Supersymmetric Yang-Mills theory in higher dimensions

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Based on: JHEP 1311 (2013) 065, arXiv:1308.0117 [hep-th]
JHEP 1404 (2014) 121, arXiv:1402.1024 [hep-th]
Phys.Lett. B 734 (2014) 111, arXiv:1404.6998 [hep-th]
To appear, 2015

Motivation

Maximal SYM

D=4 N=4
D=6 N=2
D=8 N=1
D=10 N=1

- **Partial or total cancellation of UV divergences (all bubble and triangle diagrams cancel)**
- **First UV divergent diagrams at D=4+6/L**
- **Conformal or dual conformal symmetry**
- **Common structure of the integrands**

Bern, Dixon&Co 10

Drummond, Henn, Korchemsky, Sokatchev 10

Arkani-Hammed 12

Object: Helicity Amplitudes on mass shell with arbitrary number of legs and loops

The case: Planar limit $N_c \to \infty,\ g_{YM}^2 \to 0$ and $g_{YM}^2 N_c$ - fixed

The aim: to get all loop (exact) result

UV & IR Divergences

D=4 N=4

- **No UV divergences in all loops**
- **IR & Collinear Divs on shell**

BDS conjecture

Bern, Dixon, Smirnov 05

$$\mathcal{M}_n \equiv \frac{A_n}{A_n^{tree}} = 1 + \sum_{L=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} \left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

$$\mathcal{M}_n(\epsilon) = \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \left(\frac{\gamma_{cusp}^{(l)}}{(l\epsilon)^2} + \frac{2G_0^{(l)}}{l\epsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon} + \frac{1}{4}\sum_{l=1}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^l \gamma_{cusp}^{(l)} F_n^{(1)}(0) + C(g)\right]$$

IR & Collinear Divs in dimensional regularization **Cusp anom dim**

$$M_4^{(1-loop)}(\epsilon) = A_4^{(1-loop)}/A_4^{(tree)} = \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\left[\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{s}\right)^{\epsilon} + \left(\frac{\mu^2}{-t}\right)^{\epsilon}\right) - \frac{1}{2}\log^2\left(\frac{s}{-t}\right) - \frac{\pi^2}{3}\right] + \mathcal{O}(\epsilon)$$

UV & IR Divergences

D=6 N=2

N=(1,1)

- **No IR & Collinear divergences in all loops**
- **UV Divs starting from L=6/(D-4)=3 loops**

$$[g^2] \sim \frac{1}{M^2}$$

Toy model for gravity

D=8 N=1

D=10 N=1

- **No IR & Collinear divergences in all loops**
- **UV Divs starting from L=[6/(D-4)]=1 loops**

Compactification on a torus of higher dim maximal SYM theories gives lower dimensional maximal SYM theories

Used techniques

Spinor helicity formalism

Cheung, O'Connell 09, Bern&Co 10, R.H.Boles D O' Connell 12, S.Caron-Huot D. O'Connell 10

On-shell momentum superspace for N=(1,1) SYM

Dennen, Huang, Siegel 10

. . .

Color decomposition

Color ordered amplitude

$$\mathcal{A}_n^{a_1\ldots a_n}(p_1^{\lambda_1}\ldots p_n^{\lambda_n}) = \sum_{\sigma\in S_n/Z_n} Tr[\sigma(T^{a_1}\ldots T^{a_n})]A_n(\sigma(p_1^{\lambda_1}\ldots p_n^{\lambda_n})) + \mathcal{O}(1/N_c)$$

This is what we calculate

Planar Limit $N_c \to \infty,\ g_{YM}^2 \to 0$ and $g_{YM}^2 N_c$ - fixed

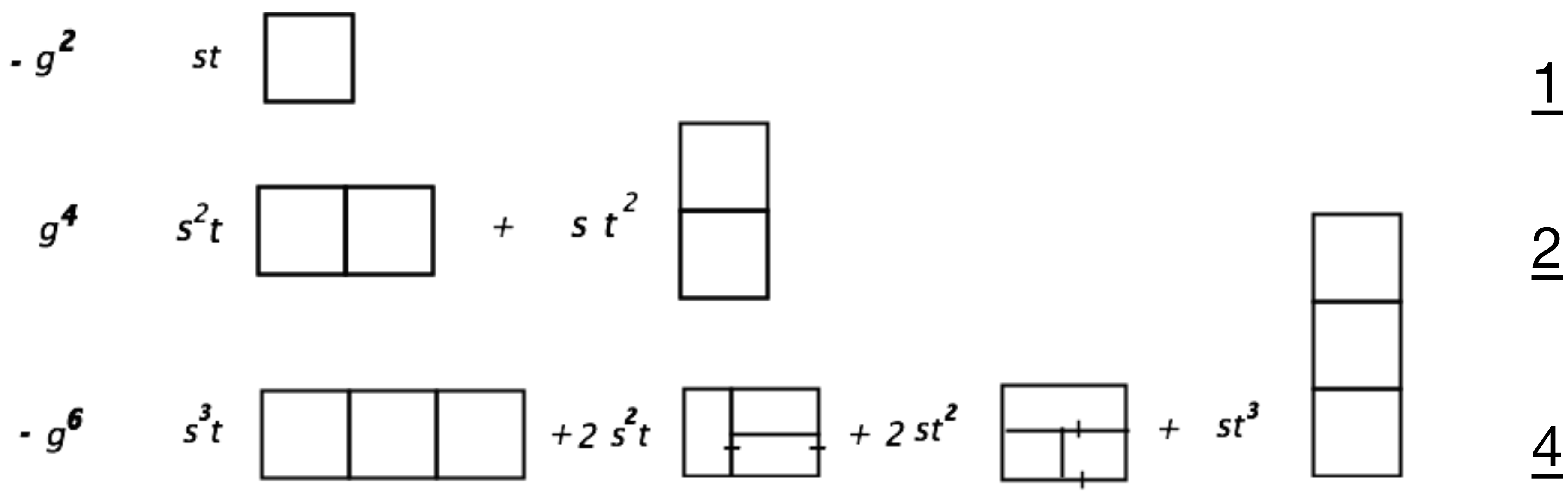
Four-point amplitude

$$\begin{aligned} A_4^{(l),phys.}(1,2,3,4) &= T^1 A_4^{(0)}(1,2,3,4)M_4^{(l)}(s,t) \\ &+ T^2 A_4^{(0)}(1,2,4,3)M_4^{(l)}(s,u) \\ &+ T^3 A_4^{(0)}(1,4,2,3)M_4^{(l)}(t,u) \end{aligned}$$

Tree level amplitude usually has a simple universal form proportional to the delta function (conservation of momenta), in SUSY case - conservation of supercharge in on shell momentum superspace

Perturbation Expansion for the Amplitudes for any D

$$A_4(s,t) = A_4^{(0)}(s,t)\,[1 + \text{loop corrections}]$$

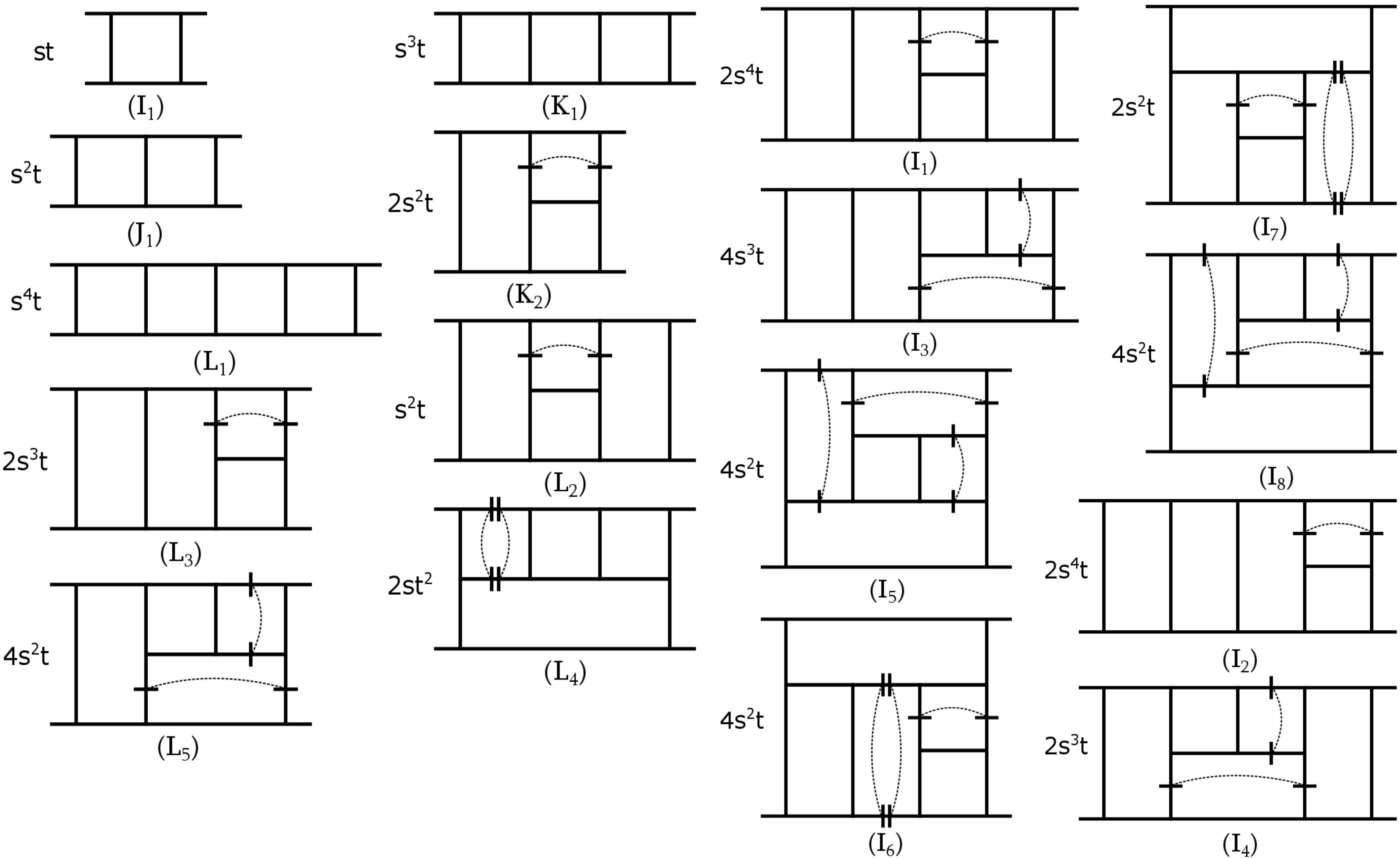


- **No Bubbles**
- **No Triangles**
- **First UV div at L=[6/(D-4)] loops**
- **IR finite**

Universal expansion for any D in maximal SYM due to **Dual conformal invariance**

T. Dennen Yu-yin Huang 10 ,
S.Caron-Huot D.O'Connell 10

Diagrams which contain leading divergence



Calculated leading divergences

MI	Comb	$D=6$	$D=8$	$D=10$
N_1	st	conv	$\frac{1}{3!\epsilon}$	$\frac{s+t}{5!\epsilon}$
J_1	s^2t	conv	$-\frac{s^2}{3!4!\epsilon^2}$	$\frac{-s^2(8s+2t)}{5!7!\epsilon^2}$
K_1	s^3t	conv	$\frac{s^3}{4!5!\epsilon^3}$	$\frac{-2s^4(135s+11t)}{5!7!7!3\epsilon^3}$
K_2	$2s^2t$	$-\frac{1}{6\epsilon}$	$\frac{s(3s^2-2st+t^2)}{3!4!5!9\epsilon^3}$	$\frac{-s^2\left(14s^4-10s^3t+\frac{33}{5}s^2t^2-\frac{19}{5}st^3+\frac{8}{5}t^4\right)}{5!7!7!9\epsilon^3}$
L_1	s^4t	conv	$-\frac{210s^4}{3!4!5!6!\epsilon^4}$	$\frac{-32s^6(99s+2t)}{5!7!7!7!3\epsilon^4}$
L_2	s^3t	$\frac{1}{48\epsilon^2}$	$\frac{s^2\left(-\frac{20}{3}s^2+\frac{8}{9}st-\frac{1}{9}t^2\right)}{3!4!5!6!\epsilon^4}$	$\frac{-28s^4\begin{pmatrix}8512s^4-1043s^3t+\frac{876}{5}s^2t^2- \\ -\frac{143}{5}st^3+\frac{16}{5}t^4\end{pmatrix}}{5!7!7!7!7!3\epsilon^4}$
L_3	$2s^3t$	$\frac{1}{24\epsilon^2}$	$\frac{s^2\left(-\frac{430}{21}s^2+\frac{4}{9}st-\frac{1}{18}t^2\right)}{3!4!5!6!\epsilon^4}$	$\frac{-2s^4\begin{pmatrix}\frac{1502144}{33}s^4-\frac{1085791}{33}s^3t \\ +\frac{2044}{5}s^2t^2-\frac{1001}{15}st^3+\frac{112}{15}t^4\end{pmatrix}}{5!7!7!7!7!\epsilon^4}$
L_4	$2s^2t$	$\sim\frac{1}{\epsilon}$	$\frac{s\begin{pmatrix}-\frac{45}{14}s^4+\frac{18}{7}s^3t-\frac{27}{14}s^2t^2 \\ +\frac{9}{7}st^3-\frac{9}{14}t^4\end{pmatrix}}{3!4!5!6!\epsilon^4}$	$\frac{-s^2\begin{pmatrix}-\frac{7504}{1287}s^7+\frac{7819}{1716}s^6t-\frac{1475}{429}s^5t^2+\frac{12745}{5148}s^4t^3 \\ -\frac{716}{429}s^3t^4+\frac{1747}{1716}s^2t^5-\frac{673}{1287}st^6+\frac{105}{572}t^7\end{pmatrix}}{5!7!7!7!\epsilon^4}$
L_5	$4s^2t$	$\frac{t-s}{3\cdot 48\epsilon^2}$	$\frac{s\begin{pmatrix}-\frac{15}{28}s^4+\frac{25}{63}s^3t-\frac{65}{252}s^2t^2 \\ +\frac{5}{42}st^3-\frac{1}{28}t^4\end{pmatrix}}{3!4!5!6!\epsilon^4}$	$\frac{-4s^2\begin{pmatrix}-\frac{95200}{143}s^7+\frac{67634}{143}s^6t-\frac{225008}{715}s^5t^2+\frac{136514}{715}s^4t^3 \\ -\frac{6608}{65}s^3t^4+\frac{6706}{143}s^2t^5-\frac{7420}{429}st^6+\frac{1715}{429}t^7\end{pmatrix}}{5!7!7!7!7!\epsilon^4}$

D=6 N=2 case

Leading Divergences for 5loop diagrams

MI	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8
comb	$2s^4t$	$2s^4t$	$4s^3t$	$2s^3t$	$4s^2t$	$4s^2t$	$2s^2t$	$4s^2t$
Int	$-\frac{1}{\epsilon^3}\frac{9}{36\cdot 40}$	$-\frac{1}{\epsilon^3}\frac{3}{36\cdot 40}$	$\frac{1}{\epsilon^3}\frac{s-t/4}{36\cdot 30}$	$\frac{1}{\epsilon^3}\frac{s-t/4}{36\cdot 15}$	$\frac{1}{\epsilon^3}\frac{s^2-st+t^2/3}{36\cdot 80}$	$-\frac{1}{\epsilon^3}\frac{s^2-st+t^2}{36\cdot 80}$	$-\frac{1}{\epsilon^3}\frac{s^2-st+t^2}{36\cdot 40}$	$\frac{1}{\epsilon^3}\frac{s^2-st+t^2/3}{36\cdot 80}$

Result up to 5 loops

$$L.P. = 2st\frac{g^4}{4}\left[\frac{g^2}{2}\frac{s+t}{6\epsilon} + \frac{g^4}{4}\frac{s^2+st+t^2}{36\epsilon^2} + \frac{g^6}{8}\frac{s^3+\frac{2}{5}s^2t+\frac{2}{5}st^2+t^3}{216\epsilon^3}\right]$$

Geom progression !?

$$-\sum_{n=1}^{\infty}\left(-\frac{g^2 s}{6\epsilon}\right)^n = \frac{\frac{g^2 s}{6\epsilon}}{1+\frac{g^2 s}{6\epsilon}} \to 1, \quad \text{when} \quad \epsilon \to +0$$

D=8,10 N=1 cases

Leading Divergences

$$L.P. = -st\left[\frac{g^2}{2}\frac{1}{3!\epsilon} + \frac{g^4}{4}\frac{s^2+t^2}{3!4!\epsilon^2} + \frac{g^6}{8}\frac{4}{3}\frac{15s^4 - s^3t + s^2t^2 - st^3 + t^4}{3!4!5!\epsilon^3}\right.$$
$$\left. + \frac{g^8}{16}\frac{1}{63}\frac{16770s^6 - 536s^5t + 412s^4t^2 - 384s^3t^3 + 412s^2t^4 - 536st^5 + 16770t^6}{3!4!5!6!\epsilon^4}\right].$$

$$L.P. = -st\left[\frac{g^2}{2}\frac{s+t}{5!\epsilon} + \frac{g^4}{4}\frac{8s^4 + 2s^3t + 2st^3 + 8t^4}{5!7!\epsilon^2}\right.$$
$$+ \frac{g^6}{8}\frac{2(2095s^7 + 115s^6t + 33s^5t^2 - 11s^4t^3 - 11s^3t^4 + 33s^2t^5 + 115st^6 + 2095t^7)}{5!7!7!45\epsilon^3}$$
$$+ \frac{g^8}{16}\frac{32(211218880s^{10} + 753490s^9t - 1395096s^8t^2 + 1125763s^7t^3 - 916916s^6t^4}{13!7!7!5!5\epsilon^4}$$
$$\left.\frac{+843630s^5t^5 - 916916s^4t^6 + 1125763s^3t^7 - 1395096s^2t^8 + 753490st^9 + 211218880t^{10})}{13!7!7!5!5\epsilon^4}\right]$$

Doesn't look like Geom progression anymore,
however, coefficients grow slowly

Leading Divergences from Generalized «Renormalization Group»

- **In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of** $1/\epsilon^n$ **in n loops in given by**

$$a_n^{(n)} = (a_1^{(1)})^n$$

- **In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation**

$$\mathcal{R}'G = 1 - \sum_{\gamma} K\mathcal{R}'_{\gamma} + \sum_{\gamma,\gamma'} K\mathcal{R}'_{\gamma}K\mathcal{R}'_{\gamma'} - ...,$$

$$\mathcal{R}'G_n = \frac{A_n(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + ... + \frac{A_1(\mu^2)^{\epsilon}}{\epsilon^n}$$

All terms like $(log\mu^2)^m/\epsilon^k$ **should cancel** →

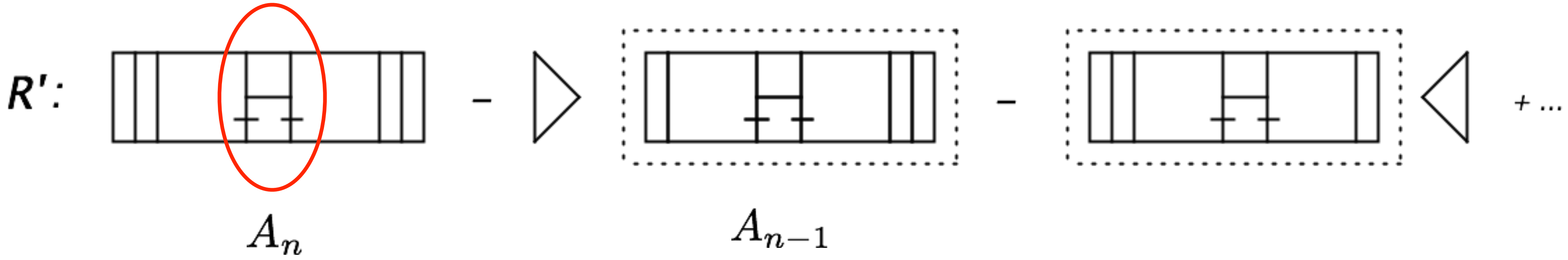
$$A_n = (-1)^{n-1}\frac{A_1}{n}$$

Leading pole — Coeff of 1 loop graph

R-operation and Recurrence Relation

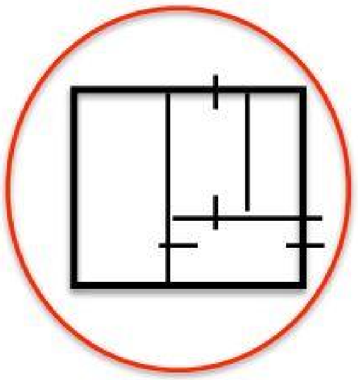
D=6 N=2

Horizontal boxes + tennis court



$$nA_n = -A_{n-1} \quad \rightarrow \quad A_n = (-1)^n \frac{2}{n!} \qquad (-g^2 s)^n$$

Horizontal boxes + double tennis court



$$nA_n^t = -\frac{1}{3}A_{n-1}^t, \qquad nA_n^s = -A_{n-1}^s + \frac{1}{3}A_{n-1}^t$$

$$\downarrow$$

$$A_n^t = \frac{(-1)^n}{3^{n-3}}\frac{1}{n!}, \qquad A_n^s = \frac{1}{2}\frac{(-1)^n}{3^{n-3}}\frac{1}{n!} - \frac{1}{2}(-1)^n\frac{1}{n!}$$

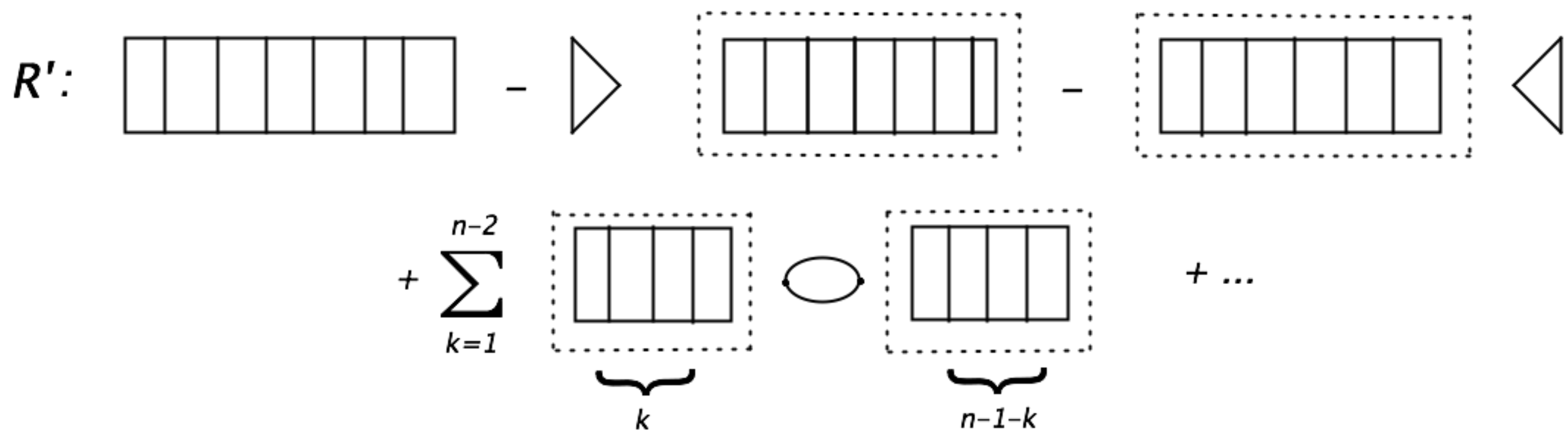
$$(-g^2 s)^{n-1}(-g^2 t) \qquad (-g^2 s)^n$$

- **Similar relations one can get for all other series**
- **All of them have 1/n! behavior**

R-operation and Recurrence Relation

D=8 N=1

Horizontal boxes



$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!}\sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3 \qquad \Sigma_k(x) = \sum_{n=k}^{\infty} A_n x^n \qquad x = \frac{-g^2 s^2}{\epsilon}$$

Diff eqn $\Sigma_3 = \Sigma_1 - A_1 x - A_2 x^2, \quad \Sigma_2 = \Sigma_1 - A_1 x, \quad A_1 = \frac{1}{3!}, \quad A_2 = -\frac{1}{3!4!} \qquad \Sigma \equiv \Sigma_1$

$$\frac{d}{dx}\Sigma_3 = -\frac{2}{4!}\Sigma_2 + \frac{2}{5!}\Sigma_1\Sigma_1$$

$$\Sigma' = \frac{1}{3!} - \frac{2}{4!}\Sigma + \frac{2}{5!}\Sigma^2$$

$$\Sigma(x) = \sqrt{5/3}\,\frac{4\tan[x/(8\sqrt{15})]}{1 + \tan[x/(8\sqrt{15})]\sqrt{5/3}}$$

$$\Sigma(x) = x/6 - x^2/144 + x^3/2880 - 7x^4/414720 + \ldots$$

All loop Exact Recurrence Relation

D=6 N=2

t-channel term $T_n(s,t)$ **s-channel term** $S_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy \, (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$$n \geq 4$$
$$t' = t(x-y) - sy$$
$$S_3 = -s/3, \; T_3 = -t/3$$

Summation $\Sigma_k^s = \sum_{n=k}^{\infty} S_n x^n \qquad \Sigma_k^t = \sum_{n=k}^{\infty} T_n x^n$

Diff eqn $\frac{d}{dz}\Sigma_3^s = 3S_3 z^2 + 2s \int (\Sigma_3^{s'} + \Sigma_3^{t'}) \qquad \Sigma_4^s = \Sigma_3^s - S_3 x^3 \qquad \Sigma^s = x^{-2}\Sigma_3^s$

$$\frac{d}{dx}\Sigma^s = s - \frac{2}{x}\Sigma^s + 2s \int (\Sigma^{s'} + \Sigma^{t'})$$

All loop Exact Recurrence Relation

D=8 N=1

t-channel term $T_n(s,t)$ **s-channel term** $S_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact relation for ALL diagrams

$$nS_n(s,t) = -2s^2 \int_0^1 dx \int_0^x dy\; y(1-x)\; (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=t(x-y)-sy}$$
$$+ \quad s^4 \int_0^1 dx\; x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!}\; \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times$$
$$\times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\; (tsx(1-x))^p$$

$$S_1 = \frac{1}{12},\; T_1 = \frac{1}{12}$$

Summation $\Sigma_k^s = \sum_{n=k}^{\infty} S_n x^n$ $\Sigma_k^t = \sum_{n=k}^{\infty} T_n x^n$ $\Sigma_3^s = \Sigma_1^s - S_2 x^2 - S_1 x,\; \Sigma_2^s = \Sigma_1^s - S_1 x$

Diff eqn

$$\frac{d}{dx}\Sigma^s = \frac{1}{12} - 2s^2 \int_0^1 dx \int_0^x dy\; y(1-x)\; (\Sigma^s + \Sigma^t)|_{t'=t(x-y)-sy}$$
$$+s^4 \int_0^1 dx\; x^2(1-x)^2 \sum_{p=0}^{\infty} \frac{1}{p!(p+2)!}(\frac{d^p}{dt'^p}(\Sigma^s + \Sigma^t)|_{t'=-sx})^2\; (tsx(1-x))^p.$$

Summation of Infinite Series

D=6 N=2

- Equation for the total sum has a fixed point $\Sigma^s = \Sigma^t = -1/2$
- It is stable when $\epsilon \to +0$ but depends on kinematics
- Having in mind all channels in full amplitude the fixed point appears to be unstable

D=8 N=1 D=10 N=1

- Due to non-linearity of equation the fixed point analysis is complicated
- Existence of a fixed point does not contradict the equation
- Example of the horizontal boxes demonstrates that the limit $\epsilon \to +0$ might be similar to a gauge theory in D=4

Conclusion

- **It might mean that in nonrenormalizable theories the finite number of PT terms has no meaning while the full theory exists.**
- **That would imply that severe UV divergences present in any given order of PT are actually artifacts of the weak coupling expansion.**

- **If this is true, one may try to apply the same arguments to quantum gravity. This would mean that one should not be confused by nonrenormalizability of PT in quantum gravity.**
- **It may well be that the full theory is meaningful, PT is just not applicable here.**

- **In order to understand the nonrenormalizable theories one has to find an alternative dual description.**
- **The result of an alternative approach might be quite different from the PT one.**

Thank you for your attention

D=6
N=2

Perturbation Expansion for the Amplitudes

Exact calculation

$$p_i^2 = 0, \; m = 0$$

$$B_1(s,t) = \frac{\pi^3}{(2\pi)^6}\frac{b_2(x)}{s+t}, \quad b_2(x) = \frac{L^2(x)+\pi^2}{2}, \quad L(x) \doteq \log(x), \quad x = \frac{t}{s}$$

$$B_2(s,t) = \left(\frac{\pi^3}{(2\pi)^6}\right)^2 \left(\frac{b_4(x)}{t} + \frac{b_3(x)}{s+t}\right)$$

Anastasiou, Tausk, Tejeda-Yeomans , 00
Bork,Kazakov,Vlasenko, 13

$$\begin{aligned} b_4(x) &= \left(2\zeta_3 - 2Li_3(-x) - \frac{\pi^2}{3}L(x)\right)L(1+x) + \left(\frac{1}{2}L(x) + \frac{\pi^2}{2}\right)L^2(1+x) \\ &+ \left(2L(x)L(1+x) - \frac{\pi^2}{3}\right)Li_2(-x) + 2L(x)S_{1,2}(-x) - 2S_{2,2}(-x) \end{aligned}$$

$$b_3(x) = -2\zeta_3 + \frac{\pi^2}{3}L(x) - \left(L(x)+\pi^2\right)L(1+x) - 2L(x)Li_2(-x) + 2Li_3(-x)$$

Regge Limit $s \to \infty, \;\; t<0, \; fixed$

$$B_1(s,t) \sim \frac{1}{2}L^2(x) \qquad\qquad B_2(s,t) \sim \frac{1}{12}L^4(x)$$

Perturbation Expansion for the Amplitudes

Leading Logarithms **UV finite**

Regge Limit $s \to \infty, \quad t < 0, \quad fixed$

$$B_n(t,s) \simeq \frac{1}{s}\frac{L^{2n}(x)}{n!(n+1)!}, \qquad L \equiv \log(s/t)$$

Bork,Kazakov,Vlasenko, 13

$$\left.\frac{A_4}{A_4^{(0)}}\right|_{L.L.} = \sum_{n=0}^{\infty} \frac{(-g^2t/2)^n L^{2n}(x)}{n!(n+1)!}, \quad \text{where} \quad g^2 \equiv \frac{g_{YM}^2 N_c}{64\pi^3}.$$

$$\sum_{n=0}^{\infty} \frac{(-g^2t/2)^n L^{2n}(x)}{n!(n+1)!} = \frac{I_1(2y)}{y}, \qquad y \equiv \sqrt{g^2|t|/2}\; L(x)$$

$$\left.\frac{A_4}{A_4^{(0)}}\right|_{L.L.} \sim \left(\frac{s}{t}\right)^{\alpha(t)-1}$$

!

Regge behaviour

Exact for $N_c \to \infty$

$$\alpha(t) = 1 + 2\sqrt{g^2|t|/2} = 1 + \sqrt{\frac{g_{YM}^2 N_c |t|}{32\pi^3}}$$

Perturbation Expansion for the Amplitudes

$$B_n(s,t) = \frac{1}{s}\left(C_n + O(t/s)\right), \quad n \geq 2$$

Kazakov, 14

Leading Powers

UV finite

Loops	1	2	3	4	5	6
Values	$\frac{\pi^2}{2}$	$\frac{\pi^2}{3}$	$-\pi^2 + \frac{31\pi^6}{1890} - 8\zeta_3 + 4\zeta_3^2$			
Numerics	4.93	3.29	2.06	2.05	2.42	3.13

Panzer,14

$$c_2 = 2\zeta_2,$$

$$c_3 = 4\zeta_3^2 + \tfrac{124}{35}\zeta_2^3 - 8\zeta_3 - 6\zeta_2,$$

$$c_4 = -56\zeta_7 - 32\zeta_2\zeta_5 + 32\zeta_3^2 + \tfrac{8}{5}\zeta_3\left(4\zeta_2^2 - 15\right) + \tfrac{992}{35}\zeta_2^3 - 8\zeta_2^2 - 18\zeta_2,$$

$$c_5 = 56\zeta_7\left(\zeta_3 - 5\right) + 26\zeta_5^2 + 4\zeta_5\left(8\zeta_2\zeta_3 + 35\zeta_3 - 40\zeta_2 - 49\right) + \tfrac{4}{5}\zeta_3^2\left(140 - 25\zeta_2 - 4\zeta_2^2\right) + 8\zeta_3\left(7\zeta_2 + 4\zeta_2^2 - 14\right) - \tfrac{1168}{385}\zeta_2^5 - \tfrac{24}{7}\zeta_2^4 + \tfrac{496}{5}\zeta_2^3 + 4\zeta_2\left(2\zeta_{3,5} - 21\right) + 20\zeta_{3,5} + 4\zeta_{3,7},$$

$$c_6 = \tfrac{18864}{35}\zeta_2^3 + 336\zeta_{3,5} - 12\zeta_9\left(20\zeta_2 + 161\right) + \tfrac{8}{5}\zeta_7\left(104\zeta_2^2 + 35\zeta_2 + 840\zeta_3 - 1120\right) + 624\zeta_5^2 + \tfrac{16}{35}\zeta_5\left(1680\zeta_2\zeta_3 - 3675 - 12\zeta_2^3 - 2240\zeta_2 + 490\zeta_2^2 + 5145\zeta_3\right) + 96\left(\zeta_2^2 + \zeta_{3,7}\right) - \tfrac{48}{5}\zeta_3^2\left(35\zeta_2 + 8\zeta_2^2 - 60\right) - \tfrac{32}{5}\zeta_3\left(105 - 32\zeta_2^2 + 3\zeta_2^3 - 75\zeta_2\right) + 24\zeta_2\left(8\zeta_{3,5} - 21\right) - \tfrac{28032}{385}\zeta_2^5 - \tfrac{288}{5}\zeta_2^4 - 1320\zeta_{11}.$$