Two-loop matching relations in the Standard Model

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- Introduction
- Relations between parameters of the SM
- Pole and running masses and couplings
- Results and discussion
- Conclusions

the Standard Model parametrization

SM parameters:

- parameters in the symmetric phase: $g, g', y_f, m_\phi, \lambda$ with scalar potential

$$V(\phi) = m_0^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

- parameters in the broken phase: e, m_W, m_Z, m_H, m_f
- spontaneous symmetry breaking does not affect the UV structure of the SM
- the running of the running parameters is governed by the Renormalization Group
  - ◇ the 3-loop RG functions are known nowdays — Mihaila, Salomon, Steinhauser '12; Bednyakov, Pikelner, Velizhanin '13
  - ◇ there are interrelations between RG functions in the broken phase and the RG functions in the symmetric phase — Jegerlehner, Kalmykov, OV '00 '01

- relationships between the running parameters and the "observables" are required at some scale (typcally of the order of electroweak scale) $\Longrightarrow$ matching relations
- for $O(\alpha^3)$ evolution we need $O(\alpha^2)$ matching conditions $\Longrightarrow$ this talk

relations between parameters

zero-order relations:

- parametes in symmetric and broken phase are related

$$e^2 = \frac{g^2 g'^2}{g^2 + g'^2}\,, \qquad \frac{4m_W^2}{v^2} = g^2\,, \qquad \frac{4m_Z^2}{v^2} = g^2 + g'^2\,, \qquad \frac{m_H^2}{2v^2} = \lambda\,, \qquad \frac{2m_f^2}{v^2} = y_f^2$$

- where we introduced the vacuum expectation value v

$$\frac{e^2}{4m_W^2(1 - m_W^2/m_Z^2)} = \frac{1}{v^2} = \frac{\lambda}{-m_\phi^2}, \qquad m_\phi^2 < 0$$

- these relations are modified by the radiative corrections

- choose what we would call “obsrevables”;
  a natural choice:
  - pole masses M_Z, M_W, M_H, M_t etc
  - couplings $\alpha_s(\mu)$ and fine-structure constant in the Thompson scattering α_{Th}
  - it is convinient to express the vacuum expectation value in terms of the Fermi constant

$$2^{1/2} G_F = \frac{1 + \Delta\bar{r}(\mu)}{v^2(\mu)}$$

relations between parameters

all-order relations:

$$\begin{aligned} g^2(\mu) &= 2^{5/2} G_F M_W^2 [1+\delta_W(\mu)] \\ g^2(\mu)+g'^2(\mu) &= 2^{5/2} G_F M_Z^2 [1+\delta_Z(\mu)] \\ \lambda(\mu) &= 2^{-1/2} G_F M_H^2 [1+\delta_H(\mu)] \\ y_f(\mu) &= 2^{3/4} G_F^{1/2} M_f [1+\delta_f(\mu)] \end{aligned}$$

$$\begin{aligned} m_Z^2(\mu) &= M_Z^2[1+\Delta\bar{r}(\mu)][1+\delta_Z(\mu)] \\ m_W^2(\mu) &= M_W^2[1+\Delta\bar{r}(\mu)][1+\delta_W(\mu)] \\ m_H^2(\mu) &= M_H^2[1+\Delta\bar{r}(\mu)][1+\delta_H(\mu)] \\ m_t(\mu) &= M_t[1+\Delta\bar{r}(\mu)]^{1/2}[1+\delta_t(\mu)] \\ m_b(\mu) &= M_b[1+\Delta\bar{r}(\mu)]^{1/2}[1+\delta_b(\mu)] \end{aligned}$$

where $\delta_x(\mu)$ and $\Delta\bar{r}(\mu) \longrightarrow 0$ at the three level

the pole mass

The pole mass M of a particle is defined by the position of the pole of its propagator, i.e. by the solution for p^2 of the equations

- Scalar particle

$$0 = p^2 - m_{H,0}^2 - \Pi_{HH}(p^2)$$

- Vector boson with possible mixing (e.g. γ-Z)

$$0 = p^2 - m_{Z,0}^2 - \frac{\Pi_{\gamma Z,T}(p^2)}{p^2 - \Pi_{\gamma\gamma(p^2)}}$$

- Fermion

$$0 = \not{p} - m_{f,0} - \not{\Sigma}_f(p)$$

with the decomposition into left and right components

$$\not{\Sigma}_f(p) = \not{p}\frac{1-\gamma_5}{2}A_L(p^2) + \not{p}\frac{1+\gamma_5}{2}A_R(p^2) + m_0 B(p^2)$$

we have (the poles for the left and right components coincide)

$$0 = p^2\Big(1 - A_L(p^2)\Big)\Big(1 - A_R(p^2)\Big) - m_0\Big(1 + B(p^2)\Big)^2$$

the pole mass (2)

Solution of the pole equation perturbatively by the Ansatz (e.g. for a fermion)

$$\not{p} = m_0\big(1 + X_1 + X_2 + \dots\big)$$

Explicitly:

$$\begin{aligned} X_1 &= B_1 + \frac{1}{2}A_{L,1} + \frac{1}{2}A_{R,1} \\ X_2 &= B_2 + \frac{1}{2}A_{L,2} + \frac{1}{2}A_{R,2} + X_1(A_{L,1} + A_{R,1} + A'_{L,1} + A'_{R,1} + 2B'_1) \\ &\quad -\frac{1}{2}X_1^2 - \frac{1}{2}A_{L,1}A_{R,1} + \frac{1}{2}B_1^2, \qquad \text{etc} \end{aligned}$$

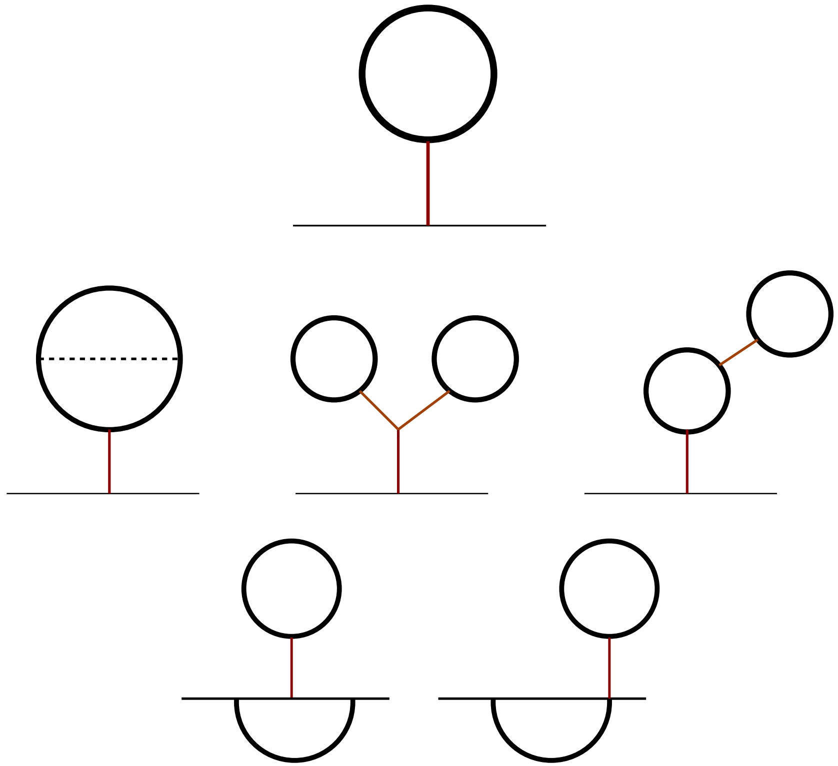
with $B = B(m_0^2)$, $B' = m_0^2 dB(m_0^2)/dm_0^2$, etc.

- $X_1, X_2, \dots$ are infrared safe and gauge invariant upon inclusion of tadpoles
- taking into account final width $\longrightarrow$ complex pole

$$p^2 = M^2 - iM\Gamma$$

- renormalization in $\overline{\text{MS}}$ scheme
  $\longrightarrow$ relation between M and $m(\mu)$

the role of tadpoles



1-loop: $T = \frac{1}{\varepsilon}\frac{g^2}{16\pi^2}\left(2N_c\frac{m_t^4}{m_W^2 m_H^2} - \frac{1}{2} - \frac{3m_W^2}{2m_H^2 c_W^4} - \frac{1}{4c_W^2} + \frac{1}{4c_W^2}(1-\xi_Z) - \frac{3m_H^2}{4m_W^2} + \frac{1}{2}(1-\xi_W) - 3\frac{m_W^2}{m_H^2}\right) + \ldots$

- gauge dependent
- large contributions, e.g. $\frac{g^2}{16\pi^2}2N_c\frac{m_t^4}{m_W^2 m_H^2} \sim 0.13$

Relation $m(\mu)/M_{\rm pole}$ is gauge invariant with the inclusion of tadpole deiagrams.

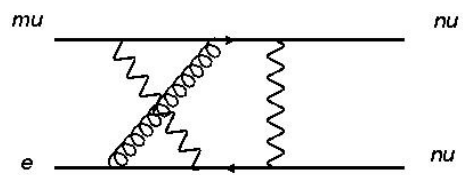
corrections to the vacuum expectation value

Parametrization of the relation between the Fermi constant G_F and the running coupling and masses via

$$G_F = \frac{\pi\alpha(\mu)}{\sqrt{2}m_W^2(\mu)[1 - m_W^2(\mu)/m_Z^2(\mu)]}\Big(1 + \Delta\bar{r}(\mu)\Big)$$

$\Delta\bar{r}$ is similar to Δr in the on-shell scheme and incorporates all radiative corrections.

It can be evaluated from any $2 \to 2$ process with charged current exchange



We can write

Awramik, Czakon, Onishchenko, OV '02

$$\frac{e^2}{8m_W^2(1 - m_W^2/m_Z^2)}(1 + \Delta\bar{r}) = \left[\sqrt{Z_{2,e}Z_{2,\nu_e}Z_{2,\mu}Z_{2,\nu_\mu}}A(e + \nu_e \to \mu + \nu_\mu)\right]_{\text{hard}}$$

$\longrightarrow$ reduced to the evaluation of the two-loop vacuum bubble diagrams!

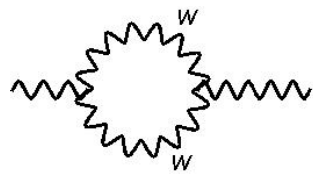
$\overline{\text{MS}}$ fine-structure constant

We need the matching relation for the running fine-structure constant α. The "usual" approach is to use the photon propagator $\Pi_\gamma(q^2)$ to define "effective" charge Degrassi, Fanchiotti, Sirlin '91

$$\alpha(q^2) = \frac{\alpha}{1-\Delta\alpha(q^2)}, \qquad \Delta\alpha(q^2) = -\mathrm{Re}\Big(\Pi'_\gamma(q^2) - \Pi'_\gamma(0)\Big)$$

Problems:

- gauge invariance is lost as the bosonic interactions are included



- $\alpha(q^2)$ is infrared sensitive
  $\longrightarrow$ hadronic contribution should be included via dispersion relation

$$\Delta\alpha_{\mathrm{had}}(M^2) = -\frac{\alpha}{3\pi}\int\limits_{4m_\pi^2}^{\infty} \frac{R_{e^+e^-}(s)}{s-M^2}\,ds, \qquad R_{e^+e^-}(s) = \frac{\sigma(e^+e^-\to\gamma^*\to \text{hadrons})}{\sigma(e^+e^-\to\gamma^*\to\mu^+\mu^-)}$$

- not possible to go beyond one-photon approximation

We need to include other objects than just $\Pi_\gamma(q^2)$ into considerations...

$\overline{\mathrm{MS}}$ fine-structure constant

A very simple consistent definition of $\alpha(\mu)$:
take the *deifnition* of the running charge $e(\mu)$ valid through all orders as

$$\frac{1}{e^2(\mu)} = \frac{1}{g^2(\mu)} + \frac{1}{g'^2(\mu)}$$

together with the quantum corrections

$$\begin{aligned}
\sqrt{2}G_F &= \frac{1+\Delta\bar{r}(\mu)}{v^2(\mu)} \\
g^2(\mu) &= 2^{5/2}G_F M_W^2\Big(1+\delta_W(\mu)\Big) \\
g^2(\mu)+g'^2(\mu) &= 2^{5/2}G_F M_Z^2\Big(1+\delta_Z(\mu)\Big)
\end{aligned}$$

$\overline{\text{MS}}$ fine-structure constant

Using the corrections to $Z-$ and $W-$boson masses and the vacuum expectation value v we can define

$$\alpha(\mu) = \frac{\sqrt{2} G_F M_W^2}{\pi} [1 + \delta_W(\mu)] \left[1 - \frac{M_W^2}{M_Z^2} \frac{1 + \delta_W(\mu)}{1 + \delta_Z(\mu)} \right] ,$$

- gauge invariant to all orders
- hadronic contributions decouple
- runs with $\overline{\text{MS}}$ β-function

$\Longrightarrow$ Implicit definition of $\alpha(\mu)$, since $\delta_W(\mu)$ and $\delta_Z(\mu)$ themselves depend on $\alpha(\mu)$.

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This can be solved: Kniehl, Pikelner, OV '15

- iteratively

$$\alpha^{-1}(M_Z) = 132.233 - \underbrace{4.741}_{\alpha} + \underbrace{0.512}_{\alpha\alpha_s} + \underbrace{0.203}_{\alpha^2} + \cdots = 128.208...$$

- perturbatively

$$\alpha^{-1}(M_Z) = 132.233 - \underbrace{4.648}_{\alpha} + \underbrace{0.641}_{\alpha\alpha_s} + \underbrace{0.127}_{\alpha^2} + \cdots = 128.354...$$

two-loop calculation

- all calculation were done twice: for the full Standard Model and for the gaugeless limit
  - ◇ for the masses m_b, m_t, m_W, m_Z, m_H and couplings $g, g', e, \lambda, y_b, t_t$
- all calculation were done in general R_ξ gauge with 4 gauge parameters $\xi_\gamma, \xi_Z, \xi_W, \xi_g$
- diagram generation QGRAF+DIANA — Nogueira '93, Tentyukov '99
  - ◇ a typical number of diagrams 2-3 thousands from which 2/3 are tadpole diagrams
- projection and Dirac γ-algebra on FORM — Vermaseren '89
  - ◇ no γ_5 problem at two loops, anticommutative γ_5 works!
- reduction of the two-loop self-energy diagrams using the shifts of the space-time dimension and Trasov's generalized recurrence relation — Tarasov '97
- using Mathematica package TARCER — Mertig, Scharf '98
- numerical library TSIL — Martin, Robertson '05
  - ◇ based on the numerical integration of the differential equations
- $\Delta\bar{r}$ evaluated with the low-energy factorization theorem — Awramik, Czakon, Onishchenko, OV '02
- C++ implimentation: program mr — Pikelner '15

stability of electroweak vacuum

Scalar potential in the Standard Model:

$$V(\phi) = m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$$

- spontanous breaking of symmetry
  $\longrightarrow$ nonzero value of $v = \langle\phi\rangle \sim 246\text{GeV}$
- quantum corrections modify potential wich could lead to instability
  $\longrightarrow$ bound of the Higgs boson mass from below $M_H^{\text{stab.}}$

Can the SM be extrapolated up to ultimate scales while still having an absolutely stable electroweak vacuum?

Cabibbo, Maiani, Parisi, Petronzio '79

Theoretical analysis requires two parts:

- evolution of the running parameters with the renormalization group
- determination of the running parameters from the physical observables

For $\text{N}^{\text{p}}\text{LO}$ evolution we need $\text{N}^{\text{p}-1}\text{LO}$ matching

stability of electroweak vacuum

Different approaches to address the SM vacuum stability problem. Assuming the stability of the SM up to the Planck scale, we can find restrictions on the Higgs and top-quark masses.

- Instability scale as a scalar field value at which the effective potential is degenerate with the electroweak vacuum. Both scalar field and electroweak vacuum are gauge dependent. A proper resummation is required at each order of PT to mantain explicit gauge-independence at the minimun.

  Nieseln '75

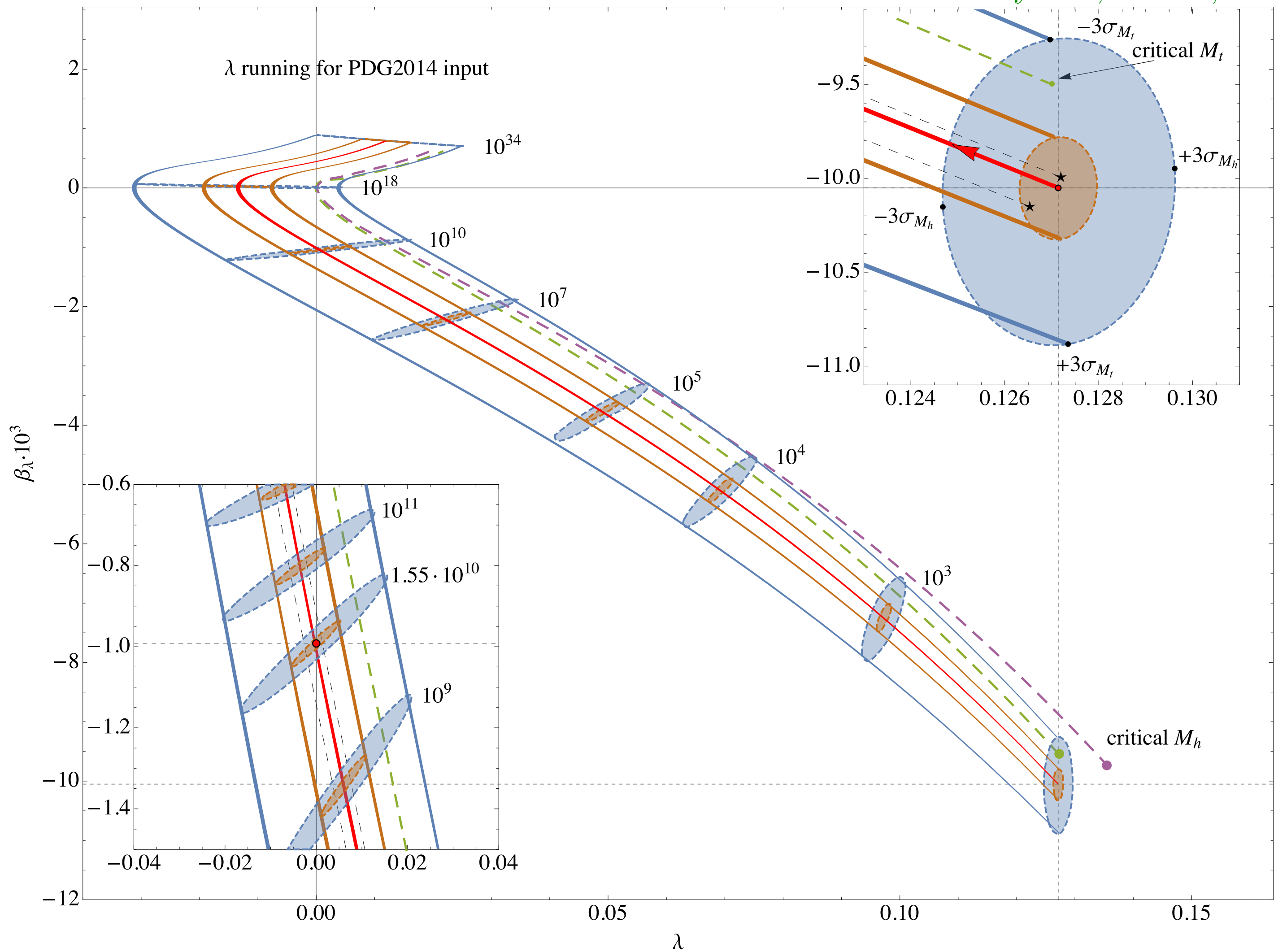
  Andreassen, Frost, Schartz '14

- Alternatively, one can study the explicit gauge-independent (but scheme dependent) running self-coupling $\lambda(\mu)$. The definition of the critical parameters and scales in this case is settled by the conditions

$$\lambda(\Lambda_{\text{crit}}) = 0, \qquad \beta_\lambda(\Lambda_{\text{crit}}) = 0,$$

running of λ

Bednyakov, Kniehl, Pikelner, OV '15



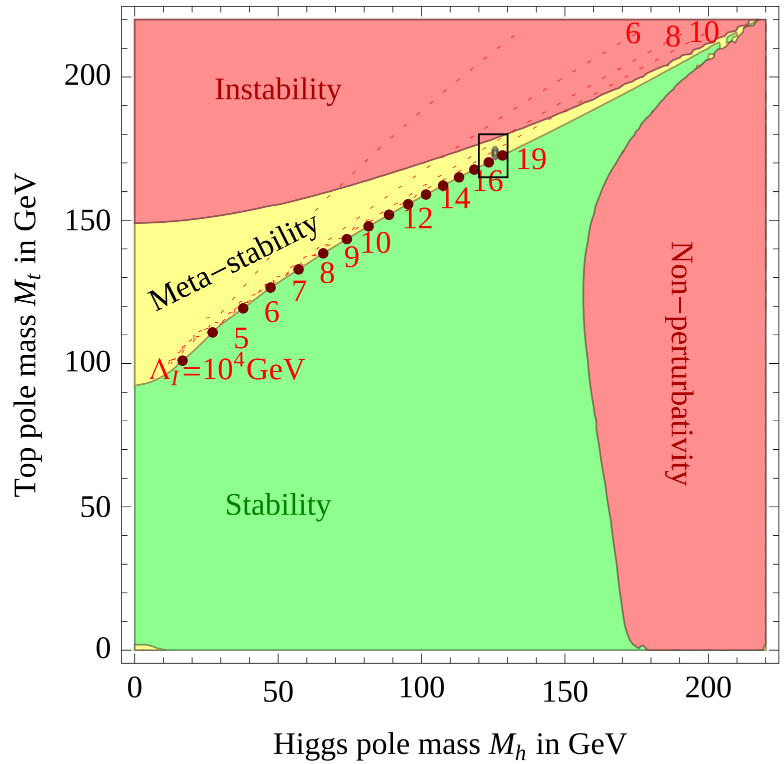
critical parameters

Preliminary:

Bednyakov, Kniehl, Pikelner, OV '15

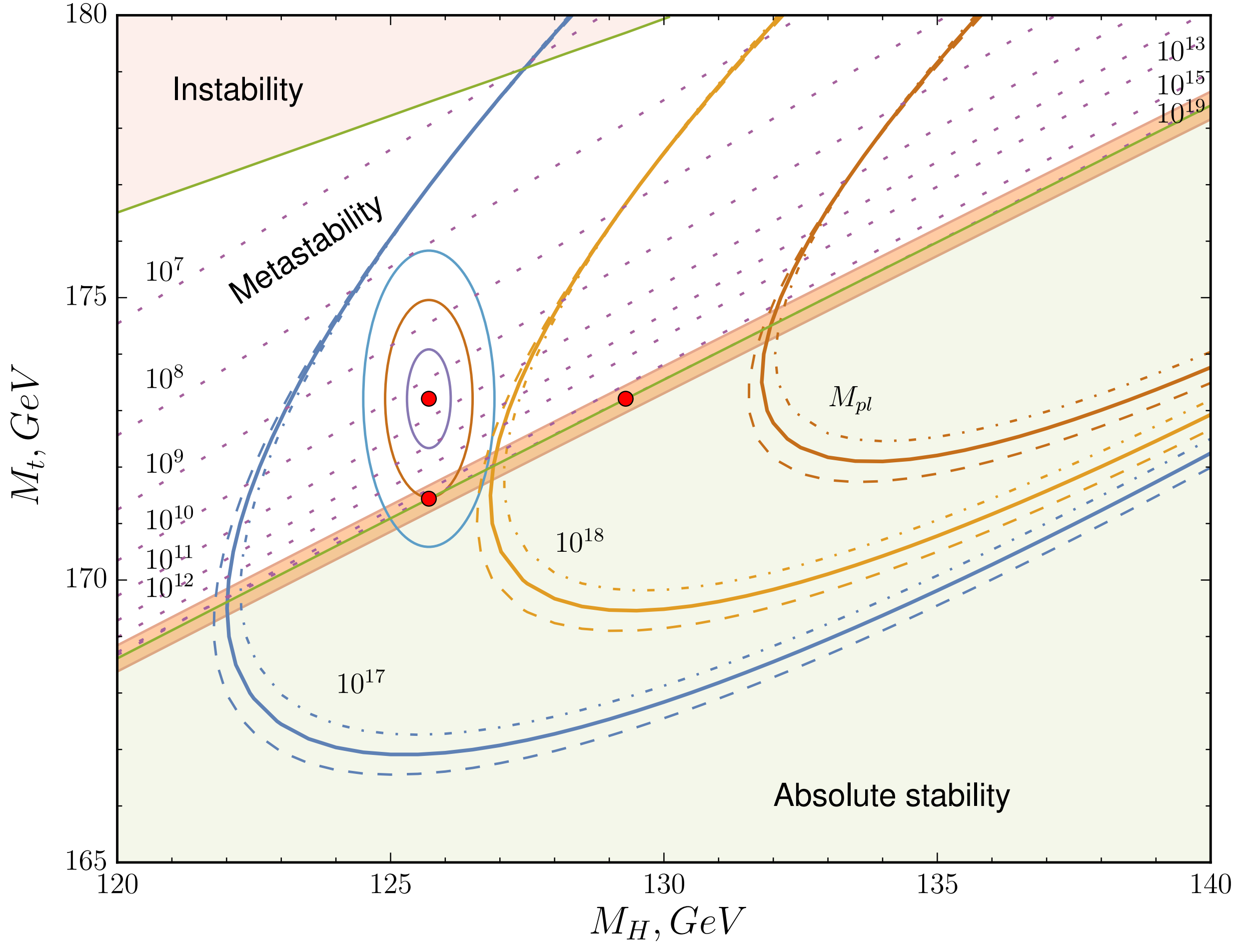
$$M_H^{\text{crit.}} = 129.30^{+0.72}_{-0.34} + 1.79\frac{M_t - 173.2}{0.87} - 0.48\frac{\alpha_s^{(5)}(M_Z^2) - 0.1185}{0.0006}$$

$$M_t^{\text{crit.}} = 171.44^{-0.36}_{+0.17} + 0.20\frac{M_h - 125.7}{0.4} + 0.23\frac{\alpha_s^{(5)}(M_Z^2) - 0.1185}{0.0006}$$



M_t-M_H phase diagram

Bednyakov, Kniehl, Pikelner, OV '15



top-quark

Various contributions to the shift of the running mass

$$m_t(M_t) - M_t$$

Contributions to the shift (in GeV).

M_H [GeV]	QCD	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha\alpha_s)$	$\mathcal{O}(\alpha^2)$	total
126	−10.38	+11.67	−0.38	−0.94 (−0.44)	−0.03

*   red — full 2-loop SM correction
** blue — gaugeless limit approximation

Various contributions to the shift of Yukawa coupling

$$y_t(\mu) = 2^{3/4} G_F M_t \Big(1 + \delta_t(\mu)\Big)$$

Contributions to $\delta_t(M_t)$ in units of 10^{-4}.

M_H [GeV]	QCD	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha\alpha_s)$	$\mathcal{O}(\alpha^2)$	total
126	−599.3	+12.9	−4.2	+2.7 (+3.1)	−587.9

*   red — full 2-loop SM correction
** blue — gaugeless limit approximation

bottom-quark

Various contributions to the shift of the running mass

$$m_b(M_b) - M_b$$

Contributions to the shift (in GeV).

M_H [GeV]	QCD	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha\alpha_s)$	$\mathcal{O}(\alpha^2)$	total
126	−0.85	-1.90	−0.53	+1.75 (+1.80)	???

*   red — full 2-loop SM correction

** blue — gaugeless limit approximation

Various contributions to the shift of Yukawa coupling

$$y_b(\mu) = 2^{3/4} G_F M_b \Big(1 + \delta_b(\mu) \Big)$$

Contributions to $\delta_t(M_b)$ in units of 10^{-4}.

M_H [GeV]	QCD	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha\alpha_s)$	$\mathcal{O}(\alpha^2)$	total
126	−1728.	-190.	−112.	+32. (+33.)	−1998

*   red — full 2-loop SM correction

** blue — gaugeless limit approximation

scalar self-coupling

Various contributions to the shift of scalar self-coupling

$$\lambda(\mu) = 2^{-1/2} G_F M_H^2 \Big(1 + \delta_H(\mu)\Big)$$

Contributions to $\delta_H(M_t)$ in units of 10^{-4}.

M_H [GeV]	QCD	$\mathcal{O}(\alpha)$	$\mathcal{O}(\alpha\alpha_s)$	$\mathcal{O}(\alpha^2)$	total
126	—	−114.1	-103.1	-26.3 (-29.3)	−243.5

* red — full 2-loop SM correction

** blue — gaugeless limit approximation

conclusions

- We have established the framework for the evaluation of the two-loop matching relations in the full SM. These include the evaluations of the pole masses of the particles and the vacuum expectation value of the scalar field (the latter is parametrized with $\Delta\bar{r}$).
- The two-loop corrections to *all* relevant matching conditions are evaluated. These include the running masses m_Z, m_W, m_H, m_t, m_b and the coupling constants g, g', λ, y_t, y_b.
- All calcualtions were done in the general R_ξ gauge and the gauge invariance of the running parameters has been shown explicitly. The inclusion of the tadpole diagrams is necessary to have the gauge-invariant parametrization of the SM.
- The evaluation of the critical Higgs boson mass and the critical top-quark mass is revised at the $O(\alpha^2)$ order. The SM can still be an absolute stable theory valid up to the Planck scale.
- The gaugeless limit approximation is tested at the two-loop level. It appears that this approximation is accurate within some 5% for all quantities, except for the running mass of the top quark m_t.