CALC2015

Dubna, JINR, Russia, July 20 - July 30 , 2015

β-expansion in QCD and generalization of BLM optimization procedure
[Based on Phys.Rev. D 91, 014007 (2015)]

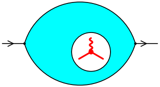
A. L. Kataev and S. V. Mikhailov

INR RAS, Moscow,Russia BLTP, JINR, Dubna, Russia

July 26, 2015

Outline

Objects:



renormalization group (RG) invariant, single-scale q^2 quantities, e.g., Adler D-function, DIS sum rules, ...

Goal: RG optimization of PT (truncated) series with the "best" μ^2

$$D(q^2/\mu^2, \alpha_s(\mu^2)) = d_0 + \sum_{n=1}^{N} \alpha_s^n(\mu^2) d_n(q^2/\mu^2)$$

.

Known example – BLM approach at NLO, [Brodsky&Lepage&Mackenzie(1983)]
"...One, therefore, has to address the question of
what is the "best" choice for μ^2 within a given scheme, usually $\overline{\mathrm{MS}}$. There is no definite answer to this question – **higher-order corrections do not "fix" the scale**, rather they render the theoretical predictions less sensitive to its variation."
(I. Hinchliffe, PDG booklet 2002)

higher-order corrections can fix the scale in accordance with RG transformation

Plan of Presentation

1. β structure of perturbative expansion for RGI quantities
   1. Introduction of Adler $\boldsymbol{D}$-function and $\boldsymbol{S}^{NS}_{Bjp}$ as the examples
   2. Expansion: from series $\{d_n\}$ to matrixes $\{D_{nl}\}$
   3. How do we identify the $\boldsymbol{\beta}$–terms D_{nl}?
2. Explicit results in N^{2}LO: Adler $\boldsymbol{D}^{NS}$-function, Bjorken sum rules $\boldsymbol{S}^{NS}_{Bjp}$
   1. The role of **generalized Crewther relation**
   2. Discussion of another $\boldsymbol{S}^{NS}_{Bjp}$ structure
3. What is “Principe of maximum conformality”, PMC ?
4. What is the optimization of PT series? A few partial results.
5. Conclusion

1.1 Adler D-function and Bjorken sum rule S^{NS}_{Bjp} in $\overline{\rm MS}$ scheme

Studies of different ways of resummation and fixation of scale-scheme uncertainties of HO QCD PT predictions $\bigoplus$ understanding of basic features and symmetries beyond these representations are important theoretically and phenomenologically.

$$D^{\rm EM}(Q^2/\mu^2, a_s(\mu^2)) = \left(\sum_i q_i^2\right) d_R D^{\rm NS}\,(Q^2/\mu^2, a_s(\mu^2)) + \left(\sum_i q_i\right)^2 d_R D^{\rm S}\,(Q^2/\mu^2, a_s(\mu^2))$$

$$R_{e^+e^-}(s) \equiv R(s, \mu^2 = s) = \frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon} \frac{D^{\rm EM}(\sigma/\mu^2; a_s(\mu^2))}{\sigma} d\sigma \,\Big|_{\mu^2=s}$$

$$D^{\rm NS}(Q^2/\mu^2, a_s(\mu^2)) \stackrel{\mu^2=Q^2}{\longrightarrow} D^{\rm NS}(a_s(Q^2)) = 1 + \sum_{l\geq 1} d_l^{\rm NS} a_s^l(Q^2) \qquad (1)$$

$$S_{Bjp}(Q^2) = \int_0^1 [g_1^{lp}(x, Q^2) - g_1^{ln}(x, Q^2)]dx = \frac{g_A}{6}\; C_{Bjp}(Q^2/\mu^2, a_s(\mu^2))$$

$$C_{Bjp}(a_s) = C^{\rm NS}_{Bjp}(a_s) + \left(\sum_i q_i\right) C^{\rm S}_{Bjp}(a_s)$$ **[Larin(2013), Baikov&Chetyrkin&Kuhn(2015)]**

$$C^{\rm NS}_{Bjp}(Q^2/\mu^2, a_s(\mu^2)) \stackrel{\mu^2=Q^2}{\longrightarrow} 1 + \sum_{l\geq 1} c_l^{\rm NS} a_s^l(Q^2) \qquad (2)$$

Coefficients $c_l^{\rm NS}$, $d_l^{\rm NS}$ are combinations of Casimirs in $\overline{\rm MS}$ scheme.

1.2 From series $\{d_n\}$ to matrixes $\{D_{nl}\}$

Instead of Scalar Representation

$$D^{NS} - 1 = \sum_{n\geq 1} a_s^n(Q^2) d_n = (\overline{a_s}\overline{d}) \quad (3)$$

we use Matrix Representation to fix the β-structure

$$D^{NS} - 1 = \sum_{n\geq 1}\sum_{l} a_s^n(Q^2) D_{nl} B_l = (\overline{a_s} D \overline{d}) \quad (4)$$

B_l-products of β-function coefficients, $\boldsymbol{d_n = D_{nl} B_l}$, elements $\boldsymbol{D_{nl}}$ do not depend on the numbers of flavours $\boldsymbol{n_f}$, they have the form

$$d_1 = d_1[0] = \frac{3}{4} C_F,$$

$\underline{d_2} = \underline{\beta_0 d_2[1] + \boldsymbol{d_2}[\boldsymbol{0}]}$, **– the Basis of BLM procedure**

$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + \boldsymbol{d_3}[\boldsymbol{0}],$$

$$d_4 = \beta_0^3 d_4[3] + \beta_1\beta_0 d_4[1,1] + \beta_2 d_4[0,0,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + \boldsymbol{d_4}[\boldsymbol{0}$$

β-expansions has been suggested in **[Mikhailov, Quarks2004, JHEP(2007)]**;
elaborated further in **[Kataev&Mikhailov, TMP(2012), PRD(2015)]**;
studied and used in **[Brodsky&Wu et al (2012-2015)]**
Such expansion should **to exist for any RGI quantity,**
it fixes the β-structure of RGI and provides New dynamical information,
terms $\boldsymbol{d_n}[\boldsymbol{0}]$ survives at conformal symmetry limit $\boldsymbol{\beta_i} \to 0$.

1.3 How do we identify the β-terms $d_n[l]$?

For first glance Casimirs and n_f dependence of d_n do **not enough** to uniquely identify the β-terms. This is a **Separate and Nontrivial task** for order $n \geq 3$. For NNLO an **additional degrees of freedom** can be used, e.g., MSSM gluino

$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + \boldsymbol{d_3[0]}$$

to disentangle β_0 and β_1, we used the number $n_{\tilde{g}}$ of MSSM gluino, together with the number of the flavours n_f of quark,
$\beta_0 \to \beta_0(n_f, n_{\tilde{g}}), \beta_1 \to \beta_1(n_f, n_{\tilde{g}})$

$$\begin{aligned}
d_1 &= 3\mathrm{C_F}; \quad d_2[1] = \frac{11}{2} - 4\zeta_3; \quad d_2[0] = \frac{\mathrm{C_A}}{3} - \frac{\mathrm{C_F}}{2} = \frac{1}{3}; \\
d_3[2] &= \frac{302}{9} - \frac{76}{3}\zeta_3 \approx 3.10345; \; d_3[0,1] = \frac{101}{12} - 8\zeta_3 \approx -1.19979; \\
d_3[1] &= \mathrm{C_A}\left(-\frac{3}{4} + \frac{80}{3}\zeta_3 - \frac{40}{3}\zeta_5\right) - \mathrm{C_F}\left(18 + 52\zeta_3 - 80\zeta_5\right) \approx 55.7005; \\
\boldsymbol{d_3[0]} &= \left(\frac{523}{36} - 72\zeta_3\right)\mathrm{C_A^2} + \frac{71}{3}\mathrm{C_A C_F} - \frac{23}{2}\mathrm{C_F^2} \approx -573.9607 \;,
\end{aligned}$$

That was obtained in [Mikhailov(2007)] using QCD + $n_{\tilde{g}}$ multiplet of massless gluino, contributing to $d_3(n_f, n_{\tilde{g}})$ **from the result of [Chetyrkin(1997)], see also [Brodsky at el(2015)].**

2.1 Explicit results in N^2LO: The role of **Crewther relation**

The β-expansion for C^{NS}_{Bjp} was obtained (in $\overline{\mathrm{MS}}$) from the **generalized Crewther relation** (CR) for D^{NS} and C^{NS}_{Bjp} **[K&M QFTHEP2010, TMP(2012)]** that provides **additional constraint**

$$\textbf{Crewther relation: } D^{NS}\cdot C^{NS}_{Bjp} = \mathbb{1} + \sum_{n\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s) \tag{5}$$

$$\begin{aligned} C^{NS}_{Bjp} &= 1 + \sum_{n\geq 1}\sum_{l} a_s^n(Q^2) C_{nl} B_l \\ D^{NS} &= 1 + \sum_{n\geq 1}\sum_{l} a_s^n(Q^2) D_{nl} B_l \end{aligned}$$

The "purely conformal" CR with $\mathbb{1}$ in (5) relates $\boldsymbol{c_n[0]}$ to $\boldsymbol{d_n[0]}$

$$\begin{aligned} c_n[0] + d_n[0] &= \sum_{l=1}^{n-1} d_l[0] c_{n-l}[0], \\ \textbf{e.g., } \boldsymbol{c_3[0]} &= -d_3[0] + 2d_1 d_2[0] - (d_1)^3, \end{aligned}$$

While the "breaking conformality" $\boldsymbol{\beta}$ terms generate the relations for other elements, e.g., $\boldsymbol{c_3[0,1]} = d_3[0,1] - d_2[1] + c_2[1]$

2.1 Explicit results in N^{2}LO: Bjorken sum rules S^{NS}_{Bjp}

from the β-expansion for $D^{\rm NS}$ with the help of $\overline{\rm MS}$-generalized Crewther relation we fix the β-expansion of C^{NS}_{Bjp} **[Kataev&Mikhailov(2010-2012)]**

c_3 elements Crewther relation

$D \to C$	$\mathbb{1}$	$\sim \beta$
all c_3 $\leftarrow$ elements	$c_3[0]$	
all c_3 $\leftarrow$ elements		$c_3[0,1]$

New prediction: we obtain C^{NS}_{Bjp} expression in QCD+$n_{\tilde{g}}$, $\beta_i \to \beta_i(n_f, n_{\tilde{g}})$

$$c_3 = \beta_0^2 c_3[2] + \beta_1 c_3[0,1] + \beta_0 c_3[1] + c_3[0]$$

$$c_1^{\rm NS} = -3\,{\rm C_F};\ c_2[1] = 2;\ c_2[0] = \left(\frac{\rm C_A}{3} - \frac{7}{2}{\rm C_F}\right);$$

$$c_3[2] = \frac{115}{18};\ c_3[0,1] = \left(\frac{59}{12} - 4\zeta_3\right);$$

$$c_3[1] = -\left(\frac{166}{9} - \frac{16}{3}\zeta_3\right){\rm C_F} - \left(\frac{215}{36} - 32\zeta_3 + \frac{40}{3}\zeta_5\right){\rm C_A};$$

$$c_3[0] = \left(\frac{523}{36} - 72\zeta_3\right){\rm C_A^2} + \frac{65}{3}{\rm C_F C_A} + \frac{\rm C_F^2}{2} \approx -560.627.$$

These results can be checked by direct analytical calculations in the QCD+$n_{\tilde{g}}$ multiplets of light gluinos in the $\overline{\rm MS}$-scheme.

2.2 Discussion of $\boldsymbol{S^{NS}_{Bjp}}$ structure from [Brodsky et al]

$$c_1 = c_1[0] = -\frac{3}{4}C_F \quad , \quad c_2 = \beta_0^2 c_2[1] + c_2[0]$$

$$c_3 = \beta_0^3 c_3[2] + \beta_1 c_3[0,1] + \boxed{\beta_0 c_3[1]} + c_3[0]$$

$$c_4 = \beta_0^3 c_4[3] + \beta_1\beta_0 c_4[1,1] + \beta_2 c_4[0,0,1] + \beta_0^2 c_4[2] + \boxed{\beta_1 c_4[0,1]} + \boxed{\beta_0 c_4[1]} + c_4[0]$$

Note that the terms in **boxes** can not be eliminated.

Without them in **[K&M(2010-2012)]** results the powers of β-function will be spoiled. Indeed the polynomial $P_n(a_s)$ at the powers of β-function contain these terms and they can not be neglected in the process of constructing Principle of Maximal Conformality by **[Brodsky et al]**

$$D^{NS} C^{NS}_{Bjp} = 1 + \sum_{n\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s)$$

$$P_1(a_s) = -a_s\left(c_2[1] + d_2[1]\right) - a_s^2\left(\boxed{c_3[1]} + \boxed{d_3[1]} + d_1(c_2[1] - d_2[1])\right) - a_s^3\delta_1$$

$$\delta_1 = c_4[1] + d_4[1] + d_1\left(\boxed{c_3[1]} - \boxed{d_3[1]}\right) + d_2[0]c_2[1] + d_2[1]c_2[0]$$

$$P_2(a_s) = a_s\left(c_3[2] + d_3[2] + a_s\left(\boxed{c_4[2]} + \boxed{d_4[2]} - d_1(c_3[2] - d_3[2])\right)\right)$$

If we neglect them the results of β expansion will not agree with the values of the factorized terms, which follow from the exact analytic calculations in the $\overline{\mathrm{MS}}$-scheme.

3. What is "Principe of maximum conformality", PMC ?

PMC by **[Brodsky et al(2011-2015)]** **in our realization:** we consider first β-expansion for $D^{NS}(a_s(t = \ln(Q^2/\Lambda^2)))$ and find $a_s(t_1, t)$ to cancel a part of β-expansion. In each new order of PT we define the new scale Q_i^2, absorbing the β-function coefficients into the scale(s). Finally we have the sequence of shifts $\{\mathbf{\Delta_0}, \mathbf{\Delta_1}, \ldots\}$ from t to t_1, $(a_s(t), t) \to (a_s(t_1), t_1)$. The general scheme to fix (a_1, t_1) looks like:

$$\begin{aligned}
\ln(Q^2/\Lambda^2) - \ln(Q_1^2/\Lambda^2) &\equiv t - t_1 = \Delta, \\
&\quad \text{Expanding } \Delta \text{ in } a_1 : \\
\Delta &= \mathbf{\Delta_0} + a_1\beta_0 \cdot \mathbf{\Delta_1} + (a_1\beta_0)^2 \cdot \mathbf{\Delta_2} + \ldots, \\
\bar{a}(t) = \bar{a}(\Delta, a_1) &= a_1 - \beta(a_1)\Delta + \beta'(a_1)\beta(a_1)\frac{\Delta^2}{2} + \ldots . \qquad (6)
\end{aligned}$$

At first time an expansion of Δ in a_s series was done [Grunberg&Kataev(1992)]

$$\begin{aligned}
\bar{a}^1 d_1 \to \quad & \bar{a}_1^1 \cdot \quad 1; \\
\bar{a}^2 d_2 \to \quad & \bar{a}_1^2 \cdot \quad \Big[d_2 - \beta_0 \mathbf{\Delta_0}\Big]; \\
\bar{a}^3 d_3 \to \quad & \bar{a}_1^3 \cdot \quad \Big[d_3 - 2{\beta_0}^2 \mathbf{\Delta_0} \cdot d_2 - \beta_1 \mathbf{\Delta_0} + (\beta_0 \mathbf{\Delta_0})^2 - {\beta_0}^2 \mathbf{\Delta_1}\Big]; \\
\bar{a}^4 d_4 \to \quad & \bar{a}_1^4 \cdot \quad \Big[d_4 - 3{\beta_0}^3 \mathbf{\Delta_0} \cdot d_3 + \Big(3{\beta_0}^3 \mathbf{\Delta_0^2} - 2\beta_1\beta_0 \mathbf{\Delta_0}\Big) d_2 \\
& \qquad -\beta_2 \mathbf{\Delta_0} + \frac{5}{2}\beta_1\beta_0 \mathbf{\Delta_0^2} - (\beta_0 \mathbf{\Delta_0})^3 + \ldots - {\beta_0}^3 \mathbf{\Delta_2}\Big] \\
\ldots \qquad & \qquad \ldots
\end{aligned}$$

That allows to get expansion in terms of $\boldsymbol{d_n}[\mathbf{0}]$

3. What is "Principe of maximum conformality", PMC ?

The final PT series contains only $\boldsymbol{d_k[0]}$ terms

$$N^2L: \quad D^{NS}(t_1) = 1 + \underline{d_1[0]a_s(t_1) + d_2[0]a_s^2(t_1)}, \quad \textbf{BLM result}$$
$$N^3L: \quad D^{NS}(t_2) = 1 + d_1[0]a_s(t_2) + d_2[0]a_s^2(t_2) + d_3[0]a_s^3(t_2),$$
$$\ldots \qquad \ldots,$$
$$N^nL: \quad D^{NS}(t_2) = 1 + d_1[0]a_s(t_{n-1}) + d_2[0]a_s^2(t_{n-1}) + \ldots + d_n[0]a_s^n(t_{n-1})$$

Now we are able to fix all scales and coefficients in these approximation of PT for $D^{NS}(a_s)$ and $C^{NS}(a_s)$ at the $O(a_s^3)$ – approximation. These coef. in opinion of **[Brodsky et al.]** respect conformal symmetry, this is the PMC. In higher-order level we have the expansion at the single scale $\boldsymbol{t_3}$

.

$$N^4L: \; D^{NS}(t_3) = 1 + d_1[0] \cdot a_s(t_3) + d_2[0] \cdot a_s^2(t_3) + d_3[0] \cdot a_s^3(t_3) + d_4[0] \cdot a_s^4(t_3) \quad (7)$$

But the convergence of such series is not improved,
by virtue of the alternating β–expansion of d_n.
In reality the conformal symmetry exhibits itself here by the Crewther Relation for $D^{NS}(a_s)$ and $C^{NS}(a_s)$.

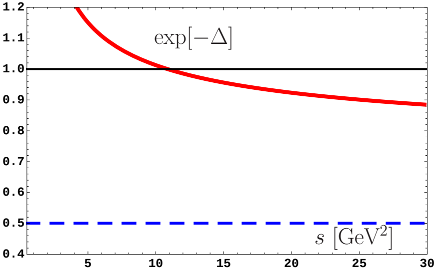
In the works of **[Brodsky et al.]** the results have different scales Q_i^2 in different orders,

$$D^{NS}(Q_i) = 1 + d_1[0] \cdot a_s(\tilde{Q}_1^2) + \tilde{d}_2[0] \cdot a_s^2(\tilde{Q}_2^2) + \tilde{d}_3[0] \cdot a_s^3(\tilde{Q}_3^2) + \tilde{d}_4[0] \cdot a_s^4(\tilde{Q}_4^2)$$

4. What is the optimization of PT series? Optimization for R^{NS}-ratio

One should not remove and absorb **all the β-terms** for the optimization, but leave a part of them for complete cancelation with the $d_n[0]$-term.
There are different ways to "optimize" perturbation series, take fist **BLM**:

$$\textbf{Example}: \quad R^{\text{NS}} = 1 + 3C_{\text{F}} \left\{ \underbrace{a_s'' + \frac{1}{3} \cdot (a_s'')^2}_{\text{BLM} \,--} + 0 \cdot (a_s'')^3 + r_4'' \cdot (a_s'')^4 + \ldots \right\}$$



$$a_s'' = a_s\left(s \cdot e^{-\Delta(Q^2)}\right)$$

$$-\Delta = -0.692 + 3.7\beta_0 a_s'(s)$$

$$a_s'' = a_s\left(s \cdot \frac{1}{2} e^{3.7\beta_0 a_s'(s)}\right)$$

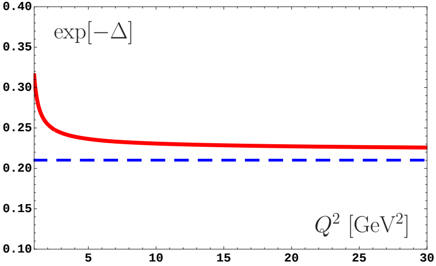
$r_4'' \approx -4740.5$ **at BLM** $r_4' \approx -8559.9$

- The convergence **become better**,
- The domain of applicability **become wider**.

4.4. What is the optimization of PT series? Optimization for $C^{\rm NS}$

One should not remove and absorb **all the β-terms** for the optimization, but leave a part of them for complete cancellation with the $d_n[0]$-term. There are different ways to "optimize" perturbation series, **at 2nd step**:

Example : $C^{\rm Bjp}_{\rm NS}(Q^2) = 1 - 3C_{\rm F}\left\{a''_s + \mathbf{0}\cdot(a''_s)^2 + \mathbf{0}\cdot(a''_s)^3 + c''_4\cdot(a''_s)^4 + \ldots\right\}$



$$a''_s = a_s\left(Q^2\cdot e^{-\Delta(Q^2)}\right)$$

$$-\Delta = -1.56 + 0.396\beta_0 a'_s(s)$$

$$c''_4 \approx 4184.6 \text{ at BLM } c'_4 \approx 6361$$

- The convergence **become better**,
- The domain of applicability **become wider**.

5. Conclusion

1. The β-expansion for RGI quantities exist and is useful in particular for perturbation series optimization.

2. The "optimal series" **are not reduced** to the series that followed to "Principe of maximum conformality" **[Brodsky et al.]**.

3. **K. Chetyrkin** has confirmed by the direct calculation our prediction for $C_{\mathbf{NS}}^{\mathrm{Bjp}}$ **with gluino at** $O(\alpha_s^3)$.

   Many Thanks to Kostja from us!

4. **A.L.K. special point of view on the uniqueness of β-expansion [Broadhurst&Kataev&Maxwell]**