A Simple Subtraction Procedure for Calculation of the Anomalous Magnetic Moment of the Electron in QED

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▪**Bogoliubov-Parasiuk theorem** [1956]
Removal of ultraviolet divergences by R-operation (defined by recurrence relations).

▪**Forest formula**: V. Scherbina [1964], O. Zavyalov, B. Stepanov [1965], W. Zimmermann [1969]. “BPHZ renormalization”.

$$f^{\text{UV-free}}=(1-K_1)(1-K_2)\ldots(1-K_n)f$$

K_j transforms Feynman amplitude of j-th divergent subgraph (G_j) into it’s Taylor expansion up to $\omega(G_j)$ order at 0 (in momentum representation). All terms with overlapping elements must be removed.
$\omega(G)$ = degree of UV divergence = $4-N_\mu-(3/2)N_e$

UV-divergences are removed in Schwinger-parametric representation point-by-point, before integration, if $i\varepsilon$ in propagator denominators is fixed ($\varepsilon>0$).

Schwinger parameters: $1/(x+i\varepsilon)=(1/i)\cdot\int_0^{+\infty}e^{i\alpha(x+i\varepsilon)}d\alpha$

$\varepsilon\to 0$ => IR-divergences.

AMM of the electron (theory and experiment)

The measured value [2011]:

a_e=0.00115965218073(28)

The most accurate prediction (T. Kinoshita et al. [2015]):

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geq 1} \left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

a_e=0.001159652181643(25)(23)(16)(763)

(α^{-1}=137.035999049(90) – from experiments with rubidium atoms)

Uncertainties come from:

$$A_1^{(8)},\ A_1^{(10)},\ a_e(hadronic) + a_e(electroweak),\ \alpha$$

T. Aoyama, M. Hayakawa, T. Kinoshita, M.Nio, Tenth-Order Electron Anomalous Magnetic Moment – Contribution of Diagrams without Closed Lepton Loops, Physical Review D, 2015, V. 91, 033006.

My method was developed for computing $A_1^{(2n)}$

Universal QED contributions

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geq 1} \left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

- J. Schwinger [1948], analytically: $A_1^{(2)} = 0.5$
- R. Karplus, N. Kroll [1949] – with a mistake

  A. Petermann [1957], C. Sommerfield [1958], analytically:

$$A_1^{(4)} = -0.328478965579193\ldots$$

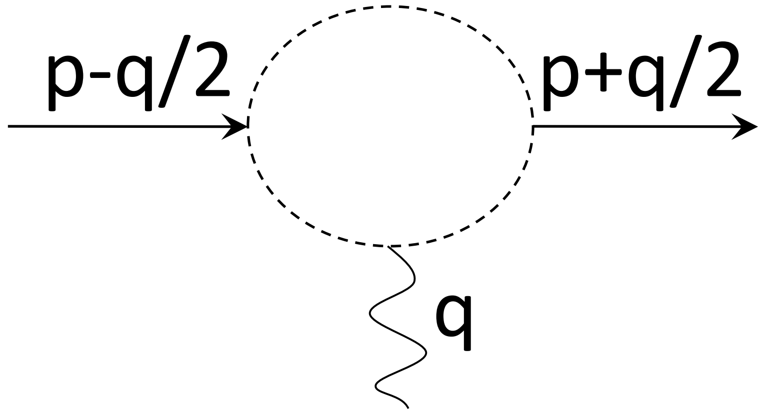
- ~1965…~1975, 3 loops, numerically:
  1. M. Levine, J. Wright.
  2. R. Carroll, Y. Yao.
  3. T. Kinoshita, P. Cvitanović.

     T. Kinoshita, P. Cvitanović [1974]: $A_1^{(6)} = 1.195 \pm 0.026$
- E. Remiddi, S. Laporta et al., ~1965…1996, analytically: $A_1^{(6)} = 1.181241456\ldots$
- T. Kinoshita et al., numerically, 1981…2015: $A_1^{(8)} = -1.91298(84)$
- T. Kinoshita et al., numerically, 2012…2015: $A_1^{(10)} = 7.795(336)$

The subtraction procedure

- FULLY AUTOMATED AT ANY ORDER OF PERTURBATION.
- UV and IR divergences are eliminated point-by-point in Feynman-parametric space for each individual Feynman diagram. No regularization is required.
- Subtraction by a forest formula with linear operators. Each operator transforms Feynman amplitude of some UV-divergent subdiagram G' (in momentum space) to the polynom with the degree that is less or equal to ω(G').
- The subtraction is equivalent to the on-shell renormalization => no residual renormalizations, no calculations of renormalization constants, no other manipulations.

Anomalous magnetic moment: definition

$\Gamma_\mu(p,q) =$ 

(renormalized)

$$\bar{u}_2 \Gamma_\mu(p,q) u_1 = \bar{u}_2 \left(f(q^2)\gamma_\mu - \frac{1}{2m} g(q^2) \sigma_{\mu\nu} q^\nu \right) u_1,$$

$$a_e = \lim_{q^2 \to 0} g(q^2)$$

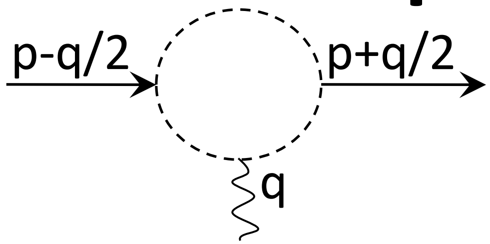
$$(p - q/2)^2 = (p + q/2)^2 = m^2$$

$$(\hat{p} - \hat{q}/2 - m) u_1 = (\hat{p} + \hat{q}/2 - m) u_2 = 0$$

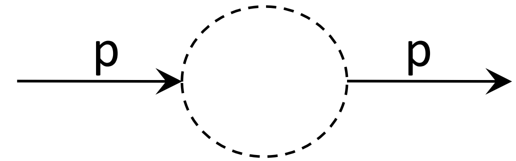
$$\sigma_{\mu\nu} = (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)/2$$

Operators

$$\Gamma_\mu(p,q) =$$



$$\Sigma(p) =$$



- A – projector of AMM

$$\bar{u}_2 \Gamma_\mu(p,q) u_1 = \bar{u}_2 \left(f(q^2)\gamma_\mu - g(q^2)\sigma_{\mu\nu} q^\nu / (2m) + h(q^2) q_\mu \right) u_1$$

$$\sigma_{\mu\nu} = (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)/2, \qquad (p - q/2)^2 = (p + q/2)^2 = m^2$$

$$(\hat{p} - \hat{q}/2 - m) u_1 = (\hat{p} + \hat{q}/2 - m) u_2 = 0$$

$$A\Gamma_\mu = \gamma_\mu \lim_{q^2 \to 0} \left[g(q^2) + Ch(q^2) \right]$$

- U – intermediate operator

$$\Gamma_\mu(p,0) = a(p^2)\gamma_\mu + b(p^2) p_\mu + c(p^2)\hat{p} p_\mu + d(p^2)(\hat{p}\gamma_\mu - \gamma_\mu \hat{p}) \qquad \Sigma(p) = r(p^2) + s(p^2)\hat{p}$$

$$U\Gamma_\mu = \gamma_\mu a(m^2) \qquad U\Sigma = r(m^2) + s(m^2)\hat{p}$$

U preserves the Ward identity!

For other types of divergent subgraphs, U=Taylor expansion at 0 up to ω order.

- L – on-shell renormalization for vertex-like subgraphs

$$L\Gamma_\mu = \gamma_\mu \left(a(m^2) + b(m^2) m + c(m^2) m^2 \right)$$

Forest formula for AMM

A set of subgraphs of a diagram is called a **forest** if any two elements of this set don't overlap.

F[G] – the set of all forests of UV-divergent subgraphs in G that contain G.

$\mathbb{I}$[G] – the set of all vertex-like UV-divergent subgraphs in G that contains the vertex that is incident to the external photon line of G.

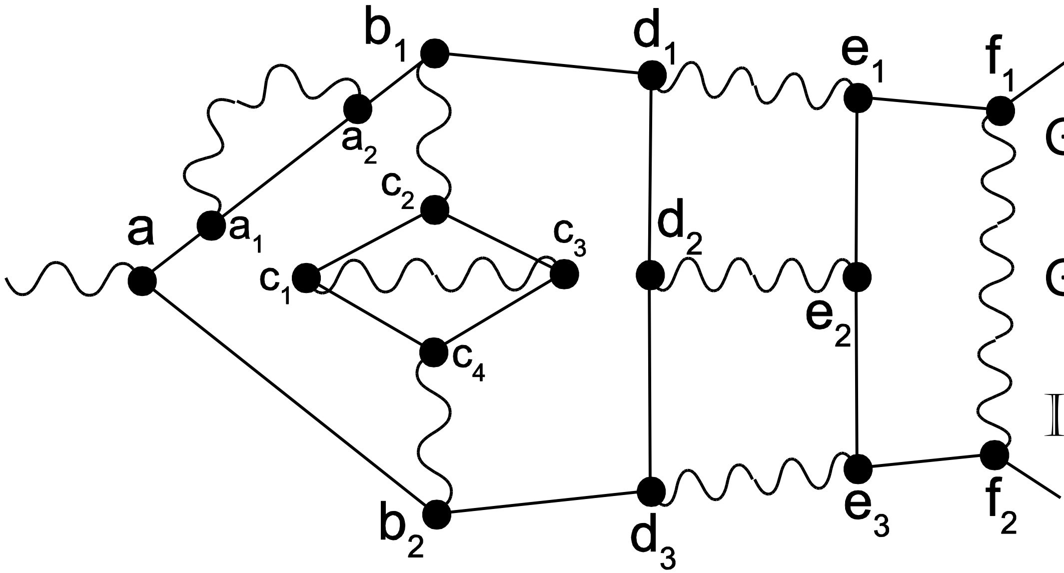
$$\tilde{f}_G = \sum_{\substack{F=\{G_1,\dots,G_n\}\in \mathrm{F}[G] \\ G'\in \mathbb{I}[G]\cap F}} (-1)^{n-1} K_{G_1}^{F,G'} \dots K_{G_n}^{F,G'} f_G$$

$$K_{G''}^{F,G'} = \begin{cases} A_{G'} \text{ for } G' = G'' \\ U_{G''} \text{ for } G'' \notin \mathbb{I}[G] \text{ or } G'' \subseteq G', G'' \neq G' \\ L_{G''} \text{ for } G'' \in \mathbb{I}[G],\ G' \subseteq G'',\ G'' \neq G,\ G'' \neq G' \\ (L_{G''} - U_{G''}) \text{ for } G'' = G,\ G' \neq G \end{cases}$$

$$\bar{f}_G = \text{coefficient before } \gamma_\mu \text{ in } \tilde{f}_G$$

$$a_e = \sum_G \bar{f}_G$$

Example



$G_c = aa_1a_2b_1b_2c_1c_2c_3c_4$

$G_e = aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3e_1e_2e_3$

$\mathbb{I}[G] = \{G_c, G_e, G\}$

Other UV-divergent subgraphs:
electron self-energy – a_1a_2, vertex-like – $c_1c_2c_3$, $c_1c_3c_4$,
photon self-energy – $c_1c_2c_3c_4$,
photon-photon scattering – $G_d = aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3$

$$\tilde{f}_G = \Big[A_G(1-U_{G_e})(1-U_{G_c}) - (L_G - U_G)A_{G_e}(1-U_{G_c}) - (L_G - U_G)(1-L_{G_e})A_{G_c}\Big]\cdot$$
$$\cdot(1-U_{G_d})(1-U_{c_1c_2c_3c_4})(1-U_{c_1c_2c_3} - U_{c_1c_3c_4})(1-U_{a_1a_2})\, f_G$$

Feynman parameters

- Assign α_j to each line.
- New propagators

photon: $$-\frac{g_{\mu\nu}}{p^2+i\varepsilon} \Rightarrow i g_{\mu\nu} e^{i\alpha_j(p^2+i\varepsilon)}$$

electron: $$\frac{i(\hat{p}+m)}{p^2-m^2+i\varepsilon} \Rightarrow (\hat{p}+m) e^{i\alpha_j(p^2-m^2+i\varepsilon)}$$

- Apply the subtraction procedure with new propagators.
- Integrate (analytically) with respect to $\lambda=\alpha_1+\ldots+\alpha_n$.
- $\varepsilon \to 0$.
- Integrate (numerically) with respect to $\alpha_1,\ldots,\alpha_n \geq 0$, $\alpha_1+\ldots+\alpha_n=1$.

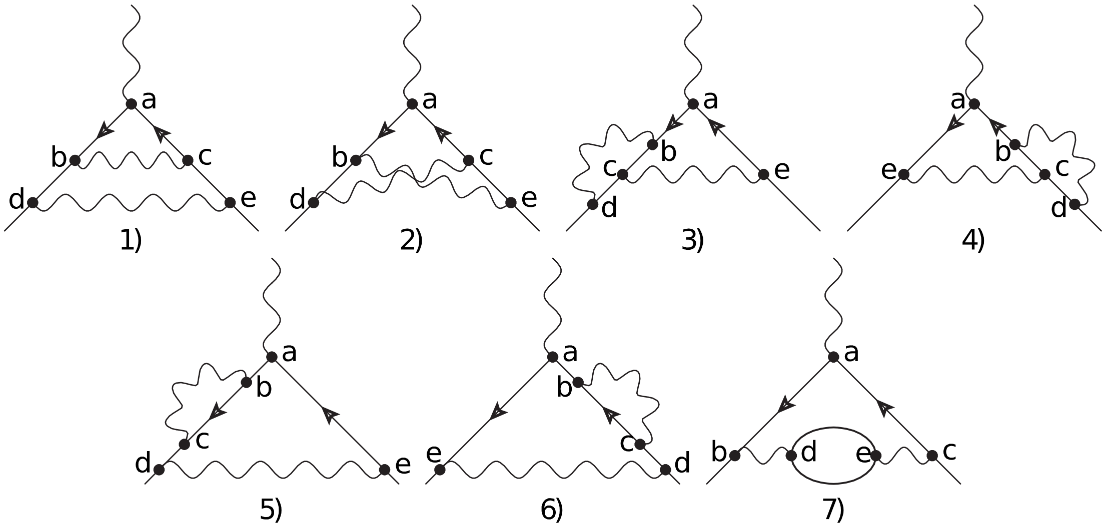
Realization

- 2 loops, 3 loops.
- Feynman gauge.
- D programming language – for the generator.
- C programming language – for the automatically generated code.
- Adaptive Monte-Carlo (homemade).
- 3 days of computation on a personal computer:

$$A_1^{(4)} = -0.328513(87)$$

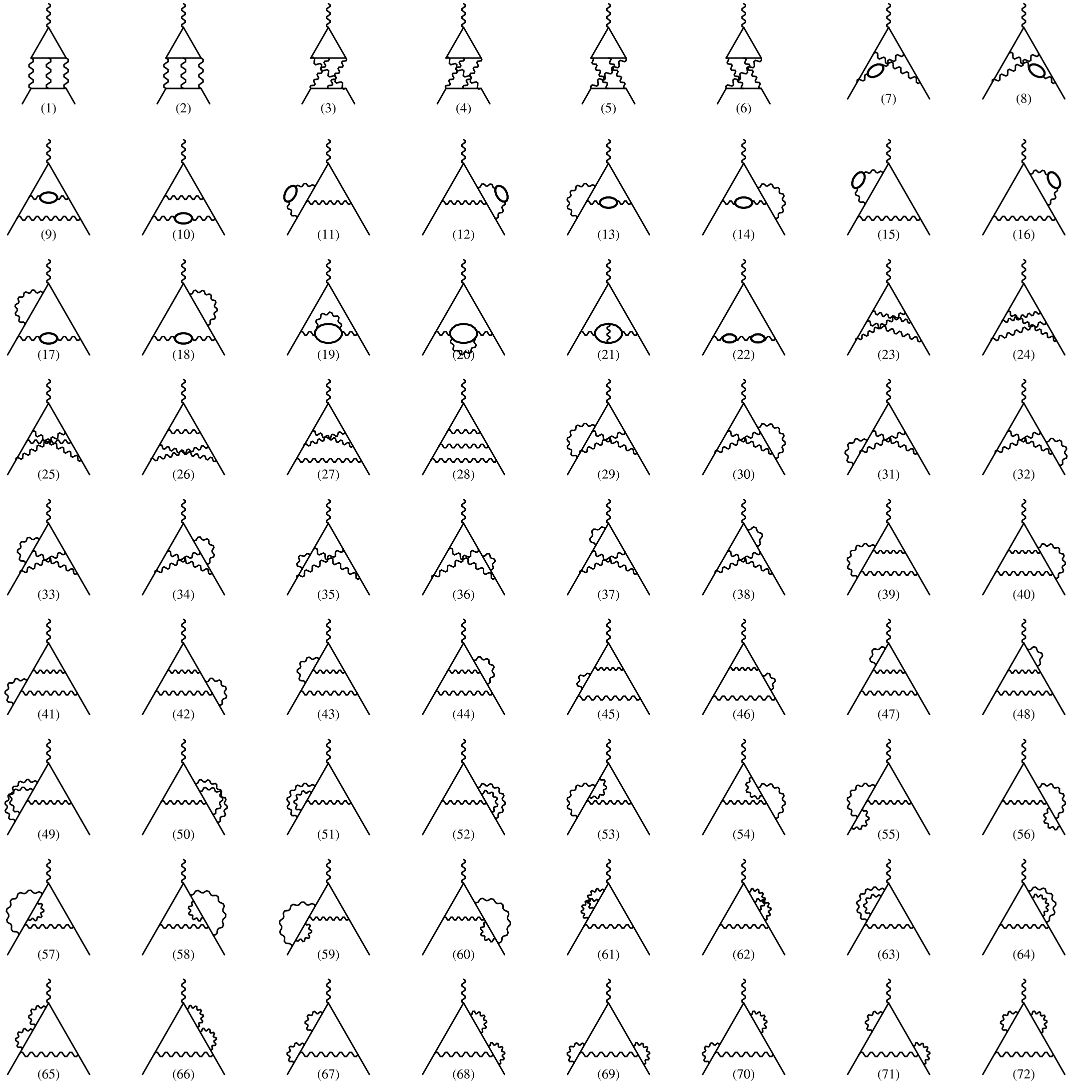
$$A_1^{(6)} = 1.1802(85)$$

2-loop case



$$B\Sigma(p) = a(m^2) + mb(m^2) +$$
$$+(\hat{p}-m)(b(m^2)+2a'(m^2)+2mb'(m^2)),$$
$$\Sigma(p) = a(p^2) + b(p^2)\hat{p}$$

#	My value	Analyt. value (Petermann, 1957)	Expression	On-shell renorm.	Difference
1	0.777455(52)	0.777478	A_G-$A_G U_{abc}$-$(L_G$-$U_G)A_{abc}$	A_G-$A_G L_{abc}$	$(L_G$-$U_G)A_{abc}$-$A_G(L_{abc}$-$U_{abc})$
2	-0.467626(44)	-0.467645	A_G	A_G	0
3	-0.032023(29)	-	A_G-$A_G U_{bcd}$	A_G-$A_G L_{bcd}$	$A_G(U_{abc}$-$L_{abc})$
4	-0.032033(29)	-	A_G-$A_G U_{bcd}$	A_G-$A_G L_{bcd}$	$A_G(U_{abc}$-$L_{abc})$
5	-0.294978(25)	-	A_G-$A_G U_{bc}$	A_G-$A_G B_{bc}$	$A_G(U_{bc}$-$B_{bc})$
6	-0.294998(24)	-	A_G-$A_G U_{bc}$	A_G-$A_G B_{bc}$	$A_G(U_{bc}$-$B_{bc})$
7	0.0156895(25)	0.0156874	A_G-$A_G U_{de}$	A_G-$A_G U_{de}$	0



3-loop Feynman diagrams for electron's AMM. Plot courtesy of F.Jegerlehner

Comparison with known analytical values

#	My value	Analyt. val.	Reference
1-6	0.3708(14)	0.3710	[10]
7-10	0.04989(20)	0.05015	[4,5]
11-12,15-16	-0.08782(15)	-0.08798	[2,4]
13-14,17-18	-0.11230(17)	-0.11234	[3,4]
19-21	0.05288(13)	0.05287	[1]
22	0.002548(20)	0.002559	[1]
23-24	1.8629(14)	1.8619	[11]
25	-0.02688(47)	-0.02680	[12]
26-27	-3.1764(22)	-3.1767	[8]
28	1.7888(19)	1.7903	[8]
29-30	-1.7579(10)	-1.7579	[12]
31-32,37-38	5.3559(27)	5.3576	[8,9]
33-34,37-38	0.4549(14)	0.4555	[8,11]
31-32,35-36	1.5436(34)	1.5416	[7,9]
33-36	-3.3573(24)	-3.3605	[7,11]
39-40	-0.33468(95)	-0.33470	[11]
41-48	-0.4030(41)	-0.4029	[6,7]
49-72	0.9529(53)	0.9541	[6-9,11,12]

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Thank you for your attention!

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http://arxiv.org/abs/1507.06435