integrals with better properties.
- Highly desirable is investigation of symmetry of arguments for integrals in FE.
- The number of functional equations is very high that requires systematic classification of these equations.
- Most probably systematic application of methods of algebraic geometry and group theory will be useful to solve above problems.

Main problem: difficult to solve polynomial equations with many variables.

It will be nice to have an algorithm to determine to which class of function belong solutions (radicals, elliptic functions, hypergeometric functions, ...)