Bjorken DIS sum rules, the CBK relation & the $\{\beta\}$ expansion: new results

based on collaborative work of "Karlsruhe-Moscow group"

Pavel Baikov (MSU), Johannes Kühn (KIT)

Konstantin Chetyrkin (KIT)

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Outline

- **Bjorken SR for unpolarized scattering at $\mathcal{O}(\alpha_s{}^4)$**

- **Bjorken SR for polarized scattering and CBK /Crewther-Broadhurst-Kataev, (1972, 1993)/ relation (not new, remainder necessary for the following 2 items)**

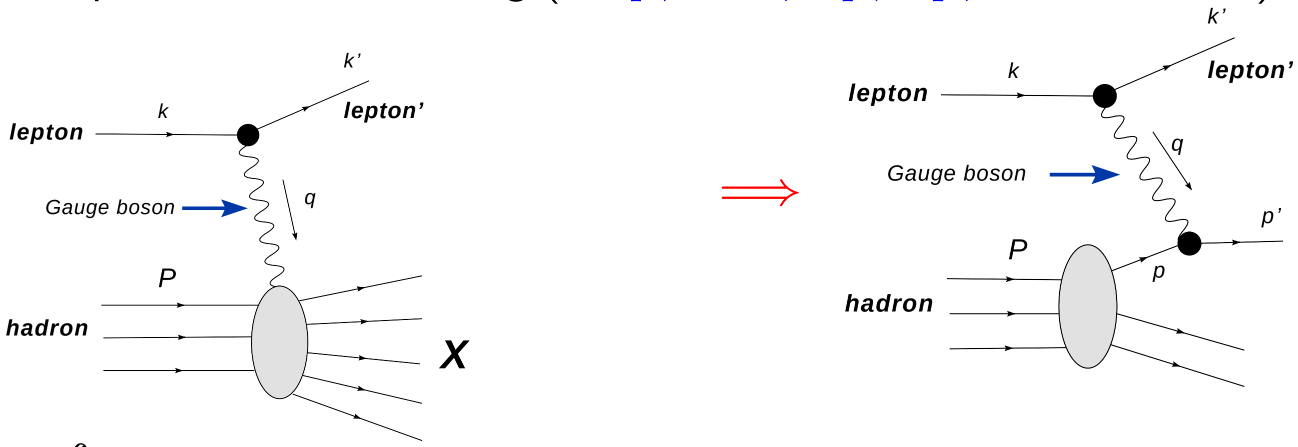
- **a new contribution★ to the the Bjorken SR for polarized scattering at $\mathcal{O}(\alpha_s{}^4)$, its (ir)relevance to physics and its compliance with the CBK relation**

- **Bjorken SR for polarized scattering at $\mathcal{O}(\alpha_s{}^3)$ in QCD with (massless) gluinos: a (successful) check of a recent non-trivial(?) prediction★★ based on the $\{\beta\}$ expansion /S. Mikhailov (2004), ...$/ and the CBK relation**

★ **ignited by:**
S. Larin, *The singlet contribition to the Bjorken SR for polarized DIS*, **arxiV:1303.4021**

★★ **due to S. Mikhaiov and A. Kataev,** *Generalization of the Brodsky-Lepage-Mackenzie optimization within the $\{\beta\}$-expansion and the Principle of Maximal Conformality*, **hep-ph/1408.0122**

- Deep-inelastic lepton-hadron scattering ($e^{\pm}p,\ e^{\pm}n,\ \nu p,\ \bar{\nu}p,\ \ldots$ - collisions)



$$
\begin{aligned}
W_{\mu\nu} &= \frac{1}{4\pi}\int d^4z e^{iqz}\langle p,s|J_\mu(z)J_\nu^+(0)|p,s\rangle \\
&= \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\boxed{F_1(x,Q^2)}+\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right)\frac{1}{p\cdot q}F_2(x,Q^2) \\
&\quad +i\epsilon_{\mu\nu\rho\sigma}q_\rho\left(\frac{s_\sigma}{p\cdot q}\boxed{g_1(x,Q^2)}+\frac{s_\sigma p\cdot q-p_\sigma q\cdot s}{(p\cdot q)^2}g_2(x,Q^2)\right)
\end{aligned}
$$

where $x = Q^2/(2p\cdot q)$ and $Q^2 = -q^2$,

J_μ is either EM $J_\mu = \sum_{i=1}^{n_f} e_i\overline{\psi_i}\gamma_\mu\psi_i \equiv \overline{\psi}E\gamma_\mu\psi$ or the weak charged current

Structure functions F_1 and g_1 appear in three DIS sum rules: the Bjorken SR's for unpolarized (F_1) and polarized DIS (g_1); both will be discussed in the talk

Parton model prediction (Bjorken sum rule for F_1) reads:

$$\int_0^1 dx\, F_1^{\bar{\nu}p-\nu p} = 1$$

in QCD the SR receives higher order PT correctons, which can be related to thec oefficient function (CF) C^{Bj}_{unp} of the corresponding Operator Product Expansion (OPE)

$$i \int dz e^{iqz} T\{J_\mu(z) J^+_\nu(0)\} \overset{Q^2\to\infty}{=\!=\!=} \tag{1}$$

$$(-g_{\mu\nu} + q_\mu q_\nu/q^2)\ C^{Bj}_{unp}\ q_\nu\, J^V_\nu + \cdots \text{(other operators)},$$

where J^V_ν is a *vector* quark current with a proper flavour structure.

The OPE above is one of the simplest ones (no epsilon tensors, no γ_5). Experimentally, this SR is presumably very difficult to deal with. On the other hand, it it has a reach history of calculations.

Bjorken SR for unpolarized scattering

$$\int_0^1 dx F_1^{\bar{\nu}p-\nu p} = 1 \quad \leftarrow \quad \text{/Bjorken (1967)/}$$

$$- \quad \frac{2}{3} a_s \quad \leftarrow \quad \text{/Bardin, Buras, Duke, Muta (1978); Altarelli, R. K. Ellis, Martinelli (1978)/}$$

$$+ \quad \underbrace{a_s^2 \left(-\frac{23}{6} + \frac{8}{27} n_f \right) \quad \leftarrow \quad \text{K.Ch., Gorishny, Larin, Tkachov (1984).}}$$

$$\boxed{\mathbf{a_s \equiv \alpha_s/\pi}}$$

⇑

First real application of the first (SCHOONSHIP) version of the legendary MINCER program for the first two loop calculation in DIS! It was also first "real life (QCD)" application of the powerfull method of projectors / Gorishny, Larin, Tkachov (1983);Gorishny, Larin (1987)/ to deal with OPE; it is now routinely being used in virtually every calculation of CF's of OPE

$$+ \underbrace{a_s^3 \left(-\frac{4075}{108} + \frac{622}{27}\zeta_3 - \frac{680}{27}\zeta_5 + n_f \left[\frac{3565}{648} - \frac{59}{27}\zeta_3 + \frac{10}{3}\zeta_5 \right] - \frac{155}{972} n_f^2 \right)} \quad \leftarrow$$

Larin, Tkachov, Vermaseren (1991)

⇑

First real application of the second (FORM 2) version of the MINCER program for the first THREE loop calculation in DIS!

We extended the above results to the four-loop level:

$$\int_0^1 F_1^{\bar{\nu}p-\nu p} \equiv C^{Bj}_{unpol} = 1 + \dots$$
$$+ \quad a_s^4 \left(-\frac{12053285}{31104} + \frac{70315}{162}\zeta_3 + \frac{67}{2}\zeta_3^2 - \frac{939995}{1296}\zeta_5 + \frac{341075}{2592}\zeta_7 \right.$$
$$+ \quad n_f \left[\frac{2756269}{31104} - \frac{256543}{3888}\zeta_3 - \frac{743}{81}\zeta_3^2 + \frac{36835}{324}\zeta_5 - \frac{49}{24}\zeta_7\right]$$
$$\left. + \quad n_f^2 \left[-\frac{548725}{93312} + \frac{29}{24}\zeta_3 + \frac{1}{3}\zeta_3^2 - \frac{55}{18}\zeta_5\right] + \frac{445}{4374} n_f^3 \right)$$

Transcendentally structure : at order α_s^4 ζ_7 does appear BUT not ζ_4, ζ_6 and $\zeta_3\,\zeta_4$! /while they do abound in separate diagrams!/

This is now well understood /Broadhurst (1999); Baikov, K.Ch.,(2010) / as a consequence of 2 facts:

- **peculiar structure of the four-loop masters**
- **the rationality (somewhat mysterious, as separate diagrams do contain ζ_3) of three-loop QCD β-function**

four-loop result for C^{Bj}_{unp}; con-ed

Numerically, the result is

$$
\begin{aligned}
C^{Bj}_{unp} &= 1. - 0.6667 a_s + a_s^2(-3.833 + 0.2962\, n_f) \\
&+ a_s^3\,(-36.155 + 6.33135\, n_f - 0.1595\, n_f^2) \\
&+ a_s^4\,(-436.768 + 111.873\, n_f - 7.115\, n_f^2 + 0.10174\, n_f^3)
\end{aligned}
$$

$$
C^{Bj}_{unp}(n_f = 3) = 1 - 0.6667\, a_s - 2.9444\, a_s^2 - 18.5963\, a_s^3 - 162.436\, a_s^4
$$

We observe two typical patterns:

(i) signifcant cancellations between n_f^0 and n_f^1 terms

(ii) almost geometrical sign-non-alternating growth of the coefficients of α_s series

It is very amusing to compare the new $\mathcal{O}(a_s^4)$ term with the old predictions due to /Broadhurst, Kataev (2002)/ based on NNA (Naive Non Abelization) procedure:

$$\left(C^{Bj}_{unp}\right)_{4,\text{exact}} = -436.768 + 111.873\, n_f - 7.115\, n_f^2 + 0.10174\, n_f^3$$

and

$$\left(C^{Bj}_{unp}\right)_{4,\text{NNA}} = -457.02 + 83.094\, n_f - 5.0360\, n_f^2 + 0.10174\, n_f^3$$

(note that $0.10174\, n_f^3$ in NNA result was used as input)

Bjorken Sum Rule (polarized) & CBK relation (remainder)

- the polarized Bjorken sum rule ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$\boldsymbol{B}jp(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{1}{6}|\frac{g_A}{g_V}|C^{Bjp}(a_s)$$

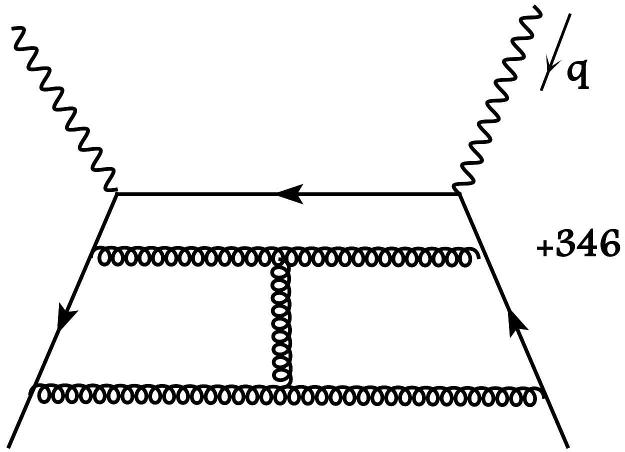
Coefficient function $C^{Bjp}(a_s)$ is fixed by OPE of two EM currents (up to power suppressed corrections)

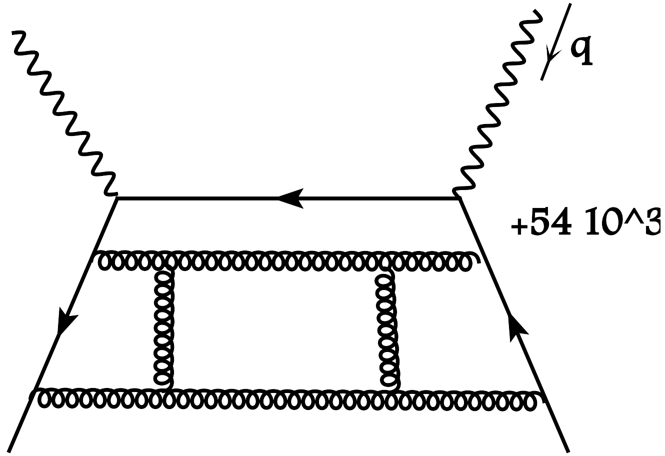
$$i\int TV_\alpha^E(x)V_\beta^E(0)e^{iqx}dx|_{q^2\to\infty} \approx \frac{q^\sigma}{Q^2}\epsilon_{\alpha\beta\rho\sigma}\times$$

$$\{Tr[E^2 t_c]\; C^{Bjp}(a_s)\;\}\; A_\rho^c(0) + \ldots$$

where $E = diag(Q_i)$, $V_\alpha^E = \overline{\psi}E\gamma_\rho\psi$ is the EM current,
$A_\rho^c = \overline{\psi}t^c\gamma_\rho\psi$ is (non-singlet!) axial current and $Q^2 = -q^2$

Typical diagrams:





At order α_s^3 the CF was computed in early nineties /Larin, Vermaseren, 91/.

The 4-loop $\mathcal{O}(\alpha_s^4)$ contribution to $C^{Bjp}(a_s)$ for *generic* color group was computed /Baikov, J.Ch. Kühn, 2010/ with two aims: to confront to exp. data (here SU(3) would be enough) and to check the $\mathcal{O}(\alpha_s^4)$ Adler function (technically significantly more complicated for evaluation) via CBK relation

The Crewther relation states that in the conformal invariant limit ($\beta \equiv 0$) $C_{Bjp}(a_s)$ is related to the (nonsinglet) Adler function via the following beautiful equality

$$\boldsymbol{C_{Bjp}(a_s) D^{NS}(a_s))|_{c-i} = 1}$$

its generalization for real QCD (the CBK relation) reads:

$$\boldsymbol{C^{Bjp}(a_s) D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K^{NS} = K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3} + .$$

Main ingredients of the derivation: the AVV 3-point function and constraints on it from (approximate) conformal invariance + Adler-Bardeen anomaly theorem

Crewther Relation: (short) bibliography

discovered: R.J. Crewther, *Phys. Rev. Lett.* **28**, 1421 (1972).
S.L. Adler, C.G. Callan, D.J. Gross and R. Jackiw, *Phys. Rev.* D **6**, 2982 (1972).

generalized for "real" QCD:

D.J. Broadhurst and A.L. Kataev, *Phys. Lett.* B 315, 179 (1993)

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"proven" (still with some hand-waving):

R.J. Crewther, *Phys. Lett.* B **397**, 137 (1997).
V. M. Braun, G. P. Korchemsky and D. Müller, Prog. Part. Nucl. Phys. **51**, 311 (2003)

Which exactly constraints come from the CBK relation?

$$C^{Bjp}(a_s)C_D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K_1\, a_s + K_2\, a_s^2 + K_3\, a_s^3 + \ldots\Big]$$

If it is valid at order a_s^n, then at the next order a_s^{n+1}, we have $(T \equiv Tr\, n_f)$

$$\left(d_{n+1} - C_{n+1}^{Bjp} + \text{interference terms}\right) a_s^{n+1} = \beta_0\, a_s\Big[K_n\, a_s^n\Big]$$

$\alpha_s^1 : (d_1 - C_1) : C_F \Longleftrightarrow K_0 \equiv 0 \leftarrow$ one constraint

$\alpha_s^2 : (d_2 - C_2) : C_F^2, T\, C_F, C_F\, C_A \Longleftrightarrow K_1 : C_F \leftarrow$ two constraints

$$\alpha_s^3 : (d_3 - C_3) : C_F^3\,, C_F^2 C_A\,, C_F C_A^2\,, C_F^2 T\,, C_F C_A T\,, C_F T^2$$

$$\Updownarrow$$

$K_2 : C_F^2, C_F C_A, C_F T \leftarrow$ three constraints

At last, at $\mathcal{O}(\alpha_s^4)$ there exist exactly 12 color strtuctures:

$$C_F^4\,,\ C_F^3 C_A\,,\ C_F^2 C_A^2\,,\ C_F C_A^3\,,\ C_F^3 T_F n_f\,,\ C_F^2 C_A T_F n_f\,,$$
$$C_F C_A^2 T_F n_f\,,\ C_F^2 T_F^2 n_f^2\,,\ C_F C_A T_F^2 n_f^2\,,\ C_F T_F^3 n_f^3\,,\ d_F^{abcd} d_A^{abcd}\,,\ n_f d_F^{abcd} d_F^{abcd}$$

while the coefficient K_3 is contributed by only **6** color structures:

$$C_F T^2\,, C_F\, C_A^2\,, C_F^2\, T\,, C_F\, C_A\, T\,, C_F^2\, C_A\,, C_F^3$$

Thus, we have 12-6 $=$ **6** constraints on the difference

$$\boldsymbol{d_4 - (C^{Bjp})_4}$$

3 of them are very simple: the above difference cannot contain

$$C_F^4,\ d_F^{abcd} d_A^{abcd} \quad n_f d_F^{abcd} d_F^{abcd}$$

remaining three are a bit more complicated

RESULT: ALL 3 (at $\mathcal{O}(\alpha_s^3)$) and 6 (at $\mathcal{O}(\alpha_s^4)$) constarints are fulfilled identically $\Longrightarrow$ an extremely strong test of our sophisticated calculations (and the very SBK relation too!)

Comments:

The CBK test is highly non-trivial:

- four-loop box-type diagrams (in propagator kinematics) versus five loop propagators

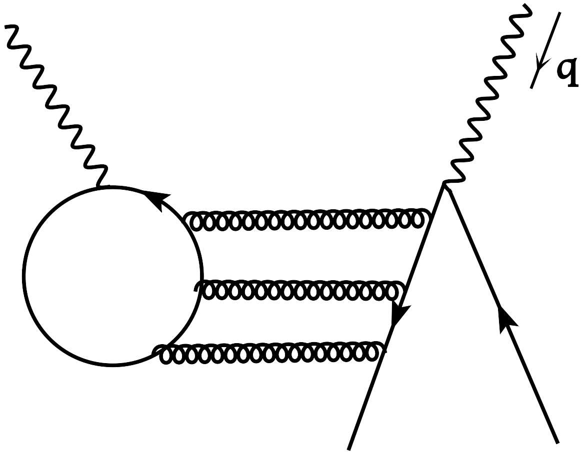
- **No IR-trickery is neccessary** in calculation of $C_{Bjp}(a_s)$

- final 4-loop p-integrals are much simpler for OPE (2 instead of 3 squared propagators inside)

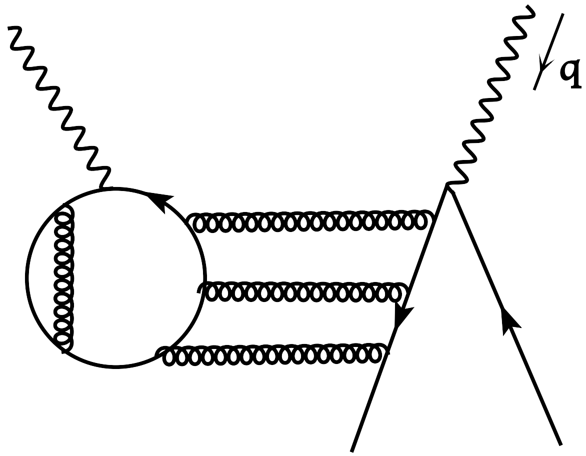
- Technical note: in the course of our calculations we have had to extend the Larin treatment of Hooft-Veltman γ_5 at 4-loop level (a natural object for the dim. reg., which really appears in the course of calculations, is $\gamma^{[\mu\nu\alpha]}$ instead of $\gamma_5\gamma^\mu$ with anticommuting γ_5; the mismatch should be corrected by the Larin factor)

Larin (arxiV:1303.4021) has drawn attention to "missing" contributions to the Bjorken sum rule:

$\mathcal{O}({\alpha_s}^3)$



$\mathcal{O}({\alpha_s}^4)$



Two important questions:

1. Is the CBK relation still valid (no missing contributions to the Adler function as far as we know)?

2. Any relevance of the new terms to the phenomenology?

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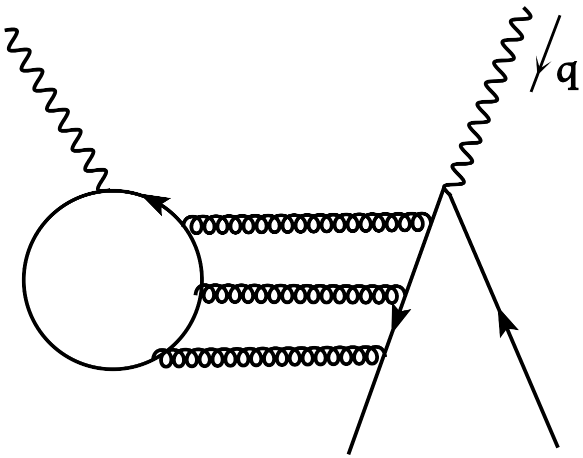
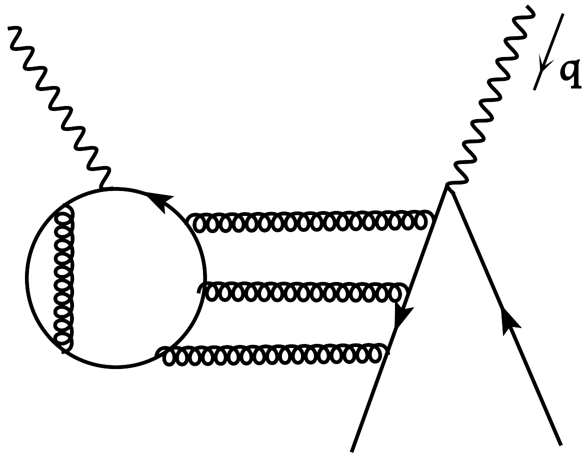




color structures: $d^{abc}\,d^{abc} = 40/3$ $(C_f|n_f \mathrm{Tr}|C_A) \times d^{abc}\,d^{abc}$

Two important questions:

1. Is the CBK relation still valid (no missing contributions to the Adler function as far as we know)? YES

2. Any relevance of the new terms to the phenomenology? /probably/ NO

Bjorken Sum Rule: new term starting from order α_s^3:

- the polarized Bjorken sum rule ($a_s \equiv \frac{\alpha_s}{\pi}$)

$$\boldsymbol{B}jp(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{1}{6}|\frac{g_A}{g_V}|C^{Bjp}(a_s)$$

Coefficient function $C^{Bjp}(a_s)$ is fixed by OPE of two EM currents (up to power suppressed corrections)

$$i\int TV_\alpha^E(x)V_\beta^E(0)e^{iqx}dx|_{q^2\to\infty} \approx \frac{q^\sigma}{Q^2}\epsilon_{\alpha\beta\rho\sigma}\times$$

$$\mathrm{Tr}[E^2 t_c]\left\{C^{Bjp}(a_s) + 3\mathrm{Tr}[E]\; C_{SI}^{Bjp}(a_s)\right\}A_\rho^c(0) + \dots$$

for c=3 (we neglect sea-quarks in nucleons but assume n_f=4 to have non-zero $\mathrm{Tr}(E)$) the content of the figure bracket would read:

$$C^{Bjp}(a_s) + 3\times\frac{2}{3}\,C_{SI}^{Bjp}(a_s) = C^{Bjp}(a_s) + 2\,C_{SI}^{Bjp}(a_s, \boldsymbol{m_c})$$

Let start from the first QQ question: under which conditions the CBK relation will survive if Larin's terms are not zero?

$$\left(C_{NS}^{Bjp}(a_s) + C_{SI}^{Bjp}(a_s)\right) D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K^{NS} = K_1\, a_s + \dots\Big]$$

with $C_{SI}^{Bjp}(a_s)$ proportional **new** color structure $d^{abc}d^{abc}$ /which is for sure absent in D^{NS}/

As $\beta_0 = \frac{11}{12}C_A - \frac{T_f\, n_f}{3}$ we conclude (following /Larin (2013)/) that

1. $\mathcal{O}({\alpha_s}^3)$ term in C_{SI}^{Bjp} must be zero (/Larin (2013)/ refers here to the original $\mathcal{O}(\alpha_s^3)$ work of /Larin+Vermaseren, 1991/ but, in fact, there the singlet term was not mentioned at all!)
2. $\mathcal{O}({\alpha_s}^4)$ term in C_{SI}^{Bjp} must have the structure

$$\text{const}\,\beta_0\; d^{abc}d^{abc}$$

Indeed, direct calculation gives (for $(n_f = 4)$ and **massless** c-quark):

$$\boldsymbol{C_{SI}^{Bjp}(a_s) = 0 \cdot a_s^3 + \frac{1}{9}\beta_0\, d^{abc}d^{abc}\left(\frac{\alpha_s}{\pi}\right)^4}$$

$$\boldsymbol{= \left(\frac{110}{27} - \frac{20}{81}\,n_f\right) a_s^4 = \frac{10}{3}\,a_s^4}$$

Thus, if we assume that c-loop is massless in SI-diagrams and set traditionally $n_f = 3$ for the rest (non-singlet) diagrams we get for full

$$C_{full}^{Bjp} = C_{NS}^{Bjp} + 2\,C_{SI}^{Bjp}$$

the following result:

$$\boldsymbol{C_{full}^{Bjp}(n_f = 3, 4)} =$$

$$\boldsymbol{1 - a_s - 3.583\,a_s^2 - 20.22\,a_s^3 + (-175.7 + 6.667 = 169.033)\,a_s^4}$$

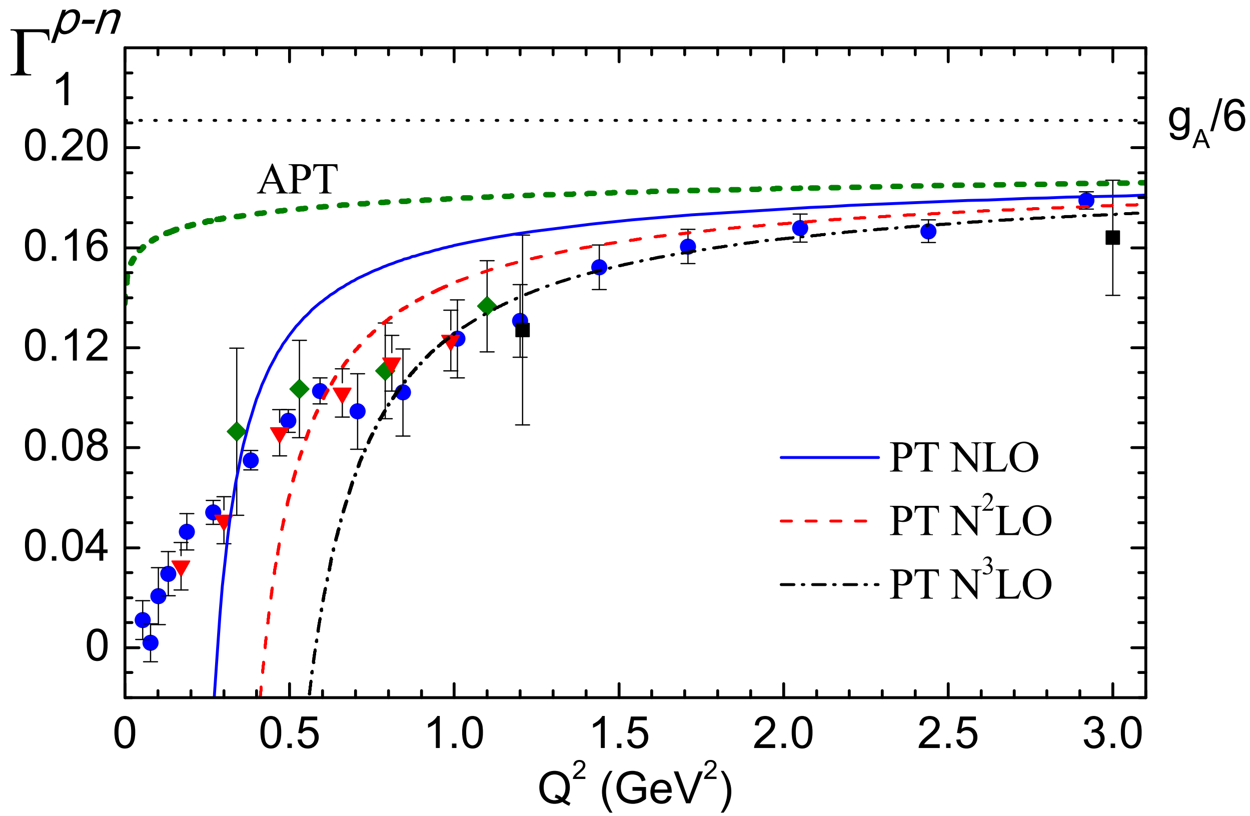
BUT! we should remember 2 things:

1. with $m_c \neq 0$ the CBK relation stops to work (completely broken conformal invariance!) and one should expect a nonzero contribution already at $\boldsymbol{\mathcal{O}(\alpha_s{}^3)}$ (but again suppressed by a factor $\boldsymbol{\frac{Q^2}{4m_c^2}}$)

2. that typical momentum scale at which experimental data are available is $Q^2 = 3\,\text{GeV}^2$, thus we should expect a "decoupling suppression" factor like $\boldsymbol{\frac{Q^2}{4m_c^2} < 1}$

3. The picture on the next slide shows a definitely "positive" effect of the α_s^4 corrections to $C^{Bjp} \equiv C^{Bjp}_{NS}$ in $n_f = 3$ massless QCD on the description of the experimantal curve

in the effective $n_f = 3$ QCD the Bjorken rule is unambiguous
/modulo higher twists!/
predictions of QCD which can be confronted with experimental data:



Perturbative part of the BSR as a function of the momentum transfer squared Q^2 in different orders in both the APT and standard PT approaches against the combined set of the Jefferson Lab (taken from V.L. Khandramai, R.S. Pasechnik, D.V. Shirkov, O.P. Solovtsova, O.V. Teryaev, *Four-loop QCD analysis of the Bjorken sum rule vs data*, Phys.Lett.B706:340-344,2012).

QCD + gluino: check of predictive power of the $\{\beta\}$-expansion

Last 10 years: S. Mikhailov together with (joined somewhat later) A. Kataev are developing a new optimization procedure to keep under controle well-known "scheme-dependence" problem of pQCD. One of the important tools: so-called $\{\beta\}$-expansion – an approach to find and isolate $\{\beta\}$-dependent contributions to pQCD predictions.

As a spin-off they published (arXiv: 1408.0122) a prediction for the gluino contribution to C_{NS}^{Bjp} at $\mathcal{O}({\alpha_s}^3)$ starting from 20-years old result for analogous contribution to the Adler function (K. Ch.,1996).

Since the very issue of "correct" optimization is rather controversial (see, e.g an ongoing battle with Principle of Maximal Conformality by S. Brodsky and Co) we decided to check the prediction with a direct calculation.

$$
\begin{aligned}
&D^{\mathrm{NS}}(a_s, n_f, n_{\tilde{g}}) = 1 + a_s \cdot (3C_F) \\
&+a_s^2 C_F \left\{ -\frac{3}{2} C_F + \left[\frac{123}{2} - 44\zeta_3 - (11 - 8\zeta_3) n_{\tilde{g}} \right] C_A - 2(11 - 8\zeta_3) n_f T_R \right\} \\
&+a_s^3 C_F \left\{ -\frac{69}{2} C_F^2 - C_F C_A \left[127 + 572\zeta_3 - 880\zeta_5 - (36 + 104\zeta_3 - 160\zeta_5) n_{\tilde{g}} \right] \right. \\
&C_A^2 \left[\frac{90445}{54} - \frac{10948}{9}\zeta_3 - \frac{440}{3}\zeta_5 - \left(\frac{33767}{54} - \frac{4016}{9}\zeta_3 - \frac{80}{3}\zeta_5 \right) n_{\tilde{g}} \right. \\
&\left. + \left(\frac{1208}{27} - \frac{304}{9}\zeta_3 \right) n_{\tilde{g}}^2 \right] - n_f T_R C_F [29 - 304\zeta_3 + 320\zeta_5] + 3 \left[\frac{302}{9} - \frac{76}{3}\zeta_3 \right] \left(\frac{4}{3} T_R n_f \right)^2 \\
&\left. - n_f T_R C_A \left[\frac{31040}{27} - \frac{7168}{9}\zeta_3 - \frac{160}{3}\zeta_5 - \left(\frac{4832}{27} - \frac{1216}{9}\zeta_3 \right) n_{\tilde{g}} \right] \right\}
\end{aligned}
$$

Which exactly constraints come from the CBK relation?

$$C^{Bjp}(a_s)C_D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s}\Big[K_1\,a_s + K_2\,a_s^2 + K_3\,a_s^3 + \ldots\Big]$$

$$\left(d_{n+1} - C_{n+1}^{Bjp} + \text{interference terms}\right) a_s^{n+1} = \beta_0\, a_s\Big[K_n\, a_s^n\Big]$$

$\alpha_s^1 : (d_1 - C_1) : C_F \Longleftrightarrow K_0 \equiv 0 \leftarrow$ one constraint

$\alpha_s^2 : (d_2 - C_2) : C_F^2, T\,C_F, C_F\,C_A \Longleftrightarrow K_1 : C_F \leftarrow$ two constraints

$$\alpha_s^3 : (d_3 - C_3) : C_F^3\,, C_F^2 C_A\,, C_F C_A^2\,, C_F^2 T\,, C_F C_A T\,, C_F T^2$$

$+$ four more structures with $T = Tr\,n_f$ replaced by $C_A\,n_{\tilde{g}}$

$$\Updownarrow$$

$K_2 : C_F^2, C_F C_A, C_F T, C_F\,C_A\,n_{\tilde{g}} \leftarrow$ 6(!) constraints

OUR RESULTS

All 6 constarints are indeed met, but the only new coefficient in front of $c_F C_A n_{\tilde{g}}$ in K_2 happens to be exactly $1/2$ times the coefficient in front of $C_F Tr n_f$!

Looks like the fermion/gluino partof K_2 is in a sense effectively contributed by the trivial 1-loop fermion insertion!

Conclusions

- **The result for Bjorken SR for unpolarized scattering is available at 4 loops ($\mathcal{O}({\alpha_s}^4)$); it is in a quite good agreement to the old NNA prediction due to Broadhurst and Kataev**

- **the new ($\mathcal{O}({\alpha_s}^4)$) contribution to $C_{pol}^{Bjorken}$ is computed (in the massless QCD!), it is a simple, purely rational number proportional to $\beta_0\, d^{abc}d^{abc}$ as is unambiguously dictated by the corresponding CBK relation (as discussed first by Larin in 2013)**

- **Physically, due to the fact that for $n_f = 3$ the current J_μ^{EM} belongs to the $SU(3)_f$ octet ($Q_u + Q_d + Q_s \equiv 0$) the Larin's term is *completely* saturated by heavy quarks: its effect on the phenomenology should be small. Its precize evaluation is difficult (m_c must be taken into account!)**

- **The predictive power (if any) of the $\{\beta\}$ expansion could be at best tested at order $O(\alpha_s^4)$; the corresponding predictions are highly welcome!**

Our tool-box

- we compute CF's of OPE with the method of projectors (cancellation of IR singularities of on-shell scattering amplitudes against UV singularities of the relevant composite operators) $\Longrightarrow$ 4-loop CF's in terms of 4-loop massless propagators

- the Baikov's way of doing reduction of resulting millions of 4-loop massless propagators with the help of $1/D$ expansion of the corresponding coefficient functions in front of masters (analytically known from **two!** independent calculations /K.Ch, P.Baikov (2010), R. Lee, V. Smirnov (2012)/).

- automatic generation of Feynman diagrams with QGRAF /Nogueira (1993)/

- the FORM program BAICER which implements $1/D$ expansion

- last but not the least: **the very FORM in two versions:**

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\,\zeta_5$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\,\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\,\zeta_3 + \frac{5}{3}\,\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\,\zeta_3 + \frac{125}{6}\,\zeta_5 + 3\,\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\,\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\,\zeta_3 - \frac{170}{9}\,\zeta_5 - \frac{1}{2}\,\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\,\zeta_3 - \frac{5}{12}\,\zeta_5 + \frac{1}{6}\,\zeta_3^2$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\,\zeta_3 - \frac{125}{4}\,\zeta_5 + \frac{105}{4}\,\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\,\zeta_3 + \frac{145}{24}\,\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\,\zeta_3 - \frac{5155}{48}\,\zeta_5 - \frac{33}{4}\,\zeta_3^2 - \frac{105}{8}\,\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\,\zeta_3 - \frac{95}{144}\,\zeta_5 - \frac{35}{4}\,\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\,\zeta_3 + \frac{18805}{288}\,\zeta_5 - \frac{11}{2}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\,\zeta_3 + \frac{1855}{288}\,\zeta_5 - \frac{11}{12}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\,\zeta_3 + \frac{2255}{32}\,\zeta_5 - \frac{1155}{16}\,\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\,\zeta_3 - \frac{1045}{48}\,\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\,\zeta_3 + \frac{6505}{48}\,\zeta_5 + \frac{1155}{32}\,\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\,\zeta_3 + \frac{55}{9}\,\zeta_5 + \frac{385}{16}\,\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{77995}{1152}\,\zeta_5 + \frac{605}{32}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{12545}{1152}\,\zeta_5 + \frac{121}{96}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$