

# Three-loop cusp anomalous dimension in QCD

A. Grozin, J. Henn, G. Korchemsky, P. Marquard

# Why

IR behavior of scattering amplitudes  $\rightarrow$  Wilson lines

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IR behavior of scattering amplitudes  $\rightarrow$  Wilson lines

- ▶  $t\bar{t}$  production at LHC and ILC
- ▶  $t \rightarrow bW$
- ▶  $b \rightarrow c$
- ▶ ...

# Wilson lines



# Wilson lines

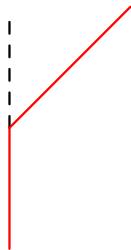


## Limiting cases

$\varphi \rightarrow 0$



$\varphi \rightarrow \infty$



$$\varphi_E = \pi - \delta$$



# HQET heavy-to-heavy current

$$J = h_{v'}^+ h_v = Z_J(\alpha_s(\mu); \varphi) J_r(\mu)$$

$$h_v = Z_h^{1/2}(\alpha_s(\mu)) h_{vr}(\mu)$$

$$\cosh \varphi = v \cdot v'$$

# Green functions

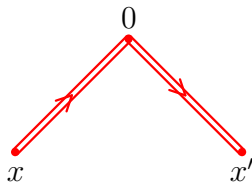


$$-i\langle h_v(x)h_v^+(0)\rangle = \delta(x_\perp)W(t) = Z_h\delta(x_\perp)W_r(t;\mu)$$

# Green functions



$$-i\langle h_v(x)h_v^+(0)\rangle = \delta(x_\perp)W(t) = Z_h\delta(x_\perp)W_r(t; \mu)$$



$$\begin{aligned}(-i)^2\langle h_{v'}(x')J(0)h_v^+(x)\rangle &= \delta(x_\perp)\delta(x'_\perp)W(t, t'; \varphi) \\ &= Z_h Z_J\delta(x_\perp)\delta(x'_\perp)W_r(t, t'; \varphi; \mu)\end{aligned}$$



# Renormalization

$$W(t, t'; 0) = W(t + t')$$

$$\log \frac{W(t, t'; \varphi)}{W(t, t'; 0)} = \log Z_J + \text{finite}$$

$$\Gamma(\alpha_s, \varphi) = \frac{d \log Z_J}{d \log \mu}$$

$$\Gamma(\alpha_s, 0) = 0$$

# Exponentiation in QED

Coordinate space, Wilson line of any shape ( $n_f = 0$ )

$$W(t) = \begin{array}{c} \text{---} \\ 0 \quad t \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} + \dots$$

$\log W(t) = ?$

# Exponentiation in QED

$$0 < t_1 < t_2 < t, 0 < t'_1 < t'_2 < t$$

The diagram illustrates the exponentiation of a propagator in QED. It shows the product of two propagators with a self-energy loop, which is then expanded into a sum of diagrams with multiple self-energy insertions.

The first row shows the product of two propagators:

- The first propagator has vertices at  $0$  and  $t_2$ , with a self-energy loop between  $t_1$  and  $t_2$ .
- The second propagator has vertices at  $0$  and  $t$ , with a self-energy loop between  $t'_1$  and  $t'_2$ .

The second row shows the expansion of the product into a sum of diagrams:

- The first diagram has vertices at  $0$  and  $t$ , with a self-energy loop between  $t_1$  and  $t_2$ .
- The second diagram has vertices at  $0$  and  $t$ , with a self-energy loop between  $t'_1$  and  $t'_2$ .
- The third diagram has vertices at  $0$  and  $t$ , with a self-energy loop between  $t_1$  and  $t_2$ .

The third row shows the expansion of the product into a sum of diagrams:

- The first diagram has vertices at  $0$  and  $t$ , with a self-energy loop between  $t_1$  and  $t_2$ .
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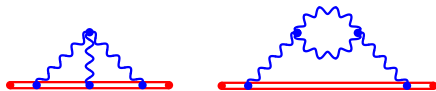
# Exponentiation in QED

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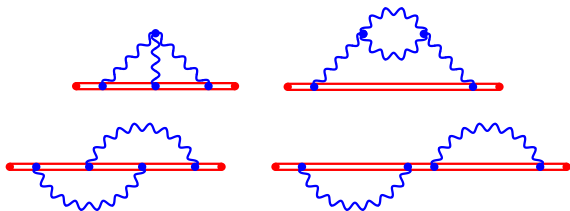
The diagram illustrates the exponentiation of a photon propagator in QED. It shows the multiplication of two diagrams, each consisting of a red double-line fermion propagator and a blue wavy photon line. The first diagram has vertices at  $0, t_1, t_2, t$  and the second at  $0, t'_1, t'_2, t$ . The result is a sum of six diagrams where the photon lines are inserted at various positions along the fermion lines. The final result is the logarithm of the sum of these diagrams, which is represented by a single diagram with a wavy photon line connecting the vertices  $0$  and  $t$ .

$$\begin{aligned} & \text{Diagram 1} \times \text{Diagram 2} \\ &= \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} \\ &+ \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} \\ &\log W(t) = \text{Diagram 9} \end{aligned}$$

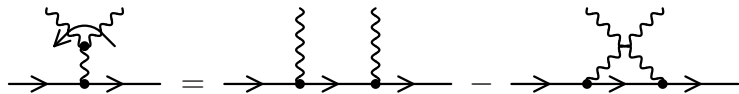
# Exponentiation in QCD



# Exponentiation in QCD

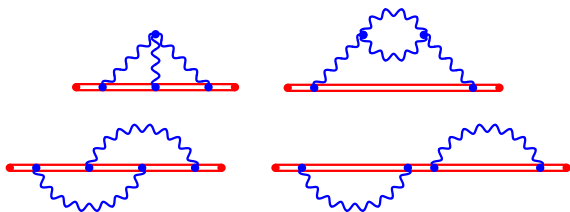


$$[t^a, t^b] = i f^{abc} t^c$$

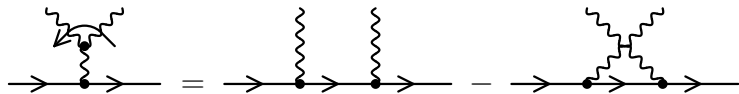


Gatheral (1983); Frenkel, Taylor (1984)

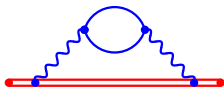
# Exponentiation in QCD



$$[t^a, t^b] = if^{abc}t^c$$



Gatheral (1983); Frenkel, Taylor (1984)



$T_F n_f \Rightarrow$  all color structures allowed

# Exponentiation in QCD

$$\log W = C_F \frac{g_0^2}{(4\pi)^{d/2}} \left[ w + (C_A w_A + T_F n_f w_f) \frac{g_0^2}{(4\pi)^{d/2}} \right. \\ \left. + (C_A^2 w_{AA} + C_F T_F n_f w_{Ff} + C_A T_F n_f w_{Af} + (T_F n_f)^2 w_{ff}) \left( \frac{g_0^2}{(4\pi)^{d/2}} \right)^2 \right]$$



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$$\Gamma = C_F \frac{\alpha_s}{\pi} \left[ \gamma + (C_A \gamma_A + T_F n_f \gamma_f) \frac{\alpha_s}{\pi} \right. \\ \left. + (C_A^2 \gamma_{AA} + C_F T_F n_f \gamma_{Ff} + C_A T_F n_f \gamma_{Af} + (T_F n_f)^2 \gamma_{ff}) \left( \frac{\alpha_s}{\pi} \right)^2 \right]$$

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Casimir scaling up to 3 loops

# Momentum space



Vertex function  $V$ : 1PI, without external-leg propagators

$$G(\omega, \omega'; \varphi) = V(\omega, \omega'; \varphi) S_v(\omega) S_{v'}(\omega')$$

$$V(\omega, \omega'; \varphi) = Z_J Z_h^{-1} V_r(\omega, \omega'; \varphi; \mu)$$

$$\log V(\omega, \omega'; \varphi) - \log V(\omega, \omega'; 0) = \log Z_J + \text{finite}$$

Convenient to set  $\omega' = \omega$

$$\varphi = 0$$

$$V(\omega, \omega'; 0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega} = Z_h^{-1} V_r(\omega, \omega'; 0; \mu)$$

$$\log V(\omega, \omega'; 0) = -\log Z_h + \text{finite}$$

$Z_h$  is gauge dependent;  $Z_J$  is gauge invariant

# History

1 loop

$$\Gamma(\alpha_s, \varphi) = C_F \frac{\alpha_s}{\pi} (\varphi \coth \varphi - 1)$$

Follows from the soft radiation function  
in classical electrodynamics

[The Guinness Book of Records](#) The anomalous  
dimension known for a longest time  
(> 100 years)

2 loops Korchemsky, Radyushkin (1987)  
Kidonakis (2009)

3 loops here

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$\gamma_h$  at 3 loops — Melnikov, van Ritbergen (2000)  
Chetyrkin, Grozin (2003)

# Abelian large $n_f$ structures

$$C_F(T_F n_f)^{L-1} \alpha_s^L \text{ and } C_F^2(T_F n_f)^{L-2} \alpha_s^L \quad (L \geq 3)$$

# Abelian large $n_f$ structures

$$C_F(T_F n_f)^{L-1} \alpha_s^L \text{ and } C_F^2(T_F n_f)^{L-2} \alpha_s^L \quad (L \geq 3)$$

QED with  $n_f$  flavors  $C_F = 1$ ,  $C_A = 0$ ,  $T_F = 1$ ,  $\beta_0 = -\frac{4}{3}n_f$

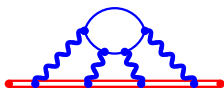
At  $L\beta_0$  and  $NL\beta_0$ , the Wilson line of any shape

$$\log W = \text{---} \img alt="Diagram of a semi-circular Wilson line with a gluon loop" data-bbox="475 345 648 415"/>$$

with



up to  $NL\beta_0$ . First broken at  $NNL\beta_0$

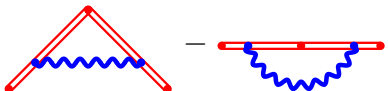


$$n_f^{L-3} \alpha_s^L \quad (L \geq 4)$$



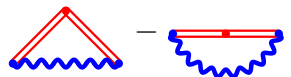
# Abelian large $n_f$ structures

$$\log W(t, t'; \varphi) - \log W(t, t'; 0)$$

$$= \text{triangle diagram} - \text{bubble diagram} = \log Z_J + \text{finite}$$


(external-leg corrections cancel). Momentum space

$$\log V(\omega, \omega; \varphi) - \log V(\omega, \omega; 0)$$

$$= \text{triangle diagram} - \text{bubble diagram} = \log Z_J + \text{finite}$$


# Abelian large $n_f$ structures

$$\begin{aligned}\Gamma &= C_F \frac{\alpha_s}{6\pi} \frac{\varphi \coth \varphi - 1}{B(2+b, 2+b)\Gamma(1+b)\Gamma(1-b)} \\ &= C_F \frac{\alpha_s}{\pi} \left[ 1 + \frac{5}{3}b - \frac{1}{3}b^2 + \left( 2\zeta_3 - \frac{1}{3} \right) b^3 + \dots \right] (\varphi \coth \varphi - 1)\end{aligned}$$

$$b = \beta_0 \frac{\alpha_s}{4\pi} \text{ Beneke, Braun (1995)}$$

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NL $\beta_0$

- ▶ NL photon self energy

$$\Pi = \text{[vacuum bubble]} + 2 \text{[vacuum bubble with fermion loop]} + \text{[vacuum bubble with photon loop]}$$

- ▶ NL  $Z_\alpha$

$C_F C_A T_F n_f$ 

$$\log Z_J = \cdots + C_F \left( \frac{\alpha_s}{\pi} \right)^3$$
$$\left[ C_A^2 z_{AA} + T_F n_f (C_F z_{Ff} + C_A z_{Af}) + (T_F n_f)^2 z_{ff} \right]$$

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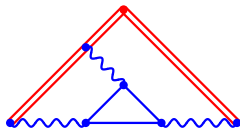
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$$N_c \rightarrow \infty \quad N_c \left( \frac{z_{Ff}}{2} + z_{Af} \right)$$

$C_F C_A T_F n_f$ 

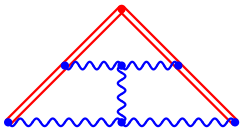
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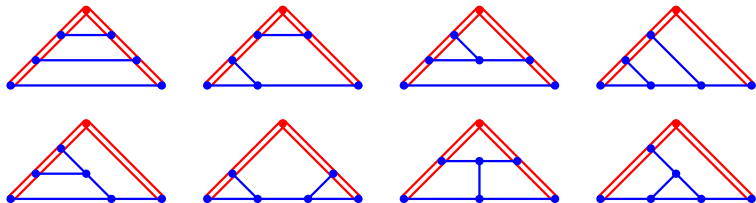
$$C_F C_A^2$$

$$N_c \rightarrow \infty \quad N_c^2 z_{AA}$$



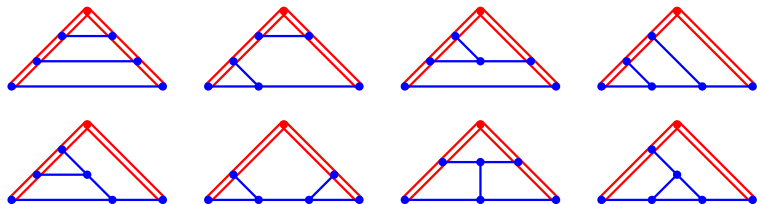
Only topologies surviving at  $N_c \rightarrow \infty$

# Topologies and master integrals





# Topologies and master integrals



71 master integrals

- ▶ 7 straight-line [Grozin (2000)]
- ▶ 8 products of lower loops
- ▶ 10 generalized triangles [Grozin, Kotikov (2011)]
- ▶ 46 nontrivial

# Differential equations

$$x = e^{-\varphi}$$

Symmetry  $x \rightarrow 1/x$

Differentiate in  $x$  and reduce to masters

Initial values at  $x = 1$

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Canonical basis  $\vec{f}$  [Henn (2013)]

$$\partial_x \vec{f}(x, \epsilon) = \epsilon \left[ \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x-1} \right] \vec{f}(x, \epsilon)$$

4 singular points

- ▶  $x = 1$  ( $\varphi \rightarrow 0$ )
- ▶  $x = 0, \infty$  ( $\varphi \rightarrow \pm\infty$ )
- ▶  $x = -1$  ( $\varphi_E \rightarrow \pi$ )

Harmonic polylogarithms  $H_{n_1 \dots n_k}(x)$  ( $n_i = 0, \pm 1$ )

[Remiddi, Vermaseren (2000)]

Uniform weight functions

# Result

$$\begin{aligned}\Gamma(\alpha_s, x) = & C_F \frac{\alpha_s}{\pi} \left\{ \tilde{A}_1 \right. \\ & + \left[ \frac{1}{2} C_A (\tilde{A}_3 + \tilde{A}_2) + \frac{1}{9} \left( \frac{67}{4} C_A - 5 T_F n_f \right) \tilde{A}_1 \right] \frac{\alpha_s}{\pi} \\ & + \left\{ \left[ \frac{1}{4} (\tilde{A}_5 + \tilde{A}_4 + \tilde{B}_5 + \tilde{B}_3) + \frac{67}{36} \tilde{A}_3 + \frac{29}{18} \tilde{A}_2 \right. \right. \\ & \quad \left. \left. + \frac{1}{24} \left( 11 \zeta_3 + \frac{245}{4} \right) \tilde{A}_1 \right] C_A^2 \right. \\ & - \left[ \frac{5}{9} (\tilde{A}_3 + \tilde{A}_2) + \frac{1}{6} \left( 7 \zeta_3 + \frac{209}{36} \right) \tilde{A}_1 \right] C_A T_F n_f \\ & \left. \left. + \left( \zeta_3 - \frac{55}{48} \right) \tilde{A}_1 C_F T_F n_f - \frac{1}{27} \tilde{A}_1 (T_F n_f)^2 \right\} \left( \frac{\alpha_s}{\pi} \right)^2 \right\}\end{aligned}$$

# Result

$$\tilde{A}_i(x) = A_i(x) - A_i(1)$$

$$A_1(x) = \frac{\xi}{2} H_1(y)$$

$$A_2(x) = \frac{1}{2} H_{1,1}(y) + \frac{\pi^2}{3} - \xi \left[ \frac{1}{2} H_{1,1}(y) - H_{1,0}(y) \right]$$

$$A_3(x) = -\xi \left[ \frac{1}{4} H_{1,1,1}(y) + \frac{\pi^2}{6} H_1(y) \right] + \xi^2 \left[ \frac{1}{4} H_{1,1,1}(y) + \frac{1}{2} H_{1,0,1}(y) \right]$$

...

$$y = 1 - x^2 \quad \xi = \frac{1 + x^2}{1 - x^2}$$

Uniform weight  $i$

$$\varphi \rightarrow \infty$$
$$x \rightarrow 0$$

$$\Gamma(\alpha_s, x) = K(\alpha_s)\varphi + \mathcal{O}(1)$$

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$$P_{q \rightarrow q}(z) = K(\alpha_s) \left( \frac{1}{1-z} \right)_+ + \dots$$

Korchemsky (1989); Korchemsky, Marchesini (1993)

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Korchemsky (1989); Korchemsky, Marchesini (1993)

$$K(\alpha_s) = C_F \frac{\alpha_s}{\pi} \left\{ 1 + \left[ \frac{1}{12} \left( \pi^2 - \frac{67}{3} \right) C_A + \frac{5}{9} T_F n_f \right] \frac{\alpha_s}{\pi} \right.$$

$$+ \left[ \frac{1}{24} \left( \frac{11}{30} \pi^4 + 11\zeta_3 - \frac{67}{9} \pi^2 + \frac{245}{4} \right) C_A^2 \right.$$

$$- \frac{1}{6} \left( 7\zeta_3 - \frac{5}{9} \pi^2 + \frac{209}{36} \right) C_A T_F n_f$$

$$\left. + \left( \zeta_3 - \frac{55}{48} \right) C_F T_F n_f - \frac{1}{27} (T_F n_f)^2 \right] \left( \frac{\alpha_s}{\pi} \right)^2 \left. \right\}$$

Moch, Vermaseren, Vogt (2004)



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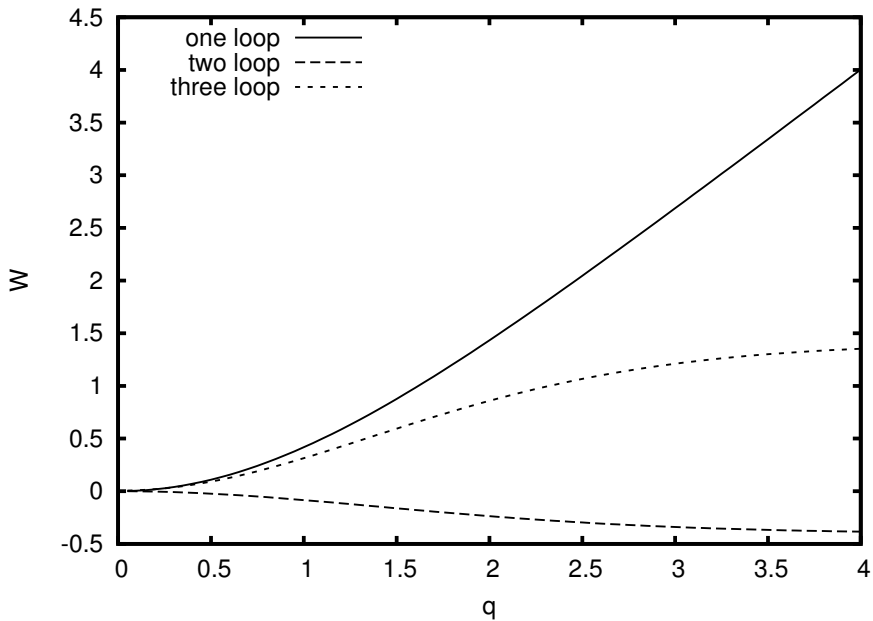
$$\left. + \left( \zeta_3 - \frac{55}{48} \right) C_F T_F n_f - \frac{1}{27} (T_F n_f)^2 \right] \left( \frac{\alpha_s}{\pi} \right)^2 \left. \right\} = C_F \frac{a}{\pi}$$

Moch, Vermaseren, Vogt (2004)

$$\Gamma(\alpha_s, x) = \Omega(a, x)$$

$$\Omega(a, x) = C_F \frac{a}{\pi} \left[ \tilde{A}_1 + \frac{1}{2} \left( \tilde{A}_3 + \tilde{A}_2 + \frac{\pi^2}{6} \tilde{A}_1 \right) C_A \frac{a}{\pi} \right. \\ \left. + \frac{1}{4} \left( \tilde{A}_5 + \tilde{A}_4 - \tilde{A}_2 + \tilde{B}_5 + \tilde{B}_3 + \frac{\pi^2}{3} \tilde{A}_3 + \frac{\pi^2}{3} \tilde{A}_2 - \frac{\pi^4}{180} \tilde{A}_1 \right) C_A^2 \left( \frac{a}{\pi} \right)^2 \right]$$

Does not contain  $n_f!$



$$\varphi_E \rightarrow \pi$$

Euclidean  $\varphi_E = \pi - \delta$

$$\Gamma = \frac{rV(r)}{\delta}$$

Kilian, Mannel, Ohl (1993)

# Conformal symmetry

Euclidean space

$$ds^2 = dx_0^2 + d\vec{x}^2$$

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Euclidean space

$$ds^2 = dx_0^2 + d\vec{x}^2$$

Spherical coordinates

$$x_0 = r \cos \delta \quad \vec{x} = r \vec{n} \sin \delta$$
$$ds^2 = dr^2 + r^2(d\delta^2 + \sin^2 \delta d\vec{n}^2)$$

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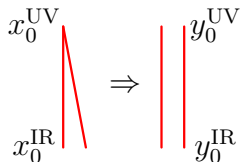
Spherical coordinates

$$x_0 = r \cos \delta \quad \vec{x} = r \vec{n} \sin \delta$$
$$ds^2 = dr^2 + r^2(d\delta^2 + \sin^2 \delta d\vec{n}^2)$$

$\delta \ll 1$

$$r = e^{y_0} \quad \vec{y} = \delta \vec{n}$$
$$ds^2 = e^{2y_0} (dy_0^2 + d\vec{y}^2)$$

# Conformal symmetry



$$\log W = \Gamma \log \frac{x_0^{\text{IR}}}{x_0^{\text{UV}}} = V(\vec{y}) (y_0^{\text{IR}} - y_0^{\text{UV}})$$

$$\Gamma = \frac{yV(y)}{\delta} = \frac{\vec{q}^2 V(\vec{q}; \alpha_s)}{4\pi\delta}$$



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3 loops

$$\delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^2 V(\vec{q}; \alpha_s(\mu))}{4\pi} = \frac{\pi}{108} \beta_0 C_F \left(\frac{\alpha_s}{\pi}\right)^3 (47C_A - 28T_F n_f)$$

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