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The property of maximal transcendentality in calculations of anomalous dimensions of the $\mathcal{N} = 4$ SYM Wilson operators and in calculations of master Feynman integrals.

OUTLINE

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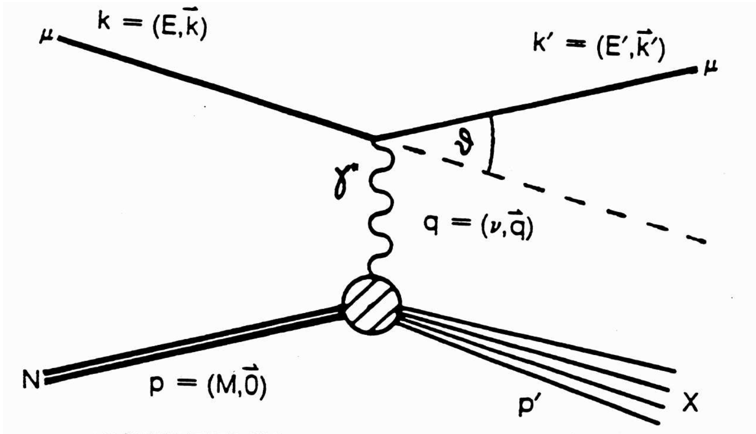
1. Introduction to DIS



A. Deep-inelastic lepton-hadron scattering (DIS) cross-section:

$$\sigma \sim L^{\mu\nu} F^{\mu\nu}$$

Hadron part $F^{\mu\nu}$ ($Q^2 = -q^2 > 0$, $x = Q^2/[2(pq)]$):

$$F^{\mu\nu} = (-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2})F_1(x,Q^2) - (p^\mu - \frac{(pq)}{q^2}q^\mu)(p^\nu - \frac{(pq)}{q^2}q^\nu)\frac{2x}{q^2}F_2(x,Q^2) + ...,$$

where $F_k(x,Q^2)$ ($k = 1,2,3,L$) - are DIS structure functions (SF) and q and p are photon and hadron (parton) momentums.

B. Wilson operator expansion: Mellin moments $M_k(j,Q^2)$

$$M_k(j,Q^2) = \int_0^1 dx\ x^{j-1} F_k(x,Q^2)$$

of DIS SF $F_k(x,Q^2)$ can be represented as sum

$$M_k(j,Q^2) = \sum_{a=ns,s,g} \underbrace{C_k^a(j,Q^2/\mu^2)}_{\text{Coeff. function}} A_a(j,\mu^2),$$

where $A_a(j,\mu^2) =< N|\mathcal{O}^a_{\mu_1,...,\mu_j}|N >$ are matrix elements of the Wilson operators $\mathcal{O}^a_{\mu_1,...,\mu_j}$.

C. The matrix elements $A_a(j,\mu^2)$ are Mellin moments of the unpolarized and polarized parton densities $f_a(x,\mu^2)$ and $\tilde{f}_a(x,\mu^2)$.

DGLAP equations:

$$\frac{d}{d\ln Q^2} f_a(x,Q^2) = \int_x^1 \frac{dy}{y} \sum_b W_{b\to a}(x/y)\, f_b(y,Q^2)$$
$$\equiv \sum_b W_{b\to a}(x) \otimes f_b(x,Q^2)\,,$$
$$\frac{d}{d\ln Q^2} \tilde{f}_a(x,Q^2) = \int_x^1 \frac{dy}{y} \sum_b \tilde{W}_{b\to a}(x/y)\, \tilde{f}_b(y,Q^2)$$
$$\equiv \sum_b \tilde{W}_{b\to a}(x) \otimes \tilde{f}_b(y,Q^2)\,. \qquad (1)$$

The anomalous dimensions (AD) $\gamma_{ab}(j)$ of the twist-2 Wilson operators $\mathcal{O}^a_{\mu_1,\ldots,\mu_j}$ (hereafter $a_s = \alpha_s/(4\pi)$)

$$\gamma_{ab}(j) = \int_0^1 dx\; x^{j-1} W_{b\to a}(x) = \sum_{m=0}^{\infty} \gamma_{ab}^{(m)}(j) a_s^m,$$
$$\tilde{\gamma}_{ab}(j) = \int_0^1 dx\; x^{j-1} \tilde{W}_{b\to a}(x) = \sum_{m=0}^{\infty} \tilde{\gamma}_{ab}^{(m)}(j) a_s^m.$$

2. Results

A. The LO AD (L.Lipatov, 1999) ($S(j) \equiv S_1(j) = \Psi(j+1) - \Psi(1)$):

for unpolarized case

$$\gamma_{gg}^{(0)}(j) = \frac{4}{j+1} - \frac{4}{j+2} - \frac{8}{j} - 4S(j-2), \qquad \gamma_{q\varphi}^{(0)}(j) = \frac{8}{j},$$

$$\gamma_{\lambda g}^{(0)}(j) = \frac{8}{j} - \frac{16}{j+1} + \frac{16}{j+2} \qquad \gamma_{\varphi g}^{(0)}(j) = \frac{12}{j+1} - \frac{12}{j+2},$$

$$\gamma_{g\lambda}^{(0)}(j) = \frac{4}{j-1} - \frac{4}{j} + \frac{2}{j+1}, \qquad \gamma_{\lambda\lambda}^{(0)}(j) = \frac{4}{j} - \frac{8}{j+1} - 4S(j-1),$$

$$\gamma_{\varphi\lambda}^{(0)}(j) = \frac{6}{j+1}, \qquad \gamma_{\varphi\varphi}^{(0)}(j) = -4S(j) \qquad \gamma_{g\varphi}^{(0)}(j) = \frac{4}{j-1} - \frac{4}{j},$$

for polarized case

$$\widetilde{\gamma}_{gg}^{(0)}(j) = \frac{8}{j} - \frac{8}{j+1} - 4S(j), \qquad \widetilde{\gamma}_{\lambda g}^{a,(0)}(j) = \frac{16}{j+1} - \frac{8}{j},$$

$$\widetilde{\gamma}_{g\lambda}^{(0)}(j) = \frac{4}{j} - \frac{2}{j+1}, \qquad \widetilde{\gamma}_{\lambda\lambda}^{(0)}(j) = \frac{4}{j+1} - \frac{4}{j} - 4S(j)$$

These matrices can be diagonalized (L.Lipatov, 1999)

$$[D\Gamma D^{-1}]^{N=4}_{\mathbf{unpol}} = \begin{vmatrix} -4S_1(j-2) & 0 & 0 \\ 0 & -4S_1(j) & 0 \\ 0 & 0 & -4S_1(j+2) \end{vmatrix}$$

$$[D\Gamma D^{-1}]^{N=4}_{\mathbf{pol}} = \begin{vmatrix} -4S_1(j-1) & 0 \\ 0 & -4S_1(j+1) \end{vmatrix},$$

The LO AD of all multiplicatively renornalized operators can be extracted through one universal function

$$\gamma^{(0)}_{uni}(j) \;=\; -\,4S(j-2) \equiv -4(\Psi(j-1)-\Psi(1)) \equiv -4\sum_{r=1}^{j-2}\frac{1}{r}$$

B. QCD results $\rightarrow \mathcal{N}=1$ SUSY results by the SUSY relations for the QCD color factors $C_F = C_A = N_c$, $T_f = N_c/2$.
The $\mathcal{N}=4$ SUSY needs contributions from scalars.

The universal AD $\gamma_{uni}(j)$ can be extracted directly from the QCD results without finding the scalar particle contribution (A.K., L.Lipatov, 2002), because

- the relation between DGLAP and BFKL dynamics in the $\mathcal{N}=4$ SYM (Lipatov idea),
- the specific properties of the kernel for the BFKL equation in this model.

C. BFKL equation

QCD in $\overline{MS}$ scheme

$$\omega^{QCD}_{\overline{MS}} = 4a_s(q^2)[\chi(n,\gamma) + \delta^{QCD}_{\overline{MS}}(n,\gamma)a_s(q^2)] \, .$$

LO part (L.Lipatov, 1986)

$$\chi(n,\gamma) = 2\Psi(1) - \Psi(\gamma + \frac{n}{2}) - \Psi(1 - \gamma + \frac{n}{2})$$

NLO part (A.K., L.Lipatov, 2000)

$$\begin{aligned}
\delta^{QCD}_{\overline{MS}}(n,\gamma) &= -\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N_c}\right)\frac{1}{2}\left(\chi^2(n,\gamma)-\Psi'(\gamma+\frac{n}{2})+\Psi'(1-\gamma+\frac{n}{2})\right) \\
&+ \left(\frac{67}{9}-2\zeta(2)-\frac{10}{9}\frac{n_f}{N_c}\right)\chi(n,\gamma)+6\zeta(3) \\
&+ \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\Big\{\left(1+\frac{n_f}{N_c^3}\right)\frac{\gamma(1-\gamma)}{2(3-2\gamma)(1+2\gamma)}\cdot\delta^2_n \\
&- \left(3+\left(1+\frac{n_f}{N_c^3}\right)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\right)\cdot\delta^0_n\Big\} \\
&+ \Psi''(\gamma+\frac{n}{2})+\Psi''(1-\gamma+\frac{n}{2})-2\Phi(n,\gamma)-2\Phi(n,1-\gamma),
\end{aligned}$$

where δ^m_n is the Kroneker symbol, and $\Psi(z)$, $\Psi'(z)$ and $\Psi''(z)$ are the Euler Ψ -function and its derivatives.

$\mathcal{N}=4$ SUSY in $\overline{DR}$ scheme $(a_s \to \hat{a}_s = a_s + \frac{1}{3}a_s^2)$

$$\omega_{\overline{DR}}^{N=4} = 4\hat{a}_s[\chi(n,\gamma) + \delta_{\overline{DR}}^{N=4}(n,\gamma)\hat{a}_s]\,.$$

$$\begin{aligned}\delta_{\overline{DR}}^{N=4}(n,\gamma) \;&=\; 6\zeta(3) + \Psi''(\gamma + \frac{n}{2}) + \Psi''(1-\gamma+\frac{n}{2}) \\ &-\; 2\Phi(n,\gamma) - 2\Phi(n,1-\gamma) - 2\zeta(2)\chi(n,\gamma)\end{aligned}$$

In the $\mathcal{N}=4$ SUSY, the eigenvalues of the BFKL equation are analytic functions of the conformal spin $|n|$.

Transcendentality principe. (A.K., L.Lipatov, 2002)

In the $\overline{\mathrm{DR}}$-scheme, there is no mixing among the functions of different transcendentality levels i, i.e. all special functions at the NLO correction contain only sums of the terms $\sim 1/\gamma^i$ $(i = 3)$.

More precisely, if we introduce the transcendentality level for the eigenvalues of integral kernels of the BFKL equations:

$$\Psi \sim 1/\gamma, \quad \Psi' \sim \beta' \sim \zeta(2) \sim 1/\gamma^2, \quad \Psi'' \sim \beta'' \sim \zeta(3) \sim 1/\gamma^3,$$

then for the BFKL kernel in LO and in NLO the corresponding levels are $i = 1$ and $i = 3$, respectively.

Lipatov idea. In $\mathcal{N} = 4$ SYM there is a relation between the BFKL and DGLAP equations: BFKL results reproduce DGLAP ones at the nonphysical values $|n| \to -(1+r), r > 0$.

Transcendentality principe. (A.K., L.Lipatov, 2002)
So, $\gamma^{(0)}_{uni}(j)$, $\gamma^{(1)}_{uni}(j)$ and $\gamma^{(2)}_{uni}(j)$ are should contain (in the $\overline{\mathrm{DR}}$-scheme) also only the functions assumed to be of the types $\sim 1/j^i$ with the levels $i = 1$, $i = 3$ and $i = 5$, respectively.

Further, the universal AD $\gamma^{(0)}_{uni}(j)$, $\gamma^{(1)}_{uni}(j)$ and $\gamma^{(2)}_{uni}(j)$ should be equal to a combination of the *most complicated contributions* to the QCD AD in LO, NLO and NNLO (i.e. the functions with a maximal value of the transcendentality levels $i = 1$, $i = 3$ and $i = 5$, respectively) with the SUSY relation for the QCD color factors $C_F = C_A = N_c$.

D. The universal anomalous dimension $\gamma_{uni}(j)$ for $\mathcal{N} = 4$ SYM is (based on (A.Moch, J.A.M.L.Vermaseren, A.Vogt, 2004))

$$\gamma(j) \equiv \gamma_{uni}(j) \;=\; \hat{a}\gamma_{uni}^{(0)}(j) + \hat{a}^2\gamma_{uni}^{(1)}(j) + \hat{a}^3\gamma_{uni}^{(2)}(j) + ...,$$

where (A.K., L.Lipatov, A.Onischenko, V.Velizhanin, 2004)

$$
\begin{aligned}
\frac{1}{4}\gamma^{(0)}_{uni}(j+2) &= -S_1, \\
\frac{1}{8}\gamma^{(1)}_{uni}(j+2) &= (S_3+\overline{S}_{-3}) - 2\,\overline{S}_{-2,1} + 2\,S_1(S_2+\overline{S}_{-2}), \\
\frac{1}{32}\gamma^{(2)}_{uni}(j+2) &= 2\,\overline{S}_{-3}\,S_2 - S_5 - 2\,\overline{S}_{-2}\,S_3 - 3\,\overline{S}_{-5} + 24\,\overline{S}_{-2,1,1,1} \\
&\quad +6(\overline{S}_{-4,1}+\overline{S}_{-3,2}+\overline{S}_{-2,3}) - 12(\overline{S}_{-3,1,1}+\overline{S}_{-2,1,2}+\overline{S}_{-2,2,1}) \\
&\quad -(S_2+2\,S_1^2)(3\,\overline{S}_{-3}+S_3-2\,\overline{S}_{-2,1}) - S_1(8\,\overline{S}_{-4}+\overline{S}^2_{-2} \\
&\quad +4\,S_2\,\overline{S}_{-2}+2\,S_2^2+3\,S_4-12\,\overline{S}_{-3,1}-10\,\overline{S}_{-2,2}+16\,\overline{S}_{-2,1,1}),
\end{aligned}
$$

$S_{\pm a} \equiv S_{\pm a}(j),\ S_{\pm a,b} \equiv S_{\pm a,b}(j),\ S_{\pm a,b,c} \equiv S_{\pm a,b,c}(j)$ are harmonic sums

$$
\begin{aligned}
S_a(j) &= \sum_{m=1}^{j}\frac{1}{m^a}, \quad S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j}\frac{1}{m^a}\,S_{b,c,\cdots}(m), \\
S_{-a}(j) &= \sum_{m=1}^{j}\frac{(-1)^m}{m^a}, \quad S_{-a,b,c,\cdots}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a}\,S_{b,c,\cdots}(m), \\
\overline{S}_{-a,b,c,\cdots}(j) &= (-1)^j\,S_{-a,b,c,\ldots}(j) + S_{-a,b,c,\cdots}(\infty)\,(1-(-1)^j)
\end{aligned}
$$

E. The N^{3}LO and N^{4}LO universal anomalous dimensions from Bethe Ansatz.

Using the algebraic Bethe Ansatz (N.Beisert, M.Staudacher, 2005), the result for the four-loop anomalous dimension can be obtained the following form:
(A.K., L.Lipatov, A.Rey, M.Staudacher, V.Velizhanin, 2007)

$$
\begin{aligned}
\frac{1}{256}\,\gamma_{uni}^{ABA}(j+2) \;=\;& \\
& \mathbf{4\,S_{-7} + 6\,S_7} + 2\,(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1}) + 3\,(-S_{-2,5} \\
+\;& S_{-2,3,-2}) + 4\,(S_{-2,1,4} + -S_{-2,-2,-2,1} - S_{-2,1,2,-2} - S_{-2,2,1,-2} - S_{1,-2,1,3} \\
-\;& S_{1,-2,2,2} - S_{1,-2,3,1}) + 5\,(-S_{-3,4} + S_{-2,-2,-3}) + 6\,(-S_{5,-2} \\
+\;& S_{1,-2,4} - S_{-2,-2,1,-2} - S_{1,-2,-2,-2}) + 7\,(-S_{-2,-5} + S_{-3,-2,-2} \\
+\;& S_{-2,-3,-2} + S_{-2,-2,3}) + 8\,(S_{-4,1,2} + S_{-4,2,1} - S_{-5,-2} - S_{-4,3} \\
-\;& S_{-2,1,-2,-2} + S_{1,-2,1,1,-2}) + 9\,S_{3,-2,-2} - 10\,S_{1,-2,2,-2} + 11\,S_{-3,2,-2} \\
+\;& 12\,(-S_{-6,1} + S_{-2,2,-3} + S_{1,4,-2} + S_{4,-2,1} + S_{4,1,-2} - S_{-3,1,1,-2} - S_{-2,2,-2,1} \\
-\;& S_{1,1,2,3} - S_{1,1,3,-2} - S_{1,1,3,2} - S_{1,2,1,3} - S_{1,2,2,-2} - S_{1,2,2,2} - S_{1,2,3,1} - S_{1,3,1,-2} \\
-\;& S_{1,3,1,2} - S_{1,3,2,1} - S_{2,-2,1,2} - S_{2,-2,2,1} - S_{2,1,1,3} - S_{2,1,2,-2} - S_{2,1,2,2} \\
-\;& S_{2,1,3,1} - S_{2,2,1,-2} - S_{2,2,1,2} - S_{2,2,2,1} - S_{2,3,1,1} - S_{3,1,1,-2} - S_{3,1,1,2} - S_{3,1,2,1} \\
-\;& S_{3,2,1,1}) + 13\,S_{2,-2,3} - 14\,S_{2,-2,1,-2} + 15\,(S_{2,3,-2} + S_{3,2,-2}) \\
+\;& 16\,(S_{-4,1,-2} + S_{-2,1,-4} - S_{-2,-2,1,2} - S_{-2,-2,2,1} - S_{-2,1,-2,2} - S_{-2,1,1,-3} \\
-\;& S_{1,-3,1,2} - S_{1,-3,2,1} - S_{1,-2,-2,2} - S_{2,-2,-2,1} + S_{-2,1,1,-2,1} + S_{1,1,-2,1,-2} \\
+\;& S_{1,1,-2,1,2} + S_{1,1,-2,2,1}) - 17\,S_{-5,2} + 18\,(-S_{4,-3} - S_{6,1} + S_{1,-3,3}) \\
+\;& 20\,(-S_{1,-6} - S_{1,6} - S_{4,3} + S_{-5,1,1} + S_{-4,-2,1} + S_{-3,-2,2} + S_{-2,-4,1} \\
+\;& S_{-2,-3,2} + S_{1,3,3} + S_{3,1,3} + S_{3,3,1} - S_{1,1,-2,3} - S_{1,2,-2,-2} - S_{2,1,-2,-2}) \\
-\;& 21\,S_{3,4} + 22\,(S_{1,-2,-4} + S_{2,2,3} + S_{2,3,2} + S_{3,-2,2} + S_{3,2,2}) + 23\,(-S_{-3,-4}
\end{aligned}
$$

$$
\begin{aligned}
&- \; S_{5,2} + S_{2,-2,-3}) + 24\,(-S_{-4,-3} + S_{1,-4,-2} - S_{1,-3,1,-2} - S_{1,1,1,4} - S_{1,1,4,1} \\
&- \; S_{1,3,-2,1} - S_{1,4,1,1} - S_{3,-2,1,1} - S_{3,1,-2,1} - S_{4,1,1,1} + S_{-2,-2,1,1,1} + S_{-2,1,-2,1,1} \\
&+ \; S_{1,-2,-2,1,1} + S_{1,-2,1,-2,1} + S_{1,1,-2,-2,1} + S_{1,1,1,-2,-2} + S_{1,1,2,-2,1} + S_{1,2,1,-2,1} \\
&+ \; S_{2,1,1,-2,1}) + 25\,S_{2,-3,-2} + 26\,(-S_{2,5} + S_{1,4,2} + S_{2,4,1} + S_{4,1,2} + S_{4,2,1}) \\
&+ \; 28\,(S_{1,2,4} + S_{2,1,4} - S_{-3,1,-2,1} - S_{-2,1,-3,1} - S_{1,-2,1,-3}) + 30\,S_{-3,1,-3} \\
&+ \; 32\,(S_{1,5,1} + S_{5,1,1} - S_{-3,-2,1,1} - S_{-2,-3,1,1} - S_{1,-3,-2,1} - S_{1,-2,-3,1} \\
&- \; S_{2,2,-2,1} + S_{1,2,-2,1,1} + S_{2,1,-2,1,1} - S_{1,1,1,-2,1,1}) + 36\,(S_{1,1,5} + S_{1,3,-3} \\
&+ \; S_{3,1,-3} - S_{1,1,-3,-2} - S_{1,1,-2,-3} - S_{1,1,2,-3} - S_{1,2,-2,2} - S_{1,2,1,-3} - S_{2,1,-2,2} \\
&- \; S_{2,1,1,-3}) + 38\,S_{-3,-3,1} + 40\,(-S_{1,-4,1,1} - S_{2,-3,1,1} + S_{1,1,1,-2,2}) \\
&- \; 41\,S_{3,-4} + 42\,(-S_{2,-5} + S_{1,-4,2} + S_{1,-3,-3}) + 44\,(S_{1,-5,1} + S_{2,-3,2} + S_{3,-3,1}) \\
&+ \; 46\,S_{2,2,-3} + 48\,S_{1,1,-3,1,1} + 60\,(S_{1,1,-5} - S_{1,1,-3,2}) + 62\,S_{2,-4,1} + 64\,S_{1,1,1,-3,1} \\
&+ \; 68\,(S_{1,2,-4} + S_{2,1,-4} - S_{1,2,-3,1} - S_{2,1,-3,1}) - 72\,S_{1,1,1,-4} - 80\,S_{1,1,-4,1} \\
&- \; \boldsymbol{\zeta(3) S_1 (S_3 - S_{-3} + 2\,S_{-2,1})}.
\end{aligned}
$$

BFKL constrains: **the above result is wrong.**

LO BFKL: 4-loop $\gamma_{uni}(j) \sim \varepsilon^{-4}$ when $j \to 1 - \varepsilon$.

Above: $\gamma_{uni}^{ABA}(j) \sim \varepsilon^{-7}$, because $S_{\pm a} \sim \pm\varepsilon^{-7}$.

Three larger poles are wrong!!!

Wrapping effects?? (they are absend in the ABA) Yes!!!

Including the wrapping effects:

(Z. Bajnok, R. Janik, T. Lukowski, 2008)

$$\gamma_{uni}(j+2) \;=\; \gamma_{uni}^{ABA}(j+2) + \gamma_{uni}^{wr}(j+2),$$

$$\frac{1}{256}\,\gamma_{uni}^{wr}(j+2) \;=\; \frac{1}{2}\,\boldsymbol{S_1^2}\,[\mathbf{2}\,\boldsymbol{S_{-5}} + \mathbf{2}\,\boldsymbol{S_5}$$
$$+4\left(S_{4,1} - S_{3,-2} + S_{-2,-3} - 2\,S_{-2,-2,1}\right)$$
$$-4\,S_{-2}\zeta(3) - 5\,\zeta(5)]$$

Now the 5-loop $\gamma_{uni}(j+2)$ has been calculated!!!

(T. Lukowski, A. Rej, V.N. Velizhanin, 2009)

3. Master Integrals

A. Calculations of master integrals (MI) with direct calcualtions and without direct calcualtions (J.Fleischer, A.K., O.Veretin, 1997, 1998), (A.K., J.H. Kuhn, O.Veretin, 2007)

Above arguments (**related with maximal complexity or maximal weight**) give a possibility to calculate a large class of Feynman diagrams, so-called master-integrals (Broadhurst, 1987).

Application of the integration-by-part (IBP) procedure (Chetyrkin, Tkachev,1981; Vassiliev et al., 1981) to loop internal momenta leads to relations between different Feynman integrals (FI) and, thus, to necessity to calculate only some of them, which in a sense, are independent (which were chosen quite arbitrary, of course). They are called the master-integrals (MI) (Broadhurst, 1987).
Shortly: IBP[FI] $\rightarrow$ MI

The application of the IBP procedure to the MI themselves leads to the differential equations (DEs) for them with the inhomogeneous terms (ITs) containing less complicated diagrams.
[**The "less complicated diagrams" contain usually less number of propagators.**]
Shortly: IBP[MI] $\rightarrow$ DEs(MI) with IT(LESS)

Analogously, IBP[LESS] $\to$ DEs(LESS) with IT(LESS^2).
And, finally, IBP[LESS^n] $\to$ DEs(LESS^n) with IT(LESS^{n+1}).

Repeating the procedure several times, at a last step one can obtain the ITs containing mostly only tadpoles.

Solving the DEs at this last step, one can reproduce the diagrams for ITs of the DEs at the previous step. Repeating the procedure several times one can obtain the results for the initial FI.

This scheme has been used successfully for variaous calculations. It is very powerful but quite complicated. There are, however, some simplifications based on the series representations of FI.

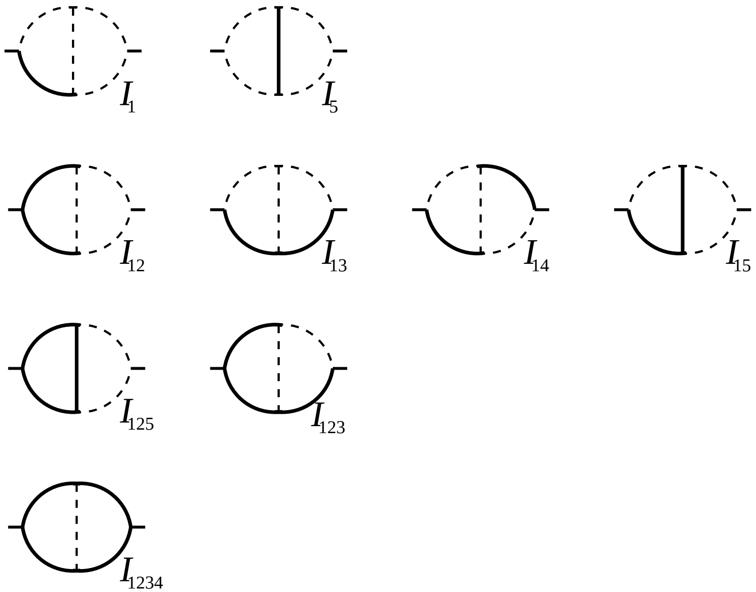


Fig. 2

Consider the inverse-mass expansion of two-loop two-point (see Fig. 2) and three-point diagrams (see Fig. 3) with one nonzero mass.

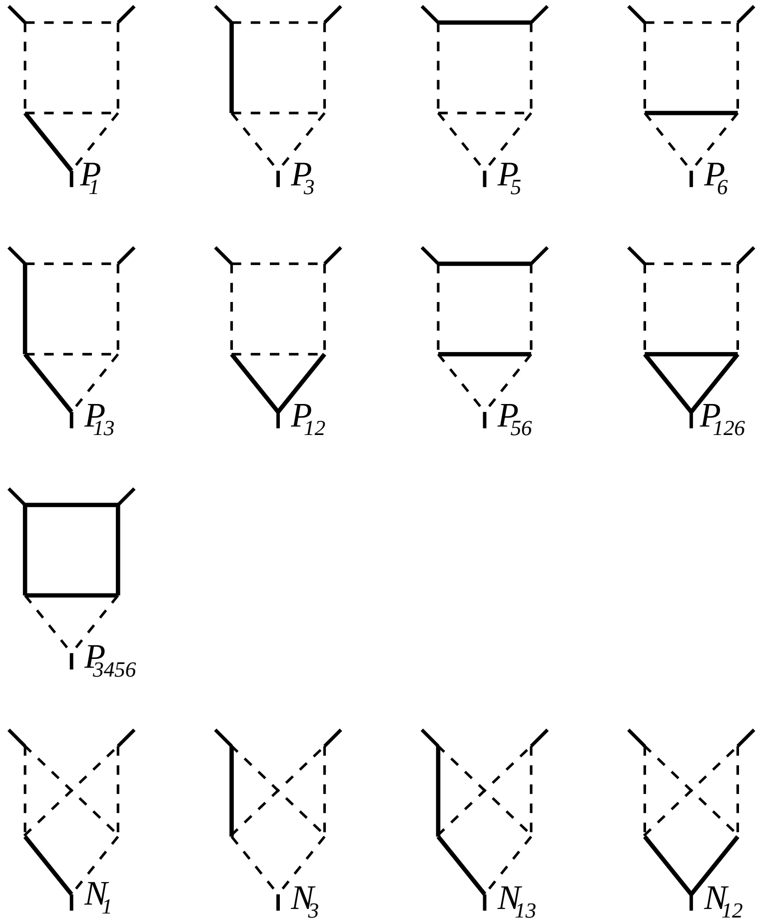


Fig. 3

[We consider only three-point diagrams with the conditions $q_1^2 = q_2^2 = 0$ **and** $(q_1 + q_2)^2 \equiv q^2 \neq 0$**.]**

The results have the floowing forms

$$\begin{aligned}
\text{FI} &= \frac{\hat{N}}{q^{2\alpha}} \sum_{n=1} C_n \frac{(\eta x)^n}{n^c} \{F_0(n) \\
&+[\ln(-x)\, F_{1,1}(n) + \frac{1}{\varepsilon} F_{1,2}(n)] \\
&+[\ln^2(-x)\, F_{2,1}(n) + \frac{1}{\varepsilon}\ \ln(-x)\, F_{2,2}(n) + \frac{1}{\varepsilon^2} F_{2,3}(n) + \zeta(2)\, F_{2,4}(n)] \\
&+[\ln^3(-x)\, F_{3,1}(n) + \frac{1}{\varepsilon}\ \ln^2(-x)\, F_{3,2}(n) + \frac{1}{\varepsilon^2}\ \ln(-x)\, F_{3,3}(n) \\
&+\frac{1}{\varepsilon^3} F_{3,4}(n) + \zeta(2)\ \ln(-x)\, F_{3,5}(n) + \frac{1}{\varepsilon} \zeta(2)\, F_{3,6}(n) + \zeta(3)\, F_{3,7}(n)] \\
&+ \cdots\}, \qquad (2)
\end{aligned}$$

where $x = q^2/m^2$, $\eta = 1$ or -1, $c = 0$, 1 and 2, and $\alpha = 1$ and 2 for two-point and three-point cases, respectively.

Here the normalization $\hat{N} = (\overline{\mu}^2/m^2)^{2\varepsilon}$, where $\overline{\mu} = 4\pi e^{-\gamma_E}\mu$ is in the standard $\overline{MS}$-scheme and γ_E is the Euler constant.

Moreover, $D = 4 - 2\varepsilon$ and

$$C_n \; = \; 1 \qquad (3)$$

for diagrams with one-massive-particle-cuts (m-cuts) and

$$C_n \; = \; 1, \quad \text{and} \quad C_n \; = \; \frac{(n!)^2}{(2n)!} \; \equiv \; \hat{C}_n \qquad (4)$$

for diagrams with two-massive-particle-cuts ($2m$-cuts).

For m-cut case, the coefficients $F_{N,k}(n)$ should have the form

$$F_{N,k}(n) \; \sim \; \frac{S_{\pm a,\ldots}}{n^b} \, ,$$

where $S_{\pm a} \equiv S_{\pm a}(j-1)$, $S_{\pm a,\pm b} \equiv S_{\pm a,\pm b}(j-1)$, $S_{\pm a,\pm b,\pm c} \equiv S_{\pm a,\pm b,\pm c}(j-1)$ are harmonic sums

$$S_{\pm a}(j) \; = \; \sum_{m=1}^{j} \frac{(\pm 1)^m}{m^a}, \quad S_{\pm a,\pm b,\pm c,\cdots}(j) \; = \; \sum_{m=1}^{j} \frac{(\pm 1)^m}{m^a} \, S_{\pm b,\pm c,\cdots}(m),$$

For $2m$-cut case, the coefficients $F_{N,k}(n)$ should have the form

$$F_{N,k}(n) \sim \frac{S_{\pm a,\ldots}}{n^b}, \frac{V_{a,\ldots}}{n^b}, \frac{W_{a,\ldots}}{n^b}$$

where

$$V_a(j) = \sum_{m=1}^{j} \frac{\hat{C}_m}{m^a}, \quad V_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{\hat{C}_m}{m^a} S_{b,c,\cdots}(m),$$

$$W_a(j) = \sum_{m=1}^{j} \frac{\hat{C}_m^{-1}}{m^a}, \quad W_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{\hat{C}_m^{-1}}{m^a} S_{b,c,\cdots}(m),$$

The terms $\sim V_{a,\ldots}$ and $\sim W_{a,\ldots}$ can come only together with the coefficients $C_n = 1$ and $C_n = \hat{C}_n$, respectively. The terms $\sim S_{\pm a,\ldots}$ can appear in combination with both C_n values.

The origin of the appearance of the terms $\sim V_{a,\ldots}$ and $\sim W_{a,\ldots}$ in the $2m$-cut case, is the product of series (2) with the different values of the coefficients $C_n = 1$ and $C_n = \hat{C}_n$.

As examples, consider two-loop two-point diagrams I_1, I_5 and I_{12} shown in Fig. 2.

$$I_1 = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n} \{\frac{1}{2}\ln^2(-x) - \frac{2}{n}\ln(-x) + \zeta(2) + 2S_2 - 2\frac{S_1}{n} + \frac{3}{n^2}\},$$

$$I_5 = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{(-x)^n}{n} \{-\ln^2(-x) + \frac{2}{n}\ln(-x) - 2\zeta(2) - 4S_{-2} - \frac{2}{n^2} - 2\frac{(-1)^n}{n^2}\}, \tag{5}$$

$$I_{12} = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n^2} \{\frac{1}{n} + \frac{(n!)^2}{(2n)!}(-2\ln(-x) - 3W_1 + \frac{2}{n})\}. \tag{6}$$

From (5) one can see that the corresponding functions $F_{N,k}(n)$ have the form

$$F_{N,k}(n) \sim \frac{1}{n^{2-N}}, \quad (N \geq 2), \tag{7}$$

if we introduce the following complexity of the sums ($\Sigma_{i=1}^{m} a_i = a$)

$$\Phi_{\eta a} \sim \Phi_{\eta a_1,\eta a_2} \sim \Phi_{\eta a_1,\eta a_2,\cdots,\eta a_m} \sim \zeta_a \sim \frac{1}{n^a}, \tag{8}$$

where $\Phi = (S, V, W)$.

In Eq. (6),

$$F_{N,k}(n) \sim \frac{1}{n^{1-N}}, \qquad (N \geq 1),$$

since now the factor $1/n^2$ has been already extracted.

So, Eqs. (6) show that the functions $F_{N,k}(n)$ should have the following form

$$\frac{1}{n^c} F_{N,k}(n) \sim \frac{1}{n^{3-N}}, \qquad (N \geq 2) \qquad (9)$$

and the number $3 - N$ defines the level of transcendentality (or complexity) of the coefficients $F_{N,k}(n)$.

The property reduces strongly the number of the possible elements in $F_{N,k}(n)$. The level of transcendentality decreases if we consider the singular parts of diagrams and/or coefficients in front of ζ-functions and of logarithm powers.

Other I-type integrals have similar form. They have been calculated exactly by differential equation method.

Now we consider two-loop three-point diagrams, P_1, P_5, P_6, P_{13} and P_{12} shown in Figure 3

$$
\begin{aligned}
P_1 = & \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{x^n}{n} \{ -\frac{1}{2\varepsilon^3} - \frac{S_1}{\varepsilon^2} + \frac{1}{2\varepsilon}[5S_2 - S_1^2 + \frac{2}{n^2} - \frac{2}{n} \ln(-x) \\
& + \frac{1}{2} \ln^2(-x) - \zeta(2)] - \frac{8}{3}\zeta_3 - (S_1 + \frac{1}{n})\zeta_2 + \frac{8}{3}S_3 + \frac{9}{2}S_1 S_2 + \frac{5}{6}S_1^3 \\
& + 4\frac{S_2}{n} + 2\frac{S_1}{n^2} + \frac{3}{n^3} + (\zeta_2 - 4S_2 - 2\frac{S_1}{n} - \frac{3}{n^2}) \ln(-x) \\
& + (S_1 + \frac{3}{2n}) \ln^2(-x) - \frac{1}{2} \ln^3(-x) \}, \\
P_5 = & \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{(-x)^n}{n} \{ -6\zeta_3 + 2(S_1\zeta_2 + 6S_3 - 2S_1 S_2 + 4\frac{S_2}{n} - \frac{S_1^2}{n} \\
& + 2\frac{S_1}{n^2} + (-4S_2 + S_1^2 - 2\frac{S_1}{n}) \ln(-x) + S_1 \ln^2(-x) \},
\end{aligned}
$$

$$P_6 = \frac{\hat{N}}{(q^2)^2} \sum_{n=1} \frac{(-x)^n}{n} \{-\frac{1}{\varepsilon^2}[\ln(-x) - \frac{1}{n}] + \frac{1}{\varepsilon}[\zeta_2 - 3S_2 - 4S_{-2}$$

$$-3\frac{S_1}{n} - \frac{3}{n^2} + (3S_1 + \frac{3}{n})\ln(-x) - \frac{3}{2}\ln^2(-x)] + 2\zeta_3 + (7S_1 + \frac{2}{n})\zeta_2$$

$$-2S_3 - 9S_1S_2 + 10S_{-3} - 12S_{-2,1} - 4S_1S_{-2} - \frac{7}{2}\frac{S_2}{n} - \frac{9}{2}\frac{S_1^2}{n}$$

$$-5\frac{S_1}{n^2} - \frac{7}{n^3} + (\frac{7}{2}S_2 - \frac{9}{2}S_1^2 + 5\frac{S_1}{n} + \frac{7}{n^2} - 2\zeta_2)\ln(-x)$$

$$+\frac{1}{2}(7S_1 + \frac{7}{n})\ln^2(-x) + \frac{7}{6}\ln^3(-x)\},$$

$$P_{13} = \frac{\hat{N}}{(q^2)^2} \sum_{n=1} x^n \{-\frac{S_2}{2\varepsilon^2} - \frac{1}{2\varepsilon}[S_3 + 4S_{1,2} - 4\frac{S_2}{n}] + \frac{S_2}{2}\zeta_2 - S_{1,3}$$

$$-3S_{3,1} + 3S_{1,1,2} + 3S_{1,2,1} - S_2^2 + (7S_3 - 8S_{1,2})S_1 + \frac{5}{2}S_1^2S_2\},$$

$$P_{12} = \frac{\hat{N}}{q^2} \sum_{n=1} \frac{x^n}{n^2}\frac{(n!)^2}{(2n)!}\{\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}(S_1 - 3W_1 + \frac{1}{n} - \ln(-x)) + 12W_2$$

$$-18W_{1,1} - 13S_2 + S_1^2 - 6S_1W_1 + 2\frac{S_1}{n} + \frac{2}{n^2} - 2(S_1 + \frac{1}{n})\ln(-x)$$
$$+ \ln^2(-x)\}\,, \tag{10}$$

Now the coefficients $F_{N,k}(n)$ have the form

$$\frac{1}{n^c} F_{N,k}(n) \;\sim\; \frac{1}{n^{4-N}}, \qquad (N \geq 3), \tag{11}$$

The diagrams P_1, P_5 and P_6 have been calculated exactly by differential equation method.

To find the results for P_{13} and P_{12} (and also all others in (J.Fleischer, A.K., O.Veretin, 1997, 1998)) we have used the knowledge of the several n terms in the inverse-mass expansion (2) (usually less than $n = 100$) and the following arguments:

- The coefficients should have the structure (11) with the rule (8). The condition (11) reduces strongly the number of possible harmonic sums. It should are related with the specific form of the differential equations for the considered master integrals, like

$$\left(\overline{k}\varepsilon + m^2\frac{d}{dm^2}\right) \ \text{FI} \ = \ \text{LESS}$$

with some $\overline{k}$ values.

- If a two-loop two-point diagram with the "similar topology" (for example, I_1 for P_1 and P_3, I_5 for P_5 and P_6, I_{12} for P_{12} an so on) has been already calculated, we should consider a similar set of basic elements for corresponding $F_{N,k}(n)$ of two-loop three-point diagrams but with the higher level of complexity.

- Let the considered diagram contain singularities and/or powers of logarithms.

  We note that every part ($\sim \ln^k x$, $\sim \zeta(k)$, $\sim \varepsilon^{-k}$) is calculated independently.

  Because in front of the leading singularity, or the largest power of logarithm, or the largest ζ-function the coefficients are very simple, they can be often predicted directly from the first several terms of expansion.

Moreover, often we can calculate the singular part using another technique. Then we should expand the singular parts, find the basic elements and try to use them (with the corresponding increase of the level of complexity) to predict the regular part of the diagram. If we have to find the ε-suppressed terms, we should increase the level of complexity for the corresponding basic elements.

Later, using the ansatz for $F_{N,k}(n)$ and several terms (usually, less than 100) in the above expression, **which can be calculated exactly in the above inverse-mass expansion (2) using O.Tarasov and V.Smirnov investigations** and compare them with the predictions of our anzatz with the above fixed coefficients.

The arguments give a possibility to find the results for many complicated two-loop three-point diagrams **without direct calculations**. Some variations of the procedure have been successfully used for calculating the Feynman diagrams for many processes.

The above series with $C_n = 1$ and $C_n = \hat{C}_n$ correspond, respectively, to the combinations of the Polylogarithms

$$
\begin{aligned}
\frac{q^2}{\hat{N}} I_1 &= (\frac{1}{2}\ln^2(-x) + \zeta_2 + 2\mathrm{Li}_2(\mathrm{x}))\mathrm{Li}_1(\mathrm{x}) - 2\ln(-\mathrm{x})\mathrm{Li}_2(\mathrm{x}) \\
&+3\mathrm{Li}_3(\mathrm{x}) - 6\mathrm{S}_{1,2}(\mathrm{x}) \\
\frac{q^2}{\hat{N}} I_5 &= -(\ln^2(-x) + 2\zeta_2 + 4\mathrm{Li}_2(\mathrm{x}))\mathrm{Li}_1(-\mathrm{x}) + 2\ln(-\mathrm{x})\mathrm{Li}_2(-\mathrm{x}) \\
&-2\mathrm{Li}_3(-\mathrm{x}) - 2\mathrm{Li}_3(\mathrm{x}) + 2\mathrm{S}_{1,2}(\mathrm{x}^2) - 2\mathrm{S}_{1,2}(\mathrm{x}) - 4\mathrm{S}_{1,2}(-\mathrm{x}) \\
\frac{q^2}{\hat{N}} I_{12} &= -\frac{1}{6}\ln^3(y) - (\zeta_2 + 4\mathrm{Li}_2(\mathrm{y}))\ln(\mathrm{y}) + 4\mathrm{Li}_3(\mathrm{y}) - 3\mathrm{Li}_3(\mathrm{y}) + \frac{1}{3}\mathrm{Li}_3(-\mathrm{y}^3) \\
&-6\zeta_3 + \mathrm{Li}_3(\mathrm{x}),
\end{aligned}
$$

where $\mathrm{S}_{\mathrm{a,b}}(\mathrm{x})$ are Nilsen Polylogarithms

$$
\mathrm{S}_{\mathrm{a+1,b}}(\mathrm{x}) = \frac{(-1)^{\mathrm{a+b}}}{\mathrm{a!b!}} \int_0^1 \frac{\mathrm{dt}}{\mathrm{t}} \ln^{\mathrm{a}}(\mathrm{t}) \ln^{\mathrm{b}}(1 - \mathrm{xt}), \;\; \mathrm{Li}_{\mathrm{a}}(\mathrm{x}) = \mathrm{S}_{\mathrm{a-1,b}}(\mathrm{x}),
$$

$$
\mathrm{Li}_1(\mathrm{x}) = -\ln(1 - \mathrm{x}), \;\; \mathrm{y} = \frac{1 - \sqrt{\frac{\mathrm{x}}{\mathrm{x+4}}}}{1 + \sqrt{\frac{\mathrm{x}}{\mathrm{x+4}}}}
$$

Elliptic Polylogariths

It is well known that the off-shall sunset contains elliptic integrals and some their generalizations. The resuls for the off-shall sunset can be expressed through Appel-functions (O.V. Tarasov, 2006).

We have studied another cases, containing elliptic integrals and some their generalizations. They are the sunsets

$$\mathrm{Sun}_{a,b,c}(m^2, M^2) \sim \int \frac{d^D k_1 d^D k_2}{(k_1^2 + M^2)^a (k_2^2 + M^2)^b ((p - k_1 - k_2)^2 + m^2)^c}$$
$$(p^2 = -m^2)$$

and also similar tree-poin and four-point diagrams. They came in the framework of nonrelativistic QCD (when $m^2 = M^2$).

The series expansions of such types of the diagrams have the form, which already considered above but with (B.A. Kniehl, A.V.K., A.I. Onischenko, O.L. Veretin, 2005)

$$C_n = \frac{[(2n)!]^3}{[n!]^2(4n)!}, \quad \text{and} \quad C_n = \frac{(n!)^2(2n)!}{(4n)!}$$

The results should be expressed as several-fold integrals of the elliptic integrals. Now there are papers, where elliptic polylogarithms were introduced (S. Bloch, P. Vanhove, 2013), (L. Adams, C. Bogner, S. Weinzierl, 2014, 2015). We hope to express our results in the form of the elliptic polylogarithms.

Popular studies

Now there is a popular extension of above results:

$$\left(\overline{k}\varepsilon + m^2 \frac{d}{dm^2}\right) \ \mathsf{FI} \ = \ \mathsf{LESS} \ ,$$

based on he representation of the LESS diagrams in the similar form

$$\left(\overline{k_1}\varepsilon + m^2 \frac{d}{dm^2}\right) \ \mathsf{LESS} \ = \ \mathsf{LESS}^2$$

and so on

$$\left(\overline{k_n}\varepsilon + m^2 \frac{d}{dm^2}\right) \ \mathsf{LESS}^n \ = \ \mathsf{LESS}^{n+1}$$

Constructing the column with the elements like

[to have same level of complexity for all terms $LESS^n$]

$F_0 = FI$, $F_n = LESS^n/\varepsilon^n$ $(n \geq 1)$, it is possible to obtain the homogeneous matrix equation

$$\left(\hat{k}\varepsilon + m^2 \frac{d}{dm^2}\right) \hat{F} \;=\; 0$$

Such type of matrix equations is popular now (see **talks of A. Grozin and V. Smirnov on the workshop**). The frans-formation to this form was discussed in **the talk of R. Lee**.

Conclusion

- I have demonstrated a way to find the universal AD $\gamma_{uni}(j)$ for the $\mathcal{N} = 4$ SUSY without a direct calculations (but based on calculations in QCD and on transcendentality principe).

- Bethe Ansatz and evaluation of the N^{3}LO andN^{4}LO universal anomalous dimensions using transcendentality principe.

- For series expansions of the master integrals with masses there is more complicated version of transcendentality (or complexity) principe.

  Similar approach has been mentioned also for the series expansions, corresponding to several-fold integrals of elliptic integrals. In our future studies we hope to express them through the Elliptic Polylogarithms.