The Nielsen Identities - A useful tool for showing gauge invariance

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What is a Nielsen Identity?

$$\partial_\xi \Gamma_1 = \Gamma_2 \Gamma_3 + \Gamma_4 \Gamma_5 + \ldots$$

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Nielsen identities

are relations between proper vertex functions, they show the dependence on the gauge parameter.

- independent on perturbation theory - they hold at all orders!
- can be used to show gauge independence!

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The Nielsen Identities for the Two-Point Functions of QED and QCD

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A word on Generating Functionals

Generating functional of correlation fuctions

$$Z[J] = \int [d\mu] \exp\left\{ i \int d^4x \, \mathcal{L} + J\phi \right\}$$

Generating functional of connected diagrams

$$W[J] = \log Z[J]$$

Generating functional of 1-PI diagrams or *proper vertex functions*

$$\Gamma[\phi] = W[J] - \int d^4x \, J(x)\phi(x)$$

The general R_ξ-Gauges

Gauge fixing in QED

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi - \bar{c}\partial \cdot \partial c - \frac{(\partial_\mu A)^2}{2\xi}$$

Convention

"Gauge independence" $:=$ "Independence on the gauge parameter"

BRST-formulation

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi - \bar{c}\partial \cdot \partial c + \frac{\xi}{2}B^2 + B\partial \cdot A$$

BRST invariance

Consider the QED Lagrangian:

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi + \frac{\xi}{2}B^2 + B\partial \cdot A - \bar{c}\partial \cdot \partial c$$

It is invariant under the BRST transformations:

$$\delta A_\mu = \epsilon \partial_\mu c \qquad \delta c = 0$$
$$\delta B = 0 \qquad \delta \bar{c} = \epsilon B$$
$$\delta \bar{\psi} = +i\epsilon c \bar{\psi}$$
$$\delta \psi = -i\epsilon c \psi$$

Imposing that the generating functional is invariant under this transformation generates the Ward identities for QED.

A generalised BRST invariance

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi + \frac{\xi}{2}B^2 + B\partial \cdot A - \bar{c}\partial \cdot \partial c + \frac{\chi}{2}\bar{c}B$$

χ is Grassmann valued.

A generalised BRST invariance

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χ is Grassmann valued.

Expand generating functional in χ:

$$Z = \int [d\mu] \exp\left\{ i \int d^4x \mathcal{L}_0 + \frac{\chi}{2}\bar{c}B \right\}$$

$$Z = \int [d\mu] \exp\left\{ i \int d^4x \mathcal{L}_0 \right\}\{i \int d^4x \, (1 + \frac{\chi}{2}\bar{c}B)\}$$

$$Z = \int [d\mu] \exp\left\{ i \int d^4x \mathcal{L}_0 \right\}$$

This additional term leaves the physics invariant!

A generalised BRST invariance

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi + \frac{\xi}{2}B^2 + B\partial\cdot A - \bar{c}\partial\cdot\partial c + \frac{\chi}{2}\bar{c}B$$

χ is Grassmann valued.

Consider a generalised BRST transformation:

$$\begin{aligned} \delta A_\mu &= \epsilon\partial_\mu c & \delta c &= 0 \\ \delta B &= 0 & \delta\bar{c} &= \epsilon B \\ \delta\bar{\psi} &= +i\epsilon c\bar{\psi} & \delta\xi &= \epsilon\chi \\ \delta\psi &= -i\epsilon c\psi & \delta\chi &= 0. \end{aligned}$$

Check invariance:

$$\delta\mathcal{L} = \frac{\epsilon\chi}{2}B^2 + \frac{\chi\epsilon}{2}B^2 = 0$$

Imposing that the generating functional is invariant under this transformation will generate the Nielsen identities for QED!

Breckenbridge, Lavelle, Steele

Nielsen identities of QED: derivation

$$Z = e^W = \int [d\mu] \exp\left\{ i \int d^4x \, \mathcal{L}_{QED} + J_\mu A^\mu + \bar{J}_\psi \psi + \bar{\psi} J_{\bar{\psi}} + B J_B - \bar{\eta}_\psi i c \psi + i c \bar{\psi} \eta_{\bar{\psi}} \right\}$$

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Notice: $\epsilon \frac{\delta \Gamma}{\delta \bar{\eta}_\psi} = -\epsilon i c \psi = \delta \psi$ and $\epsilon \frac{\delta \Gamma}{\delta \eta_{\bar{\psi}}} = \epsilon i c \bar{\psi} = \delta \bar{\psi}$

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Consider the generating functional of proper (i.e. amputated) vertex functions:

$$\Gamma(A^\mu, \psi, \bar{\psi}, B, c, \bar{c}, \chi, \xi, \bar{\eta}_\psi, \eta_{\bar{\psi}}) =$$

$$W(J_\mu, J_{\bar{\psi}}, \bar{J}_\psi, J_B, \eta_{\bar{\psi}}, \bar{\eta}_\psi, \chi, \xi) - \int d^4x\, J_\mu A^\mu + J_B B + \bar{J}_\psi \psi + \bar{\psi} J_{\bar{\psi}}$$

Nielsen identities of QED: derivation

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The variation of Γ is

$$\delta\Gamma = \delta A_\mu \frac{\delta\Gamma}{\delta A_\mu} + \delta\psi \frac{\delta\Gamma}{\delta\psi} + \delta\bar{\psi} \frac{\delta\Gamma}{\delta\bar{\psi}} + \delta\bar{c} \frac{\delta\Gamma}{\delta\bar{c}} + \delta\xi \frac{\partial\Gamma}{\partial\xi} + \delta B \frac{\delta\Gamma}{\delta B} + \delta\chi \frac{\delta\Gamma}{\delta\chi} + \delta c \frac{\delta\Gamma}{\delta c}$$

Nielsen identities of QED: derivation

Using the generalised BRST symmetry...

$$\delta A_\mu = \epsilon \partial_\mu c$$
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$$\delta \psi = -i\epsilon c \psi = \frac{\delta \Gamma}{\delta \bar{\eta}_{\psi}}$$

$$\delta c = 0$$
$$\delta \bar{c} = \epsilon B$$
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$$\delta \chi = 0.$$

...we can obtain the identity

$$0 = \partial_\mu c \frac{\delta \Gamma}{\delta A_\mu} + \frac{\delta \Gamma}{\delta \bar{\eta}_\psi} \frac{\delta \Gamma}{\delta \psi} + \frac{\delta \Gamma}{\delta \eta_{\bar{\psi}}} \frac{\delta \Gamma}{\delta \bar{\psi}} + B \frac{\delta \Gamma}{\delta \bar{c}} + \chi \frac{\partial \Gamma}{\partial \xi}$$

...and after some more simplifications...

$$0 = \frac{\partial \Gamma}{\partial \xi} + B \frac{\delta^2 \Gamma}{\delta \chi \delta \bar{c}} + \frac{\delta^2 \Gamma}{\delta \chi \delta \bar{\eta}_\psi} \frac{\delta \Gamma}{\delta \psi} + \frac{\delta^2 \Gamma}{\delta \chi \delta \eta_{\bar{\psi}}} \frac{\delta \Gamma}{\delta \bar{\psi}}$$

Nielsen identities of QED: derivation

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QED- Nielsen identities for generating functionals (fermionic part)

$$-\partial_\xi\Gamma = \Gamma_{\chi\bar{\eta}_\psi}\Gamma_\psi + \Gamma_{\chi\eta_{\bar{\psi}}}\Gamma_{\bar{\psi}}$$

The Mass of the Electron

The Mass of the Electron

QED- Nielsen identities for generating functionals (fermionic part)

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Nielsen idetintity for the fermion two point function

$$-\frac{\partial}{\partial\xi}\Gamma_{\psi\bar{\psi}} = \Gamma_{\psi\bar{\eta}_\psi\chi}\Gamma_{\bar{\psi}\psi} + \Gamma_{\psi\bar{\psi}}\Gamma_{\bar{\psi}\eta_{\bar{\psi}}\chi}$$

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On shell

$$\frac{\partial}{\partial\xi}\Gamma_{\psi\bar{\psi}}|_{p^2=M^2} = 0$$

The Mass of the Electron

In general: $\Gamma_{\psi\bar{\psi}} = A(p^2)\not{p} - B(p^2)$ and $A(M^2)M = B(M^2)$.

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$$\left(\frac{\partial A(p^2)}{\partial \xi}\not{p} - \frac{\partial B(p^2)}{\partial \xi}\right)\Big|_{p^2=M^2} = M\frac{\partial A(M^2)}{\partial \xi} - \frac{\partial B(M^2)}{\partial \xi} = 0$$

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$$\left[A + \frac{\partial A}{\partial M} - \frac{\partial B}{\partial M}\right]\frac{\partial M}{\partial \xi} = \left[\frac{\partial B(M^2)}{\partial \xi} - M\frac{\partial A(M^2)}{\partial \xi}\right] = 0 \rightarrow \frac{\partial M}{\partial \xi} = 0$$

First application of the Nielsen identities

The on-shell mass of the electron is gauge independent.

BRST-Lagrangian for the Standard Model

$$\begin{aligned}\mathcal{L}_\chi &= -\frac{\chi_g}{2\xi_g}\,\bar{c}^a\,\partial_\mu G^{a,\mu} - \frac{\chi_A}{2\xi_A}\,\bar{c}^A\,\partial_\mu A^\mu - \frac{\chi_Z}{2\xi_Z}\,\bar{c}^Z\left(\partial_\mu Z^\mu + \xi_Z M_Z G_0\right)\\ &- \frac{\chi_W}{2\xi_W}\left[\bar{c}^+\left(\partial^\mu W^-_\mu - i\xi_W M_W G^-\right) + \bar{c}^-\left(\partial^\mu W^+_\mu + i\xi_W M_W G^+\right)\right]\end{aligned}$$

Gambino, Grassi 2000

BRST-Lagrangian for the Standard Model

$$
\begin{aligned}
\mathcal{L}_{BRST} &= \gamma_3^{\mu}\left\{c_W\partial_\mu c^Z - s_W\partial_\mu c^A - ig\left[W_\mu^+ c^- - W_\mu^- c^+\right]\right\} \\
&+ \gamma_W^{\mp\mu}\left\{\partial_\mu c^\pm \mp ieW_\mu^\pm\left(c^A - \frac{c_W}{s_W}c^Z\right) \pm iec^\pm\left[A_\mu - \frac{c_W}{s_W}Z_\mu\right]\right\} \\
&+ \gamma^{a\mu}\left\{\partial_\mu c^a - g_s f^{abc}G_\mu^b c^c\right\} - \gamma_{c^3}\left\{igc^+c^-\right\} + \gamma_{c^\mp}\left\{\mp\frac{ie}{2}c^\pm\left(c^A - \frac{c_W}{s_W}c^Z\right)\right\} \\
&+ \gamma_{c^a}\left\{\frac{g_s}{2}f^{abc}c^bc^c\right\} + \gamma^H\left\{\frac{ig}{2}\left[G^+c^- - G^-c^+\right] + \frac{g}{2c_W}G^0c^Z\right\} \\
&+ \gamma^{\mp}\left\{\pm\frac{ig}{2}\left[H + v \pm iG^0\right]c^\pm \mp ieG^\pm\left(c^A - \frac{c_W^2 - s_W^2}{2c_Ws_W}c^Z\right)\right\} \\
&+ \gamma^0\left\{\frac{g}{2}\left[G^+c^- + G^-c^+\right] - \frac{g}{2c_W}\left(H + v\right)c^Z\right\} \\
&+ i\left(\overline{\eta}_\nu, \overline{\eta}_l^L\right)\begin{pmatrix} \frac{g}{\sqrt{2}}\,l^L\,c^+ + \frac{g}{2}\,\frac{c^Z}{c_W}\nu \\ \frac{g}{\sqrt{2}}\nu c^- - e\left[Q_lc^A + \left(\frac{1}{2s_W} + Q_ls_W\right)\frac{c^Z}{c_W}\right]l^L \end{pmatrix} \\
&+ i\left(\overline{\eta}_u^L, \overline{\eta}_d^L\right)\begin{pmatrix} \frac{gV_{ud}}{\sqrt{2}}d^Lc^+ - e\left[Q_uc^A - \left(\frac{1}{2s_W} - Q_us_W\right)\frac{c^Z}{c_W}\right]u^L + g_s\frac{\lambda^a}{2}u^Lc_a \\ \frac{gV_{ud}^*}{\sqrt{2}}u^Lc^- - e\left[Q_dc^A + \left(\frac{1}{2s_W} + Q_ds_W\right)\frac{c^Z}{c_W}\right]d^L + g_s\frac{\lambda^a}{2}d^Lc_a \end{pmatrix} \\
&- i\,\overline{\eta}_l^R\left\{eQ_l\left(c^A + \frac{s_W}{c_W}c^Z\right)l^R\right\} + i\,\overline{\eta}_u^R\left\{-e\,Q_u\left(c^A + \frac{s_W}{c_W}c^Z\right)u^R + g_s\frac{\lambda^a}{2}u^Rc_a\right\} \\
&+ i\,\overline{\eta}_d^R\left\{-e\,Q_d\left(c^A + \frac{s_W}{c_W}c^Z\right)d^R + g_s\frac{\lambda^a}{2}d^Rc_a\right\} + \text{h.c.}
\end{aligned}
$$

Gambino, Grassi 2000

Nielsen identities of the Standard Model

Recall: NI of QED/QCD

$$-\partial_\xi \Gamma = \Gamma_{\chi \bar{\eta}_\psi} \Gamma_\psi + \Gamma_{\chi \eta_{\bar{\psi}}} \Gamma_{\bar{\psi}}$$

Generalising this procedure we can obtain the following relation (in the fermionic sector):

Standard Model Nielsen identities for generating functionals

$$-\partial_\xi \Gamma = \sum_{I \text{ fermions}} \left[\Gamma_{\chi \bar{\eta}_I} \Gamma_{\psi_I} + \Gamma_{\chi \eta_I} \Gamma_{\bar{\psi}_I} \right]$$

The actual Nielsen identities for proper vertex functions can now be obtained by functional differentiation with respect to some Standard Model fields. The corresponding BRST- invariant Lagrangian density is rather complicated!

Quark self-energies

Goal

We want to find gauge invariant combinations of quark self-energies!

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$$-\partial_\xi \Gamma_{u_j \bar{u}_i} = \frac{\delta}{\delta u_j} \frac{\delta}{\delta \bar{u}_i} \sum_I \left[\Gamma_{\chi \eta_I} \Gamma_{\bar{\psi}_I} + \Gamma_{\chi \bar{\eta}_I} \Gamma_{\psi_I} \right]$$

Quark self-energies

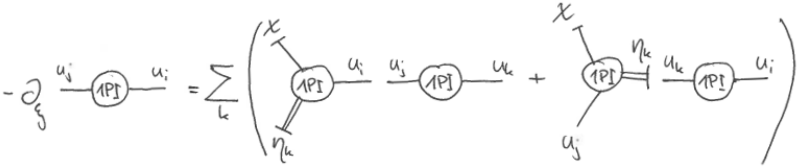
Imposing fermion, lepton and ghost conservation, we find:

$$-\partial_\xi \Gamma_{u_j \bar{u}_i} = \sum_l \left[\Gamma_{\bar{\psi} u_j} \Gamma_{\chi \eta_l \bar{u}_i} + \Gamma_{\chi \bar{\eta}_l u_j} \Gamma_{\psi_l \bar{u}_i} \right]$$

Also, quark number and isospin are conserved → all ψ's must be up-type quarks.

Nielsen identity for quark 2-point function

$$-\partial_\xi \Gamma_{u_j \bar{u}_i}(p) = \sum_{k=1,2,3} \left[\Gamma_{\chi \bar{u}_i \eta_k}(p) \Gamma_{\bar{u}_k u_j}(p) + \Gamma_{\chi \bar{\eta}_k u_j}(p) \Gamma_{\bar{u}_i u_k}(p) \right]$$



Perturbation theory

Expanding in the number of loops...

$$\Gamma_{\chi\bar{u}\eta} = \cancel{\Gamma^{(0)}_{\chi\bar{u}\eta}} + \Gamma^{(1)}_{\chi\bar{u}\eta} + \Gamma^{(2)}_{\chi\bar{u}\eta} + \dots$$

$$\Gamma_{u\bar{u}} = \Gamma^{(0)}_{u\bar{u}} + \Gamma^{(1)}_{u\bar{u}} + \Gamma^{(2)}_{u\bar{u}} + \dots \qquad \text{etc.}$$

...gives the first-order (and also the higher-order) Nielsen identity for the fermion 2-point function:

$$\partial_\xi \Sigma^{u(1)}_{ij} = \sum_{k=1,2,3} \left[\Gamma^{(1)}_{\chi\bar{u}_i\eta_k} \Gamma^{(0)}_{\bar{u}_k u_j} + \Gamma^{(0)}_{\bar{u}_i u_k} \Gamma^{(1)}_{\bar{\eta}_k u_j \chi} \right]$$

At tree level, $\Gamma_{u_i\bar{u}_j}$ is the inverse propagator: $\Gamma^{(0)}_{u_i\bar{u}_j} = (\not{p} - m^u_j)\delta_{ij}$

Nielsen identity @ 1-loop

$$\partial_\xi \Sigma^{u(1)}_{ij} = \Gamma^{(1)}_{\chi\bar{u}_i\eta_j}(\not{p} - m^u_j) + (\not{p} - m^u_i)\Gamma^{(1)}_{\bar{\eta}_i u_j \chi}$$

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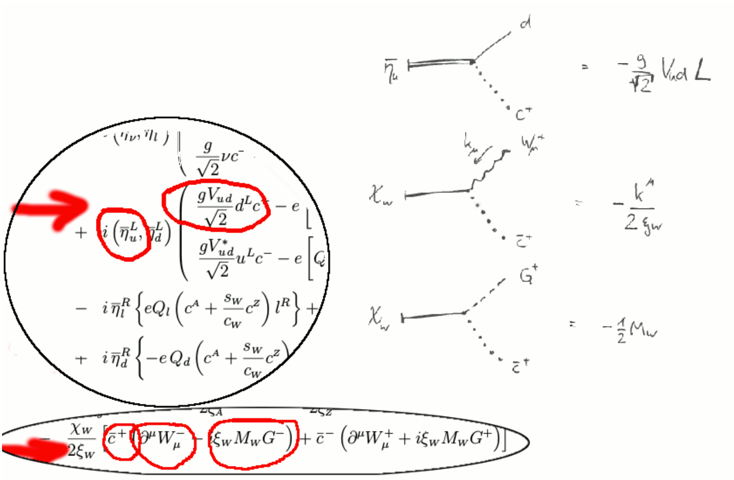
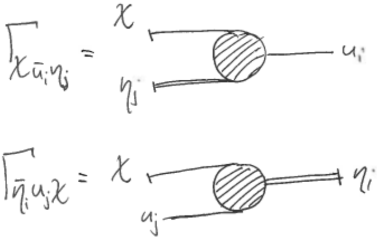
Need to calculate $\Gamma^{(1)}_{\chi\bar{u}_i\eta_j}$ and $\Gamma^{(1)}_{\bar{\eta}_i u_j \chi}$!

Espriu, Mazano, Talavera 2002

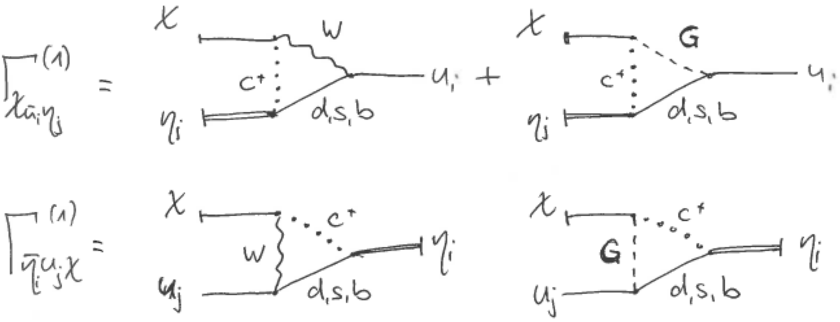
Feynman rules



Proper vertex functions



Relevant diagrams



The calculation was done both manually and using computer algebra (*Mathematica*, *FeynArts*,*FormCalc*).

Calculation of the diagrams

Results

$$\Gamma^{(1)}_{\chi\bar{u}_i\eta_j} = \left(\not{p} m_i B^u_{ij}(p^2) + C_{ij}(p^2) + A_{ij}(p^2)\right) R$$

$$\Gamma^{(1)}_{\bar{\eta}_i u_j \chi} = L\left(\not{p} B_{ij}(p^2) m_j + C_{ij}(p^2) + A_{ij}(p^2)\right)$$

Here:
A_{ij}: W-boson in the loop
B_{ij}, C_{ij}: Goldstone boson in the loop

Espriu, Mazano, Talavera 2002

Gauge invariance

$$\Gamma^{(1)}_{\chi\bar{u}_i\eta_j} = \left(\not{p} m_i B^u_{ij}(p^2) + C_{ij}(p^2) + A_{ij}(p^2)\right) R$$

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...plugging these into Nielsen idetity...

$$\partial_\xi \Sigma^{u(1)}_{ij} = \Gamma^{(1)}_{\chi\bar{u}_i\eta_j}(\not{p} - m^u_j) + (\not{p} - m^u_i)\Gamma^{(1)}_{\bar{\eta}_i u_j\chi}$$

...and using the chiral decomposition:

$$\Sigma_{ij}(\not{p}) = \not{p} L \Sigma^{\gamma L}_{ij}(p^2) + \not{p} R \Sigma^{\gamma R}_{ij}(p^2) + L\Sigma^L_{ij}(p^2) + R\Sigma^R_{ij}(p^2)$$

Gauge invariance

$$\Gamma^{(1)}_{\chi\bar{u}_i\eta_j} = \left(\not{p} m_i B^u_{ij}(p^2) + C_{ij}(p^2) + A_{ij}(p^2)\right) R$$

$$\Gamma^{(1)}_{\bar{\eta}_i u_j \chi} = L\left(\not{p} B_{ij}(p^2) m_j + C_{ij}(p^2) + A_{ij}(p^2)\right)$$

...plugging these into Nielsen idetity...

$$\partial_\xi \Sigma^{u(1)}_{ij} = \Gamma^{(1)}_{\chi\bar{u}_i\eta_j}(\not{p} - m^u_j) + (\not{p} - m^u_i)\Gamma^{(1)}_{\bar{\eta}_i u_j \chi}$$

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We find gauge invariant combinations:

$$\partial_\xi \left(m_i \Sigma^R_{ij}\left(p^2\right) - \Sigma^L_{ij}\left(p^2\right) m_j \right) = 0 \qquad (1)$$

$$\partial_\xi \left(p^2 \Sigma^{\gamma R}_{ij}\left(p^2\right) + \Sigma^{\gamma L}_{ij}\left(p^2\right) m_i m_j + m_i \Sigma^R_{ij}\left(p^2\right) + \Sigma^L_{ij}\left(p^2\right) m_j \right) = 0 \qquad (2)$$

Espriu, Mazano, Talavera 2002

Checking NIs @ two loops

Recall the Nielsen identity for the quark self-energies:

$$-\partial_\xi \Gamma_{u_j \bar{u}_i}(p) = \sum_{k=1,2,3} \left[\Gamma_{\chi \bar{u}_i \eta_k}(p) \Gamma_{\bar{u}_k u_j}(p) + \Gamma_{\chi \bar{\eta}_k u_j}(p) \Gamma_{\bar{u}_i u_k}(p) \right]$$

Checking NIs @ two loops

Recall the Nielsen identity for the quark self-energies:

$$-\partial_\xi \Gamma_{u_j \bar{u}_i}(p) = \sum_{k=1,2,3} \left[\Gamma_{\chi \bar{u}_i \eta_k}(p) \Gamma_{\bar{u}_k u_j}(p) + \Gamma_{\chi \bar{\eta}_k u_j}(p) \Gamma_{\bar{u}_i u_k}(p) \right]$$

Expanding this at two loops gives

$$-\partial_\xi \Gamma^{(2)}_{u_j \bar{u}_i}(p) = \sum_{k=1,2,3} [\Gamma^{(2)}_{\chi \bar{u}_i \eta_k}(p) \Gamma^{(0)}_{\bar{u}_k u_j}(p) + \Gamma^{(2)}_{\chi \bar{\eta}_k u_j}(p) \Gamma^{(0)}_{\bar{u}_i u_k}(p)$$
$$+\Gamma^{(1)}_{\chi \bar{u}_i \eta_k}(p) \Gamma^{(1)}_{\bar{u}_k u_j}(p) + \Gamma^{(1)}_{\chi \bar{\eta}_k u_j}(p) \Gamma^{(1)}_{\bar{u}_i u_k}(p)]$$

Need to calculate $\Gamma^{(2)}_{\chi \bar{u}_i \eta_j}$ and $\Gamma^{(2)}_{\bar{\eta}_i u_j \chi}$!

Conclusion

Nielsen Identities...

- show dependence on the gauge parameter:

$$\partial_\xi \Gamma_1 = \Gamma_2 \Gamma_3 + \Gamma_4 \Gamma_5 + \ldots$$

- can be derived from a generalised BRST-invariance of the SM (they generalise the Slavnov-Taylor IDs)
- can be used to show gauge independence both in general and in perturbation theory

References

- Kniehl, Sirlin 2009 arXiv:0811.0366
- Espriu, Manzano, Talavera 2002 hep-ph/0204088
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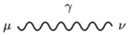
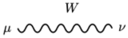
- Breckenridge, Lavelle, Steele 1994 hep-th/9407028
- Itzykson, Zuber: Quantum Field Theory, McGraw-Hill 1980

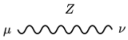
R_ξ-gauge Feynman Rules



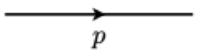
$$-i\left[\frac{g_{\mu\nu}}{k^2+i\epsilon}-(1-\xi_A)\frac{k_\mu k_\nu}{(k^2)^2}\right]$$



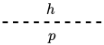
$$-i\,\frac{1}{k^2-m_W^2+i\epsilon}\left[g_{\mu\nu}-(1-\xi_W)\frac{k_\mu k_\nu}{k^2-\xi_W m_W^2}\right]$$



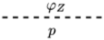
$$-i\,\frac{1}{k^2-m_Z^2+i\epsilon}\left[g_{\mu\nu}-(1-\xi_Z)\frac{k_\mu k_\nu}{k^2-\xi_Z m_Z^2}\right]$$



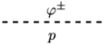
$$\frac{i(\not{p}+m_f)}{p^2-m_f^2+i\epsilon}$$



$$\frac{i}{p^2-m_h^2+i\epsilon}$$



$$\frac{i}{p^2-\xi_Z m_Z^2+i\epsilon}$$



$$\frac{i}{p^2-\xi_W m_W^2+i\epsilon}$$