Static potential in perturbative QCD: the results and applications

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Abstract

The current knowledge on the three-loop approximation of the static potential in QCD is reviewed. This results are used for the determination of the 4-th order approximation of the QCD β-function and for the e^+e^--annihilation R-ratio in the V-scheme. The final results are compared with the ones, obtained in the $\overline{\mathrm{MS}}$-scheme and minimal MOM-scheme in the Landau gauge. The common features of the QED expression for the β-function in the V-scheme and MOM-scheme are summarized.

Plan

- The third order perturbative QCD static potential
- QCD β-function in $\overline{\mathrm{MS}}$-scheme at four-loop level
- Definition of the gauge- and scheme-independent V-scheme
- The QCD β^{V}-function at the fourth order approximation in the case of the $SU(N_c)$ group
- R-ratio for e^+e^--annihilation into hadrons in V, $\overline{\mathrm{MS}}$ and mMOM in Landau gauge schemes: comparison and analysis
- QED-limit: the relation of the $\beta^{\mathrm{V}}(\alpha_{\mathrm{V}})$-function with the Gell-Mann–Low function
- Conclusion

The third order perturbative QCD static potential

The static potential in QCD is introduced as a potential of interaction between static quark and antiquark at a distance r with using Wilson loop:

$$\mathrm{V}_{QCD}(\mu^2 r^2, \alpha_s(\mu^2)) = - \lim_{T\to\infty} \frac{1}{iT} \ln \frac{\langle 0 \mid \mathrm{Tr}\ \mathrm{P} e^{ig\oint_C dx^\mu A^a_\mu T^a} \mid 0\rangle}{\langle 0 \mid \mathrm{Tr}\ 1 \mid 0\rangle} =$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\vec{r}} \mathrm{V}(\vec{q}^{\,2}, \mu^2, \alpha_s(\vec{q}^{\,2})) \quad ,$$

where α_s is the strong coupling constant in $\overline{\mathrm{MS}}$-scheme, $\alpha_s/4\pi = g^2/16\pi^2$, g is the strong coupling constant of the QCD Lagrangian, C is a rectangular loop of time extent T and spatial extent r, T^a is the generator of $SU(N_c)$ group, A^a_μ is the gluon field, P is the ordering operator along the way.

The third order perturbative QCD static potential

In momentum space the static potential can be written into the form

$$\mathrm{V}(\vec{q}^{\,2},\mu^2,\alpha_s(\vec{q}^{\,2})) = \frac{-4\pi C_F \alpha_s(\vec{q}^{\,2})}{\vec{q}^{\,2}}\Bigg(1 + a_1^{\overline{\mathrm{MS}}}\frac{\alpha_s(\vec{q}^{\,2})}{4\pi} + \\ + a_2^{\overline{\mathrm{MS}}}\left(\frac{\alpha_s(\vec{q}^{\,2})}{4\pi}\right)^2 + \left(a_3^{\overline{\mathrm{MS}}} + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^{\,2}}\right)\left(\frac{\alpha_s(\vec{q}^{\,2})}{4\pi}\right)^3 + \dots\Bigg) \quad .$$

$[T^a, T^b] = i f^{abc} T^c$, $f^{acd} f^{bcd} = C_A \delta^{ab}$, $(T^a T^a)_{ij} = C_F \delta_{ij}$, C_A and C_F are the Casimir operators, $C_A = N_c$, $C_F = (N_c^2 - 1)/2N_c$. The additional term $8\pi^2 C_A^3 L$ appears due to the infrared (IR) divergences, which begin to manifest themselves in the the static potential at the three-loop level. We will neglect it in our RG-based analysis.

Colour structures of SU(N_c)-group

N_A is the number of the generators of the Lie algebra of the $SU(N_c)$, $n_l{=}n_f - 1$, n_f is the number of quark flavours, $d_F^{abcd} = Tr(T^a T^{(b} T^c T^{d)})/6$ and $d_A^{abcd} = Tr(C^a C^{(b} C^c C^{d)})/6$ are the total symmetric tensors, $(C^a)_{bc} = -if^{abc}$, where C^a are the generators of the adjoint representation of the Lie algebra of the $SU(N_c)$-group and

$$N_A = N_c^2 - 1 \; , \frac{d_A^{abcd} d_A^{abcd}}{N_A} = \frac{N_c^2(N_c^2+36)}{24} \; ,$$

$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c(N_c^2+6)}{48} \; , \frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{N_c^4 - 6N_c^2 + 18}{96 N_c^2} \; .$$

High order PT QCD corrections to the static potential

The coefficients $a_i^{\overline{\rm MS}}$ are calculated from the concrete Feynman diagrams and equal

$$a_1^{\overline{\rm MS}} = \frac{31}{9} C_A - \frac{20}{9} T_F n_l \ ,$$

(W. Fischler, A. Billoire, 1977)

$$a_2^{\overline{\rm MS}} = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3) \right) C_A^2 - \left(\frac{1798}{81} + \frac{56}{3}\zeta(3) \right) C_A T_F n_l$$

$$- \left(\frac{55}{3} - 16\zeta(3) \right) C_F T_F n_l + \left(\frac{20}{9} T_F n_l \right)^2 \ ,$$

(M. Peter, Y. Schroder, 1997)

High order PT QCD corrections to the static potential

The three-loop constant perturbative contribution to the static potential in the $\overline{\mathrm{MS}}$-scheme can be presented as

$$a_3^{\overline{\mathrm{MS}}} = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)} \ .$$

$$a^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3 \ ,$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368}{3}\zeta(3) + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416}{3}\zeta(3)\right) C_F T_F^2$$

$$a_3^{(1)} = -709.717 C_A^2 T_F + \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5)\right) C_A C_F T_F$$

$$+\left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5)\right) C_F^2 T_F - 56.83(1)\frac{d_F^{abcd} d_F^{abcd}}{N_A}$$

where the error of numerical calculation of the $C_A^2 T_F$-coefficient is not indicated (A. Smirnov, V. Smirnov, M. Steinhauser, 2008) .

High order PT QCD corrections to the static potential

The numerical expressions of the n_l-independent contributions were obtained by A. Smirnov, V. Smirnov and M. Steinhauser in 2010 and read

$$a_3^{(0)} = 502.24(1) C_A^3 - 136.39(12) \frac{d_F^{abcd} d_A^{abcd}}{N_A}$$

These results should be compared with the independent calculation of C. Anzai, Y. Kiyo and Y. Sumino in 2010

$$a_3^{(0)} = 502.22(12) C_A^3 - 136.8(14) \frac{d_F^{abcd} d_A^{abcd}}{N_A} \quad (1)$$

which have greater inaccuracies. Recent the more accurate result in Eq.(1) was obtained by Y. Sumino:

$$a_3^{(0)} = 502.22(12) C_A^3 - 136.6(2) \frac{d_F^{abcd} d_A^{abcd}}{N_A}$$

QCD β-function in $\overline{\mathrm{MS}}$-scheme at four-loop level

The evolution of the strong coupling constant α_s is determined by the QCD β-function in $\overline{\mathrm{MS}}$-scheme:

$$\mu^2 \frac{\partial(\alpha_s/4\pi)}{\partial\mu^2} = \beta^{\overline{\mathrm{MS}}}(\alpha_s) = -\sum_{i=0}^{\infty} \beta_i \left(\frac{\alpha_s}{4\pi}\right)^{i+2} ,$$

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_l ,$$

(G.'t Hooft, D. Gross, F. Wilczek, H. Politzer, 1973)

$$\beta_1 = \frac{34}{3} C_A^2 - 4 C_F T_F n_l - \frac{20}{3} C_A T_F n_l ,$$

(D. Jones, W. Caswell, 1974; E. Egorian, O. V. Tarasov, 1979)

$$\beta_2^{\overline{\mathrm{MS}}} = \frac{2857}{54} C_A^3 + 2 C_F^2 T_F n_l - \frac{205}{9} C_F C_A T_F n_l - \frac{1415}{27} C_A^2 T_F n_l + \\ + \frac{44}{9} C_F T_F^2 n_l^2 + \frac{158}{27} C_A T_F^2 n_l^2 ,$$

O. Tarasov, A. Vladimirov, A. Zharkov, (80); S. Larin, J. Vermaseren, (93)

QCD β-function in $\overline{\mathrm{MS}}$-scheme at four-loop level

$$\beta_3^{\overline{\mathrm{MS}}} = \left(\frac{150653}{486} - \frac{44}{9}\zeta(3)\right) C_A^4 + \left(-\frac{39143}{81} + \frac{136}{3}\zeta(3)\right) C_A^3 T_F n_l +$$

$$+ \left(\frac{7073}{243} - \frac{656}{9}\zeta(3)\right) C_A^2 C_F T_F n_l + \left(-\frac{4204}{27} + \frac{352}{9}\zeta(3)\right) C_A C_F^2 T_F n_l +$$

$$+46 C_F^3 T_F n_l + \left(\frac{7930}{81} + \frac{224}{9}\zeta(3)\right) C_A^2 T_F^2 n_l^2 +$$

$$+ \left(\frac{1352}{27} - \frac{704}{9}\zeta(3)\right) C_F^2 T_F^2 n_l^2 + \left(\frac{17152}{243} + \frac{448}{9}\zeta(3)\right) C_A C_F T_F^2 n_l^2 +$$

$$+\frac{424}{243} C_A T_F^3 n_l^3 + \frac{1232}{243} C_F T_F^3 n_l^3 + \left(-\frac{80}{9} + \frac{704}{3}\zeta(3)\right) \frac{d_A^{abcd} d_A^{abcd}}{N_A} +$$

$$\left(\frac{512}{9} - \frac{1664}{3}\zeta(3)\right) \frac{d_F^{abcd} d_A^{abcd}}{N_A} n_l + \left(-\frac{704}{9} + \frac{512}{3}\zeta(3)\right) \frac{d_F^{abcd} d_F^{abcd}}{N_A} n_l^2$$

(T. Ritbergen, J. Vermaseren, S. A. Larin, 1997; M. Czakon, 2005)

Definition of the gauge- and scheme-independent V-scheme

In the QCD static potential the renormalization group logarithms L=$\ln(\mu^2/\vec{q}^{\,2})$ can be recovered with help of the solution of RG-equation for β-function in $\overline{\mathrm{MS}}$-scheme at three-loop level:

$$\alpha_s(\vec{q}^{\,2}) = \alpha_s(\mu^2)\Bigg(1 + \beta_0 L \frac{\alpha_s(\mu^2)}{4\pi} + (\beta_0^2 L^2 + \beta_1 L)\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 + \\ +(\beta_0^3 L^3 + 2.5\beta_0\beta_1 L^2 + \beta_2^{\overline{\mathrm{MS}}})\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^3 + \ldots\Bigg)$$

We will consider a V-scheme i.e. a scheme in which the static potential in momentum space will be as the Coulomb potential

$$\mathrm{V}(\vec{q}^{\,2}, \mu^2, \alpha_s(\mu^2)) = -4\pi C_F \frac{\alpha_{s,\mathrm{V}}(\vec{q}^{\,2})}{\vec{q}^{\,2}}$$

Definition of the gauge- and scheme-independent V-scheme

Hence one can obtain the perturbative expansion of the strong coupling constant $\alpha_{s,\mathrm{V}}(\vec{q}^{\,2})$ in V-scheme

$$\alpha_{s,\mathrm{V}}(\vec{q}^{\,2}) = \alpha_s(\mu^2)P(\alpha_s(\mu^2), L) = \alpha_s(\mu^2)\sum_{n=0}^{\infty} P_n^{\overline{\mathrm{MS}}}(L)\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^n ,$$

where polynomials $P_n^{\overline{\mathrm{MS}}}(L)$ are expressed through higher order PT QCD corrections to the static potential and equal:

$$P_0^{\overline{\mathrm{MS}}} = 1 ,$$

$$P_1^{\overline{\mathrm{MS}}}(L) = a_1^{\overline{\mathrm{MS}}} + \beta_0 L ,$$

$$P_2^{\overline{\mathrm{MS}}}(L) = a_2^{\overline{\mathrm{MS}}} + (2a_1^{\overline{\mathrm{MS}}}\beta_0 + \beta_1)L + \beta_0^2 L^2 ,$$

$$P_3^{\overline{\mathrm{MS}}}(L) = a_3^{\overline{\mathrm{MS}}} + (3a_2^{\overline{\mathrm{MS}}}\beta_0 + 2a_1^{\overline{\mathrm{MS}}}\beta_1 + \beta_2^{\overline{\mathrm{MS}}})L+$$

$$+(3a_1^{\overline{\mathrm{MS}}}\beta_0^2 + \frac{5}{2}\beta_0\beta_1)L^2 + \beta_0^3 L^3 .$$

Definition of the gauge- and scheme-independent V-scheme

For definition of the renormalized parameter μ_{V} and expansion of the QCD coupling constant in V-scheme through coupling constant in $\overline{\mathrm{MS}}$-scheme at $\vec{q}^{\,2}=\mu_{\mathrm{V}}^2$ is used the effective charge method ECH, developed by G. Grunberg.

At the first step, following the NLO definition of the ECH scheme, we define the effective scale of the V-scheme as

$$\mu_V^2 = \exp[\mathrm{a}_1^{\overline{\mathrm{MS}}}/\beta_0]\mu_{\overline{\mathrm{MS}}}^2$$

At the next step we fix $\vec{q}^{\,2} = \mu_V^2$ and get the following relation between the effective charge of the V-scheme and the QCD coupling constant $\alpha_{s,\overline{\mathrm{MS}}}$:

$$\alpha_{s,V}(\mu_V^2) = \alpha_{s,\overline{\mathrm{MS}}}(\mu_V^2)\left(1 + a_2^{\overline{\mathrm{MS}}}\left(\frac{\alpha_{s,\overline{\mathrm{MS}}}(\mu_V^2)}{4\pi}\right)^2 + a_3^{\overline{\mathrm{MS}}}\left(\frac{\alpha_{s,\overline{\mathrm{MS}}}(\mu_V^2)}{4\pi}\right)^3\right)$$

The QCD $\beta^{\rm V}$-function at the fourth order approximation in the case of the $SU(N_c)$ group

Now it is possible to define the ECH β-function of the static potential, which is the RG β-function in the $\rm V$-scheme

$$\mu_V^2 \frac{\partial(\alpha_{s,V}/4\pi)}{\partial\mu_V^2} = \beta^{\rm V}(a_{s,V}) = -\sum_{i=0}^{\infty} \beta_i^V \left(\frac{\alpha_{s,V}}{4\pi}\right)^{i+2}$$

where $a_{s,V}{=}\alpha_{s,V}/4\pi$. The standard RG equation relates $\beta^{\rm V}$-function to the β-function in the MS-like schemes:

$$\beta^{\rm V}(a_{s,V}(a_{s,\overline{\rm MS}}(\mu_{\rm V}^2)) = \beta^{\overline{\rm MS}}(a_{s,\overline{\rm MS}}(\mu_{\rm V}^2))\frac{da_{s,V}(a_{s,\overline{\rm MS}}(\mu_{\rm V}^2))}{da_{s,\overline{\rm MS}}(\mu_{\rm V}^2)} \ .$$

The QCD $\beta^{\rm V}$-function at the fourth order approximation in the case of the $SU(N_c)$ group

Using the perturbative expansion of the strong coupling constant $\alpha_{s,\rm V}(\vec{q}^{\,2})$ in $\rm V$-scheme through coupling constant in $\overline{\rm MS}$-scheme, one can get the transformation laws of the β-function from one gauge-invariant renormalization scheme to another:

$$\beta_0^{\rm V} = \beta_0^{\overline{\rm MS}} = \frac{11}{3} C_A - \frac{4}{3} T_F n_l \ ,$$

$$\beta_1^{\rm V} = \beta_1^{\overline{\rm MS}} = \frac{34}{3} C_A^2 - 4 C_F T_F n_l - \frac{20}{3} C_A T_F n_l \ ,$$

$$\beta_2^{\rm V} = \beta_2^{\overline{\rm MS}} - a_1^{\overline{\rm MS}} \beta_1 + (a_2^{\overline{\rm MS}} - (a_1^{\overline{\rm MS}})^2)\beta_0 =$$

$$= \left(\frac{206}{3} + \frac{44\pi^2}{3} - \frac{11\pi^4}{12} + \frac{242}{9}\zeta(3)\right) C_A^3 + 2 C_F^2 T_F n_l -$$

$$\left(\frac{445}{9} + \frac{16\pi^2}{3} - \frac{\pi^4}{3} + \frac{704}{9}\zeta(3)\right) C_A^2 T_F n_l - \left(\frac{686}{9} - \frac{176}{3}\zeta(3)\right) C_A C_F T_F n_l$$

$$+ \left(\frac{2}{9} + \frac{224}{9}\zeta(3)\right) C_A T_F^2 n_l^2 + \left(\frac{184}{9} - \frac{64}{3}\zeta(3)\right) C_F T_F^2 n_l^2 \ ;$$

The QCD β^{V}-function at the fourth order approximation in the case of the $SU(N_c)$ group

$$\beta_3^{\mathrm{V}} = \beta_3^{\overline{\mathrm{MS}}} - 2a_1^{\overline{\mathrm{MS}}}\beta_2^{\overline{\mathrm{MS}}} + (a_1^{\overline{\mathrm{MS}}})^2\beta_1 + (2a_3^{\overline{\mathrm{MS}}} - 6a_1^{\overline{\mathrm{MS}}}a_2^{\overline{\mathrm{MS}}} + 4(a_1^{\overline{\mathrm{MS}}})^3)\beta_0 =$$

$$= \left(\frac{-5914367}{4374} + \frac{22}{3}\cdot 502.24(1) - \frac{2728\pi^2}{9} + \frac{341\pi^4}{18} - \frac{15136}{27}\zeta(3)\right)C_A^4 +$$

$$+\left(\frac{4841537}{2187} - \frac{22}{3}\cdot 709.717 - \frac{8}{3}\cdot 502.24(1) + \frac{2752\pi^2}{9} - \frac{172\pi^4}{9} +\right.$$

$$\left.+\frac{18184}{9}\zeta(3)\right)C_A^3 T_F n_l + \left(\frac{-15290}{9} + \frac{1952}{3}\zeta(3) + \frac{1760}{3}\zeta(5)\right)C_A^2 C_F T_F n_l$$

$$+\left(\frac{572}{9} + \frac{2288}{3}\zeta(3) - \frac{3520}{3}\zeta(5)\right)C_A C_F^2 T_F n_l + 46 C_F^3 T_F n_l +$$

$$+\left(\frac{-740860}{729} + \frac{8}{3}\cdot 709.717 - \frac{640\pi^2}{9} + \frac{3208\pi^4}{405} - \frac{5696}{9}\zeta(3)\right)C_A^2 T_F^2 n_l^2 +$$

$$+\left(-\frac{232}{9} - \frac{1024}{3}\zeta(3) + \frac{1280}{3}\zeta(5)\right)C_F^2 T_F^2 n_l^2 +$$

The QCD β^{V}-function at the fourth order approximation in the case of the $SU(N_c)$ group

$$+\left(\frac{9328}{9}-448\zeta(3)-\frac{640}{3}\zeta(5)\right)C_A C_F T_F^2 n_l^2+$$

$$+\left(\frac{9376}{81}-\frac{512\pi^4}{405}+\frac{128}{27}\zeta(3)\right)C_A T_F^3 n_l^3+\left(\frac{256}{3}\zeta(3)-128\right)C_F T_F^3 n_l^3+$$

$$+\left(\frac{-80}{9}+\frac{704}{3}\zeta(3)\right)\frac{d_A^{abcd}d_A^{abcd}}{N_A}+\left(\frac{512}{9}-\frac{1664}{3}\zeta(3)\right)\frac{d_F^{abcd}d_A^{abcd}}{N_A}n_l+$$

$$+\left(\frac{-704}{9}+\frac{512}{3}\zeta(3)\right)\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_l^2-$$

$$-\frac{22}{3}\cdot 56.83(1)\,C_A\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_l-\frac{22}{3}\cdot 136.39(12)\,C_A\frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

$$+\frac{8}{3}\cdot 56.83(1)\frac{d_F^{abcd}d_F^{abcd}}{N_A}T_F n_l^2+\frac{8}{3}\cdot 136.39(12)\frac{d_F^{abcd}d_A^{abcd}}{N_A}T_F n_l \quad .$$

The QCD β^{V}-function at the fourth order approximation in the case of the $SU(3)$ group

$$\beta_0^{\mathrm{V}} = 11 - 0.666666 n_l \ ,$$
$$\beta_1^{\mathrm{V}} = 102 - 12.66666 n_l \ ,$$
$$\beta_2^{\mathrm{V}} = 4224.181 - 746.0062 n_l + 20.87191 n_l^2 \ ,$$
$$\beta_3^{\mathrm{V}} = 43175.06(6.43) - 12951.700(390) n_l + 706.9658(6) n_l^2 - 4.87214 n_l^3 .$$

The errors of the first three terms in equality for β_3^{V} are defined as the mean square error $\sigma = \sqrt{\sum_{i=1}^{k} \sigma_i^2}$, where σ_i are the numerical errors that arise from multiplication of the factor $2\beta_0$ by computed errors of the corresponding $\overline{\mathrm{MS}}$-scheme numbers for $a_3^{(1)}$ and $a_3^{(0)}$. It should note that the property of the *scheme-independence* of the coefficients β_i^{V} within the gauge-independent MS-like schemes is the consequence of application of the ECH approach to the static potential.

The guess about analytical representation of the numerical terms in the $SU(N_c)$ expression for $\beta_3^{\rm V}$

There is the general rule, that the rate of transcendentality structure is increasing with rising order of PT calculations. Following this general rule and considering the terms in the expressions for $\beta_2^{\rm V}$ and $\beta_3^{\rm V}$, we claim that the evaluated numerically contributions in the expressions for the $a_3^{(1)}$ and $a_3^{(0)}$-coefficients, can be decomposed in terms of rational and transcendental numbers in the following way

$$
\begin{aligned}
709.717 &= R_1 + R_2\pi^2 + R_3\pi^4 + R_4\zeta(3) + R_5\pi^2\zeta(3) + R_6\zeta(5)\ , \\
502.24(1) &= R_7 + R_8\pi^2 + R_9\pi^4 + R_{10}\zeta(3) + R_{11}\pi^2\zeta(3) + R_{12}\zeta(5)\ , \\
56.83(1) &= R_{11} + R_{12}\pi^2 + R_{13}\pi^4 + R_{14}\zeta(3)\ , \\
136.39(12) &= R_{15} + R_{16}\pi^2 + R_{17}\pi^4 + R_{18}\zeta(3)\ ,
\end{aligned}
$$

where R_i are still unknown rational numbers. There are indications, that R_{12} and R_{16} may be really zero.

To the question about e^+e^- R-ratio

The asymptotic structure of the PT series for the β-function in the V-scheme has the non-regular behaviour and differs from the asymptotic structure for the β-function in the $\overline{\mathrm{MS}}$-scheme. In view of this it is interesting whether this non-regular behaviour will manifest itself in the process of studies of scheme-dependence of high-order coefficients for the characteristics of typical physical QCD processes, e.g. for the e^+e^--annihilation R-ratio in the region of direct production of the pair of heavy quarks and antiquarks with $n_f = 4, 5$ number of flavours. We will not consider the case of $n_f = 6$, related to the direct production of the pair of $t\bar{t}$-quarks in the process $e^+e^- \to hadrons$, which may be studies in future if ILC will be built. Indeed, the total cross-section of this process is dominated by the subprocess $e^+e^- \to Z^0 \to hadrons$ and not by the subprocess $e^+e^- \to \gamma \to hadrons$, we are interested in.

R-ratio for e^+e^--annihilation into hadrons

We remind that the e^+e^--annihilation R-ratio is defined as

$$R(s) = \frac{\sigma(\mathrm{e}^+\mathrm{e}^- \to \gamma \to \text{hadrons})}{\sigma_0(\mathrm{e}^+\mathrm{e}^- \to \gamma \to \mu^+\mu^-)} = 12\pi \mathrm{Im}\, \Pi(s + i\varepsilon)$$

where s is the transferred energy,
$\sigma_0(\mathrm{e}^+\mathrm{e}^- \to \gamma \to \mu^+\mu^-) = 4\pi^2\alpha/3\mathrm{s}$, $\Pi(q^2)$ is the QCD expression for the photon vacuum polarization function

$$\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - g_{\mu\nu} q^2)\Pi(q^2) = i \int d^4x\ e^{iqx} \langle 0| T\, j_\mu(x) j_\nu(0)|0\rangle$$

and $j_\mu = \sum\limits_f Q_f \bar{\psi}_f \gamma_\mu \psi_f$ is the electromagnetic hadronic current.
Since the e^+e^--annihilation R-ratio is the RG-invariant quantity, it obeys the RG equation without anomalous dimension term, namely

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right) R(s) = 0\ ,$$

R-ratio for e^+e^--annihilation into hadrons in $\overline{\mathrm{MS}}$-scheme

In the $\overline{\mathrm{MS}}$-scheme the $O(\alpha_s^4)$ approximation for the e^+e^- R-ratio is

$$R^{\overline{\mathrm{MS}}}(s) = 3\sum_f Q_f^2\left(1+4\frac{\alpha_s}{4\pi}+r_1^{\overline{\mathrm{MS}}}\left(\frac{\alpha_s}{4\pi}\right)^2+r_2^{\overline{\mathrm{MS}}}\left(\frac{\alpha_s}{4\pi}\right)^3+r_3^{\overline{\mathrm{MS}}}\left(\frac{\alpha_s}{4\pi}\right)^4\right)$$

For $SU(3)$ group the coefficients $r_i^{\overline{\mathrm{MS}}}$ have following numerical form

$$r_1^{\overline{\mathrm{MS}}} = -1.84472 n_f + 31.7713 \ ,$$

(K. Chetyrkin, A. Kataev, F. Tkachov; M. Dine, J. Sapirstein, 1979)

$$r_2^{\overline{\mathrm{MS}}} = -0.33139 n_f^2 - 76.8085 n_f - 424.763 - 26.4435\delta_f \ ,$$

(S. Gorishny, A. Kataev, S. Larin, 1991; L. Surguladze, M. Samuel, 1991)

$$r_3^{\overline{\mathrm{MS}}} = 5.50812 n_f^3 - 204.1431 n_f^2 + 4806.339 n_f - 40091.67 \\ +(49.0568 n_f - 1521.214)\delta_f .$$

(P. Baikov, K. Chetyrkin, J. Kuhn, 2008). Where the terms with $\delta_f = (\sum_f Q_f)^2/(\sum_f Q_f^2)$, are the singlet contributions (2012).

R-ratio for e^+e^--annihilation into hadrons in V-scheme

In the V-scheme the PT expression for the e^+e^- R-ratio is defined as

$$R^{\rm V} = 3\sum_f Q_f^2\left(1+4\frac{\alpha_{s,\rm V}}{4\pi}+r_1^{\rm V}\left(\frac{\alpha_{s,\rm V}}{4\pi}\right)^2+r_2^{\rm V}\left(\frac{\alpha_{s,\rm V}}{4\pi}\right)^3+r_3^{\rm V}\left(\frac{\alpha_{s,\rm V}}{4\pi}\right)^4\right)$$

Using the ECH approach and the V-scheme relations, we obtain the following general expressions for r_i^V :

$$r_1^{\rm V} = r_1^{\overline{\rm MS}} - 4a_1^{\overline{\rm MS}} = 2.59972n_f - 9.5620,$$

$$r_2^{\rm V} = r_2^{\overline{\rm MS}} - 4a_2^{\overline{\rm MS}} - 2a_1^{\overline{\rm MS}}r_1^{\rm V} =$$

$$= 0.50749n_f^2 + 113.6320n_f - 2054.140 - 26.4435\delta_f \ ,$$

$$r_3^{\rm V} = r_3^{\overline{\rm MS}} - 4a_3^{\overline{\rm MS}} - 3a_1^{\overline{\rm MS}}r_2^{\rm V} - (2a_2^{\overline{\rm MS}} + (a_1^{\overline{\rm MS}})^2)r_1^{\rm V} =$$

$$= 3.05815n_f^3 - 144.9455n_f^2 + 3455.279(2)n_f - 20387.90(1.17) -$$

$$-(39.0881n_f + 701.466)\delta_f$$

mMOM-scheme in QCD

For the renormalization constants with arbitrary covariant gauge:

$$A_0^{a\mu} = \sqrt{Z_{\rm A}} A^{a\mu},\ c_0^a = \sqrt{Z_{\rm c}} c^a,\ g_0 = Z_{\rm g} g,\ \lambda_0 = Z_{\rm A} Z_\lambda^{-1} \lambda$$

where A_μ^a, c^a are the gluons and ghosts fields correspondingly, λ is the gauge parameter. The renormalization constant of the gluon-ghost-ghost vertex is $Z_{\rm ccg} = Z_{\rm g} Z_{\rm A}^{1/2} Z_{\rm c}$. The definition of the $\rm mMOM$-scheme is based on this relation. Hence

$$\alpha_s^{\rm mMOM}(\mu^2) = \frac{Z_{\rm A}^{\rm mMOM}(\mu^2)(Z_{\rm c}^{\rm mMOM}(\mu^2))^2}{(Z_{\rm ccg}^{\rm mMOM}(\mu^2))^2} \alpha_s^0 \ .$$

The most important requirements of the mMOM scheme are

$$Z_{\rm ccg}^{\rm mMOM}(\alpha_s^{\rm mMOM}) = Z_{\rm ccg}^{\overline{\rm MS}}(\alpha_s^{\overline{\rm MS}}) \ ,$$

and as a consequence

$$\alpha_s^{\rm mMOM}(\mu^2) = \frac{Z_{\rm A}^{\rm mMOM}}{Z_{\rm A}^{\overline{\rm MS}}} \left(\frac{Z_{\rm c}^{\rm mMOM}}{Z_{\rm c}^{\overline{\rm MS}}} \right)^2 \alpha_s^{\overline{\rm MS}}(\mu^2) \ .$$

R-ratio for e^+e^--annihilation into hadrons in V, $\overline{\rm MS}$ and mMOM in Landau gauge schemes

The energy dependence of coupling constant $a_s = \alpha_s^{\overline{\rm MS}}/(4\pi)$ at the NLO, NNLO, $\rm N^3LO$ approximations is defined through the powers of logarithmic terms $Log = \ln(s/\Lambda^2(n_f))$ as

$$a_s^{NLO} = \frac{1}{\beta_0 Log} - \frac{\beta_1 \ln(Log)}{\beta_0^3 Log^2} \ ,$$

$$a_s^{NNLO} = a_s^{NLO} + \Delta a_s^{NNLO} \ ,$$

$$\Delta a_s^{NNLO} = \frac{1}{\beta_0^5 Log^3}[\beta_1^2 \ln^2(Log) - \beta_1^2 \ln(Log) + \beta_2\beta_0 - \beta_1^2] \ ,$$

$$a_s^{N^3LO} = a_s^{NNLO} + \Delta a_s^{N^3LO} \ ,$$

$$\Delta a_s^{N^3LO} = \frac{1}{\beta_0^7 Log^4}[\beta_1^3(-\ln^3(Log) + \frac{5}{2}\ln^2(Log) + 2\ln(Log) - \frac{1}{2}) - 3\beta_0\beta_1\beta_2 \ln(Log) + \beta_0^2\frac{\beta_3}{2}] \ .$$

K. Chetyrkin, B. Kniehl, M. Steinhauser, 1997

R-ratio for e^+e^--annihilation into hadrons in V, $\overline{\mathrm{MS}}$ and mMOM in Landau gauge schemes

According to the ECH approach, scale parameters $\Lambda(n_f)$ should be chosen so

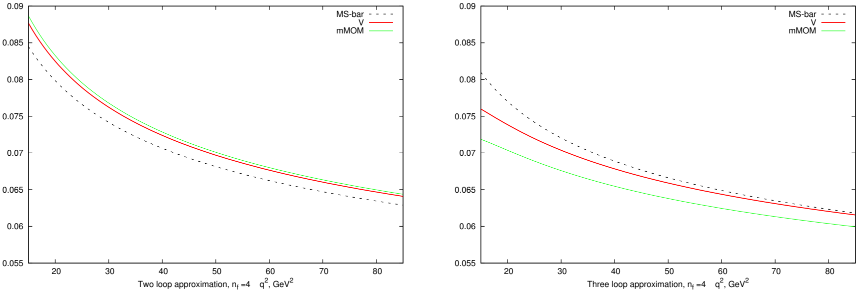
$$\Lambda_{\mathrm{V}}^{(n_f)2} = \Lambda_{\overline{\mathrm{MS}}}^{(n_f)2}\exp[a_1^{\overline{\mathrm{MS}}}(n_f)/\beta_0(n_f)] \ ,$$
$$\Lambda_{\mathrm{mMOM}}^{(n_f)2} = \Lambda_{\overline{\mathrm{MS}}}^{(n_f)2}\exp[(r_1^{\overline{\mathrm{MS}}}(n_f) - r_1^{\mathrm{mMOM}}(n_f))/4\beta_0(n_f)] \ .$$

The numerical values of the Λ_{QCD} in different schemes, MeV				
n_f	the order of approximation ν	$\Lambda_{\overline{\mathrm{MS}}}^{(n_f)}$	$\Lambda_{\mathrm{V}}^{(n_f)}$	$\Lambda_{\mathrm{mMOM}}^{(n_f)}$
4	2	350	500	625
4	3	335	475	600
4	4	330	470	590
5	2	250	340	435
5	3	245	335	430
5	4	240	330	420

R-ratio for e^+e^--annihilation into hadrons in V, $\overline{\mathrm{MS}}$ and mMOM in Landau gauge schemes

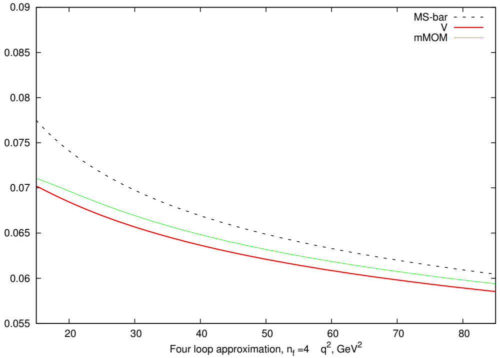
In the cases of $n_f = 4$ of active flavours and $\nu = 2, 3, 4$ the values for $\Lambda^{(n_f=4)}_{\overline{\mathrm{MS}}}$ are fixed from the results (A. Kataev, G. Parente, A. Sidorov, 2003) fits of the Fermilab Tevatron experimental data for the xF_3 structure function of the neutrino-nucleon deep-inelastic scattering process at the $N^{(\nu-1)}LO$ of the theoretical PT results. In the case of $n_f = 5$ the values of $\Lambda^{(n_f=5)}_{\overline{\mathrm{MS}}}$ at $\nu = 2, 3, 4$ were obtained by A. Kataev and V. Kim in 2008 from the related results for $\Lambda^{(n_f=4)}_{\overline{\mathrm{MS}}}$ using the the NLO, NNLO and $\mathrm{N}^3\mathrm{LO}$ matching conditions for the analysis of $\Gamma(H \to b\bar{b})$ PT uncertaintes. The matching point was fixed by the on-shell b-quark mass values, extracted in at different orders of PT from the analysis of heavy quarkonium spectrum with taking into account the Pade estimated value of the coefficient a_3 (A. Penin, M. Steinhauser, 2002).

R-ratio for e^+e^--annihilation into hadrons in V, $\overline{\mathrm{MS}}$ and mMOM in Landau gauge schemes



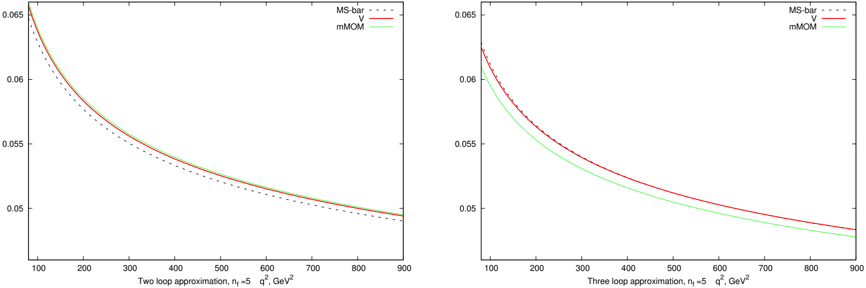
The energy and scheme dependence of the NLO and NNLO approximations for the function $r(s) = R(s)/(3\sum_f Q_f^2) - 1$. It depends on $s = q^2$, where s is measured in GeV^2. For $n_f = 4$ plots are presented for the energy region above the threshold of charmonium production and below threshold of the bottomonium.

R-ratio for e^+e^--annihilation into hadrons in V, $\overline{\rm MS}$ and mMOM in Landau gauge schemes



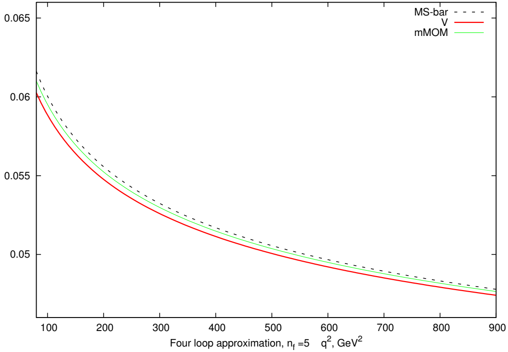
The energy and scheme dependence of the $\rm N^3LO$ approximations for $r(s)$ with $n_f = 4$.

R-ratio for e^+e^--annihilation into hadrons in V, $\overline{\mathrm{MS}}$ and mMOM in Landau gauge schemes



The energy and scheme dependence of the NLO and NNLO approximations for $r(s)$ with $n_f = 5$. We consider the energy region above the threshold of bottomonium production and up to the energies $s = 900\ \mathrm{GeV}^2$, where the subprocess $e^+e^- \to Z^0 \to hadrons$, starts to dominate.

R-ratio for e^+e^--annihilation into hadrons in V, $\overline{\mathrm{MS}}$ and mMOM in Landau gauge schemes



The energy and scheme dependence of the $\mathrm{N}^3\mathrm{LO}$ approximations for $r(s)$ with $n_f = 5$.

QED-limit: the relation of the $\beta^{\mathrm{V}}(\alpha_{\mathrm{V}})$-function with the Gell-Mann–Low function

In QED case $C_A = 0$, $C_F = 1$, $T_F = 1$, $d_A^{abcd} = 0$, $d_F^{abcd} = 1$, $N_A = 1$ and $n_f = N$ – number of leptons. Thus, making the limiting transition to the QED case, we obtain from β^{V} in QCD the β-function in QED in V-scheme:

$$\beta^{\mathrm{V}}_{QED}(a_{\mathrm{V}}) = \frac{4}{3}Na_{\mathrm{V}}^2 + 4Na_{\mathrm{V}}^3 + \Big(-2N + (\frac{64}{3}\zeta(3) - \frac{184}{9})N^2\Big)a_{\mathrm{V}}^4 +$$
$$+\Big(-46N + (104 + \frac{512}{3}\zeta(3) - \frac{1280}{3}\zeta(5) - \frac{8}{3}\cdot 56.83(1))N^2 +$$
$$+(128 - \frac{256}{3}\zeta(3))N^3\Big)a_{\mathrm{V}}^5 + O(a_{\mathrm{V}}^6)$$

QED-limit: the relation of the $\beta^{\rm V}(\alpha_{\rm V})$-function with the Gell-Mann–Low function

Comparing this result with the evaluated analytically by S. Gorishny, A. Kataev, S. Larin and L. Surguladze in 1991 the four-loop approximation of the QED β-function in the MOM-scheme, i. e. of the Gell-Man–Low $\Psi(a_{\rm MOM})$-function, namely with

$$\Psi = \frac{4}{3}Na^2_{\rm MOM} + 4Na^3_{\rm MOM} + \Big(-2N + (\frac{64}{3}\zeta(3) - \frac{184}{9})N^2\Big) a^4_{\rm MOM} +$$
$$\Big(-46N + (104 + \frac{512}{3}\zeta(3) - \frac{1280}{3}\zeta(5))N^2 + (128 - \frac{256}{3}\zeta(3))N^3\Big) a^5_{\rm MOM}$$

we conclude that at the third order of PT $\beta^{\rm V}_{QED}(a_{\rm V})$ and $\Psi(a_{\rm MOM})$ coincide, and start to differ from the fourth order of PT due to contributing to $O(a^5_{\rm V})$ coefficient of the $\beta^{\rm V}$-function from diagrams with additional light-by-light-type scattering contributions through the QED analog of the coefficient $a_3^{(1)}$ in the $\overline{\rm MS}$-scheme, which enter in the definition of the N^2-term of the $\beta^{\rm V}_3$-coefficient.

QED-limit: the relation of the $\beta^{\rm V}(\alpha_{\rm V})$-function with the Gell-Mann–Low function

It is possible to clarify that kinds of N-dependent high-order coefficients of the QED β-function in the V-scheme

$$\beta^{\rm V}(a_{\rm V}) = \sum_{i=0}^{\infty} \beta_i^{\rm V}\left(\frac{\alpha_{\rm V}}{4\pi}\right)^{i+2} = \beta_0^{\rm V[1]} N \left(\frac{\alpha_{\rm V}}{4\pi}\right)^2 + \sum_{i=1}^{\infty}\sum_{l=1}^{i} \beta_i^{\rm V[l]} N^l \left(\frac{\alpha_{\rm V}}{4\pi}\right)^{i+2}$$

will also receive additional contributions and what kinds of the N-dependent coefficients of the QED $\beta^{\rm V}$-function will coincide with the similar expressions for the Ψ-function, which we will define as

$$\Psi(a_{\rm MOM}) = \Psi_0^{[1]} N \left(\frac{\alpha_{\rm MOM}}{4\pi}\right)^2 + \sum_{i=1}^{\infty}\sum_{l=1}^{i} \Psi_i^{[l]} N^l \left(\frac{\alpha_{\rm MOM}}{4\pi}\right)^{i+2} ,$$

$$\beta_i^{\rm V[l]} = \Psi_i^{[l]} + \Delta\beta_i^{\rm V[l]}$$

where extra terms $\Delta\beta_i^{\rm V[l]}$ in the N-dependent contributions to the coefficients of the QED β^V-function appear in the following region of indexes $[i, l] = [i \geq 3, 2 \leq l \leq i-1]$.

QED-limit: the relation of the $\beta^{\mathrm{V}}(\alpha_{\mathrm{V}})$-function with the Gell-Mann–Low function

In the cases of $[i, l] = [i \geq 3, l = 1 \text{ or } i]$ the proportional to $N^{[l]}$-coefficients of the β^{V} and Ψ-function are the same. For $i = 4$ it can read

$$\beta_4^{\mathrm{V}[1]} = \Psi_4^{[1]} = \frac{4157}{6} + 128\zeta(3) \ ,$$

$$\beta_4^{\mathrm{V}[4]} = \Psi_4^{[4]} = -\frac{8756}{9} + \frac{3584}{9}\zeta(3) + \frac{5120}{9}\zeta(5)$$

P. Baikov, K. Chetyrkin, J. Kuhn, J. Rittinger, 2012.

Note, that scheme-independence of the linear in N-contribution is the consequence of the "quenched" QED *conformal symmetry* property.

QED-limit: the relation of the $\beta^{\mathrm{V}}(\alpha_{\mathrm{V}})$-function with the Gell-Mann–Low function

We present the QED numerical results for β-functions

$$
\begin{aligned}
\beta_2^{\overline{\mathrm{MS}}} &= -2N + 4.88888N^2 \ , \\
\beta_3^{\overline{\mathrm{MS}}} &= -46N + 82.9753N^2 + 5.06995N^3 \ , \\
\Psi_2 &= -2N + 5.19943N^2 \ , \\
\Psi_3 &= -46N + 133.2714N^2 - 25.42447N^3 \ , \\
\beta_2^{\mathrm{V}} &= -2N + 5.19943N^2 \ , \\
\beta_3^{\mathrm{V}} &= -46N + 284.818(26)N^2 - 25.42447N^3 \ .
\end{aligned}
$$

Note once more, that the first three coefficient of the β^{V}-function and of the Ψ-function are the same and start to differ at the fourth order of PT on extra sizable contribution

$$\beta_3^{\mathrm{V}} = \Psi_3 - 151.54(2)N^2$$

This additional contribution arises from the light-by-light-type scattering contribution, which is typical to the V-scheme.

Conclusion

- The fourth term of the PT expression for β^{V}-function in the general case of $\mathrm{SU(N_c)}$ group is obtained
- The comparison between the fourth-order expressions for the e^+e^- annihilation R-ratio, obtained in the $\overline{\mathrm{MS}}$, in the Landau-gauge variant of the mMOM and in the gauge-independent V-scheme leads to drastical decrease of the scheme-dependence for the case of $n_f = 5$ number of active flavours in particular
- Considering the QED limit of the $\mathrm{SU(N_c)}$-group β^{V}_{QCD}-function we observe that its perturbative expression is starting to differ from the perturbative expression for the Gell-Mann–Low Ψ-function at $O(\alpha_{\mathrm{V}}^5)$-level
- Starting from the fourth-order perturbative approximation two N-dependent terms in the coefficients of the perturbative expansions of the β^{V}- and Ψ-functions will always coincide

Thank you for your attention!