Color Structure for Parton Showers and Resummation

LUND UNIVERSITY

- Pros and cons of various color bases:
  - Trace bases
  - DDM bases
  - Color flow bases
  - Multiplet bases
- Computational tools:
  - ColorFull C++
  - ColorMath Mathematica
- Summary

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Motivation

- With the LHC there is an increased interest the treatment of color structure for processes with many colored partons
- This is applicable to fixed order calculations as well as parton showers and resummation

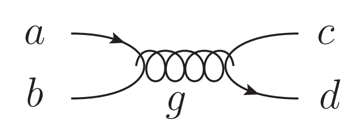
Dealing with color space

- We never observe individual colors
  → we are only interested in color summed/averaged quantities
- For given external partons, the color space is a finite dimensional vector space equipped with a scalar product

$$\langle A, B \rangle = \sum_{a,b,c,\ldots} (A_{a,b,c,\ldots})^* B_{a,b,c,\ldots}$$

Example: If

$$A = \sum_g (t^g)^a{}_b (t^g)^c{}_d =$$



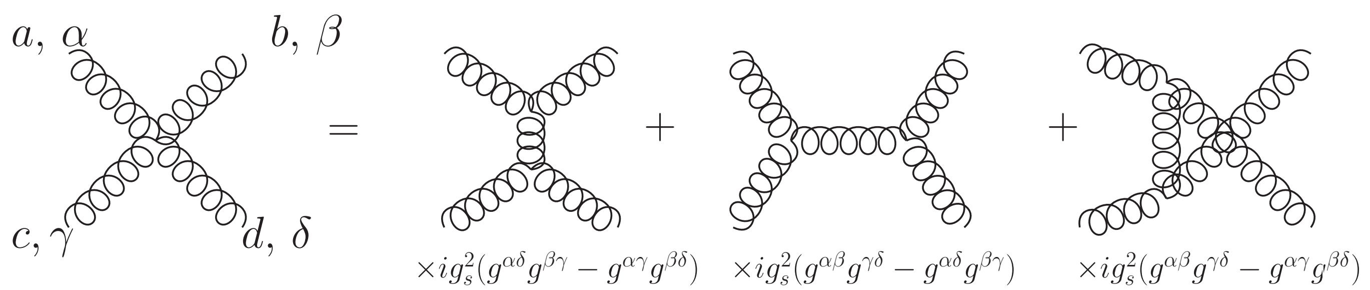
,

then $\langle A|A \rangle = \sum_{a,b,c,d,g,h} (t^h)^b{}_a (t^h)^d{}_c (t^g)^a{}_b (t^g)^c{}_d$

- One way of dealing with color space is to just square the amplitudes as one encounters them
- Alternatively, we may use any basis (spanning set)

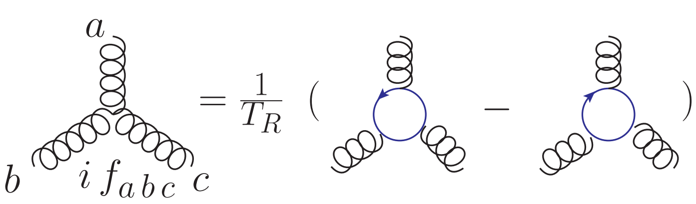
The most popular bases: Trace bases

- Every 4g vertex can be replaced by 3g vertices:

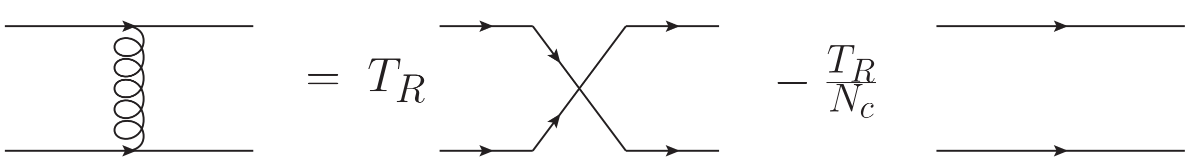


(read counter clockwise)

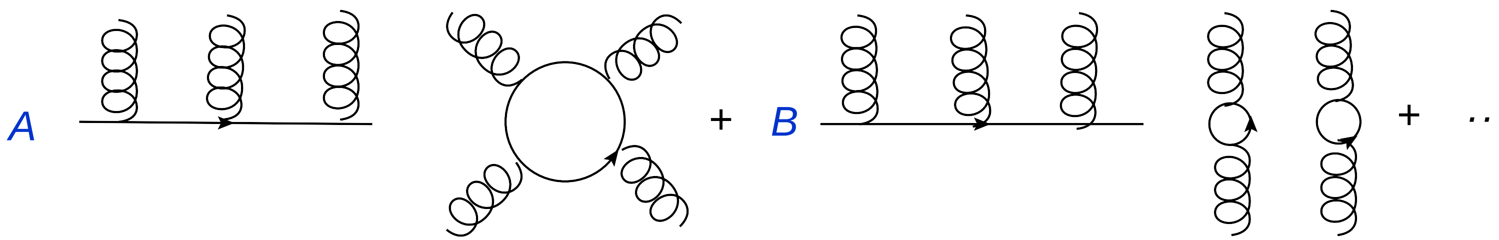
- Every 3g vertex can be replaced using:



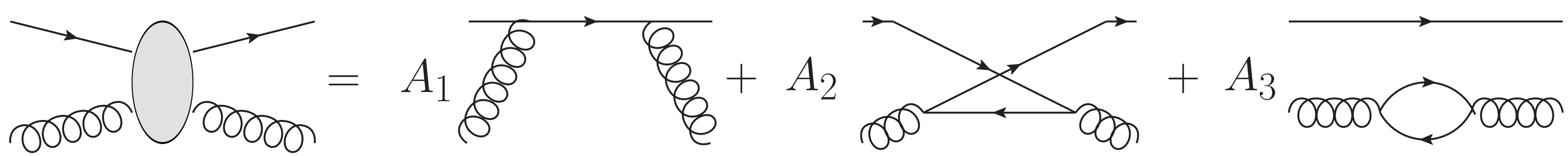
- After this every internal gluon can be removed using:



- This can be applied to any QCD amplitude, tree level or beyond
- In general an amplitude can be written as linear combination of different color structures, like



- For example for 2 (incoming + outgoing) gluons and one $q\bar{q}$ pair



(an incoming quark is the same as an outgoing anti-quark)

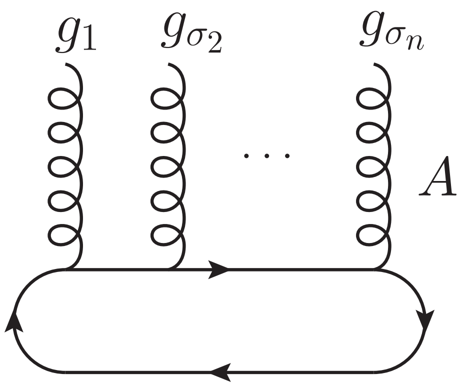
- The above type of color structures can be used as a spanning set, a trace basis

These bases have some nice properties

- Conceptual simplicity
- Can be made minimal to a given *order* in perturbation theory, for example, for tree-level N_g-gluon amplitudes we have $(N_g - 1)!$ color structures of form

$$\mathcal{M}(g_1, g_2, \ldots, g_n) = \sum_{\sigma \in S_{N_g-1}} \mathrm{Tr}(t^{g_1} t^{g_{\sigma_2}} \ldots t^{g_{\sigma_n}}) A(\sigma)$$

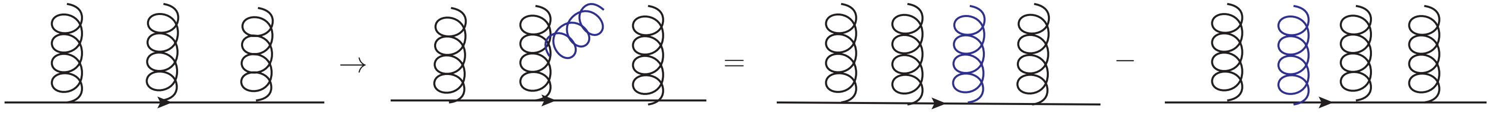
$$= \sum_{\sigma \in S_{N_g-1}}$$



$A(\sigma)$

whereas for higher orders we also have products of traces.

- Taking the leading N_c limit is trivial an results in a flow of colors
- The basis vectors are orthogonal when $N_c \to \infty$
- The effect of gluon emission is easily described:

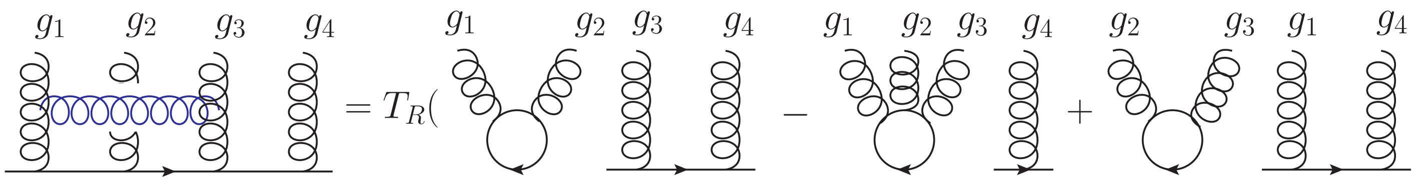


Convention: + when inserting after, minus when inserting before.

We get just one new basis vector if the emitter is an (anti-)quark and two if the emitter is a gluon $\rightarrow$ extremely sparse matrices $\mathbf{T_i}$ describing radiation

(Z. Nagy & D. Soper, JHEP 0807 (2008) 025)

- So is the effect of gluon exchange:



Convention: + when inserting after, - when inserting before

We get at most four terms $\rightarrow$ extremely sparse soft anomalous dimension matrices describing gluon exchange

(M.S., JHEP 0909 (2009) 087 JHEP, arXiv:0810.5756)

For these reasons trace bases are commonly used:

- ColorFull (C++ code for color space, more later)
  (On hepforge since 2013, http://colorfull.hepforge.org/, M.S. arXiv:1412.3967, accepted for publication in EPJC)
- MadGraph (fixed order calculations)
  (J. Alwall, M. Herquet, F. Maltoni, O. Mattelaer, T. Stelzer,JHEP 1106 (2011) 128, arXiv:1106.0522)
- $N_c = 3$ parton showers by M.S and S. Plätzer , and by D. Soper and Z. Nagy
  (D. Soper and Z. Nagy JHEP 0709 (2007) 114, arXiv:0706.0017 M.S. and S. PLätzer JHEP 07(2012)042, arXiv:1201.0260)
- Resummation
  (E. Gerwick, S Höche, S. Marzani, S. Schumann, JHEP 1502 (2015) 106, arXiv:1411.7325)

ColorFull

For the purpose of treating a general QCD color structure (any number of partons, any order) I have written a C++ color algebra code, ColorFull, which:

- Automatically creates trace bases for any number and kind of partons, and to arbitrary order in $\alpha_{\rm s}$
- Squares color amplitudes in various ways
- Describes the effect of gluon emission, calculates "radiation matrices", $\mathbf{T_i}$, which gives the vectors obtained when emitting a gluon from parton i decomposed in the larger basis

- Describes the effect of gluon exchange, automatically calculates soft anomalous dimension matrices
- Interfaces to Herwig++ ($\geq$ 2.7) via Simon Plätzer's Matchbox code and will be shipped along with the next major Herwig release

ColorFull can be downloaded from `colorfull.hepforge.org`, and the corresponding paper is accepted for publication in EPJC (arXiv:1412.3967)

ColorMath

- I have also written a Mathematica package, ColorMath, (Eur. Phys. J. C 73:2310 (2013), arXiv:1211.2099)
- ColorMath is an easy to use Mathematica package for color summed calculations in QCD, SU(N_c)
- Repeated indices are implicitly summed

```
In[2]:= Amplitude = I f[g1, g2, g] t[{g}, q1, q2]
Out[2]= i t^{g}q1_q2 f^{g1,g2,g}

In[3]:= CSimplify[Amplitude Conjugate[Amplitude /. g → h]]
Out[3]= 2 Nc (-1 + Nc^2) TR^2
```

- ColorMath does not automatically construct bases
- but given a basis (constructed by the user) it can calculate the soft anomalous dimension matrix automatically

- The ColorMath package and tutorial can be downloaded from
http://library.wolfram.com/infocenter/MathSource/8442/
or www.thep.lu.se/~malin/ColorMath.html

But there are also drawbacks with trace bases

- Not orthogonal

  $\rightarrow$ When squaring amplitudes (almost) all cross terms have to be taken into account $\rightarrow N^2_{\text{basis}}$ terms
- Overcomplete

  For $N_g + N_{q\bar{q}} > N_c$ the bases are also overcomplete
- The size of the vector space asymptotically grows as an exponential in the number of gluons/$q\bar{q}$-pairs for finite N_c

- For general N_c the basis size grows as a factorial

$$N_{\text{vec}}[N_q, N_g] = N_{\text{vec}}[N_q, N_g - 1](N_g - 1 + N_q) + N_{\text{vec}}[N_q, N_g - 2](N_g - 1)$$

where

$$\begin{aligned} N_{\text{vec}}[N_q, 0] &= N_q! \\ N_{\text{vec}}[N_q, 1] &= N_q N_q! \end{aligned}$$

- For general N_c and gluon only amplitudes (to all order) the size is given by Subfactorial(N_g)$\approx N_g!/e$
- For tree-level gluon amplitudes traces may be used as spanning vectors giving $(N_g - 1)!$ spanning vectors

Number of spanning vectors for N_g gluons (without imposing charge conjugation invariance)

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$	LO Vectors $N_c \to \infty$
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880
11	614 000	14 684 570	3 628 800
12	3 609 760	176 214 841	39 916 800

(Y. Du, M.S. & J. Thorén, arXiv:1503.00530, accepted by JHEP)

The dimension of the full vector space (all orders) for $N_c = 3$

N_g	$N_{q\overline{q}} = 0$	N_g	$N_{q\overline{q}} = 1$	N_g	$N_{q\overline{q}} = 2$
4	8	3	10	2	13
5	32	4	40	3	50
6	145	5	177	4	217
7	702	6	847	5	1 024
8	3 598	7	4 300	6	5 147
9	19 280	8	22 878	7	27 178
10	107 160	9	126 440	8	149 318
11	614 000	10	721 160	9	847 600
12	3 609 760	11	4 223 760	10	4 944 920

(M.S. & Johan Thorén, work in preparation)

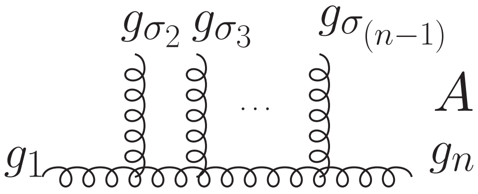
- For a tree-level gluon parton shower, we can get away with the tree-level color structures giving $(N_g-1)!^2$ terms when squaring amplitudes
- For NLO gluon processes we need more color structures
- For all order resummation all color structures will appear $\rightarrow N_{\text{basis}}^2 \approx (N_g!/e)^2$ when squaring
- Note however, that for finding the soft anomalous dimension matrix we need *not* invert the matrix of scalar products since we can easily derive the result of gluon exchange in the trace basis
- On the other hand if we really want to exponentiate the soft anomalous dimension matrix this scales as $N_{\text{basis}}^3 \approx (N_g!/e)^3$
- Numbers for processes with quarks are comparable. (For every gluon you can alternatively treat one $q\bar{q}$-pair)
- Hard to go beyond ~ 8 gluons plus $q\bar{q}$-pairs

DDM bases

- The DDM bases (adjoint bases) are based on the observation that tree-level gluon-only color structures can be expressed as

$$\mathcal{M}(g_1, g_2, \ldots, g_n) = \sum_{\sigma \in S_{N_g-2}} i f^{g_1 g_{\sigma_2} i_1} i f^{i_1 g_{\sigma_3} i_2} \ldots i f^{i_{n-3} g_{\sigma_{n-1}} g_n} A(\sigma)$$

$$= (-1)^{N_g} \sum_{\sigma \in S_{N_g-2}}$$



$$A(\sigma).$$

- In this way we only need $(N_g - 2)!$ spanning vectors
- Charge conjugation symmetry is manifest
- For higher order color structures additional basis vectors are needed

Color flow bases

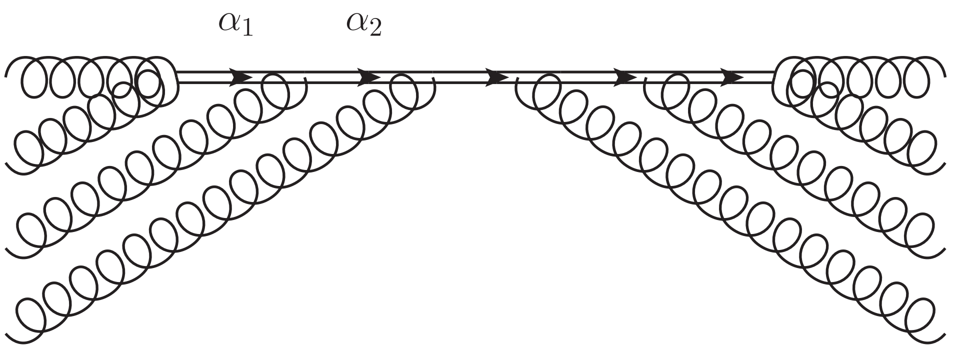
- One way out is to rewrite all vertices in terms of color flows (Maltoni, Stelzer, Paul, Willenbrock, Phys.Rev. D67 (2003), hep-ph/0209271)
- Explicit colors (r, g, or b) are then assigned to the lines, and one may run a Monte Carlo sum over colors to sample color space
- This is not exact but much quicker

(C. Duhr, S. Hoeche, F. Maltoni, JHEP 0608 (2006) 062, hep-ph/0607057
Comix, T. Gleisberg, S. Hoeche, JHEP 0812 (2008) 039, 0808.3674)

Multiplet bases

- QCD is based on SU(3) $\rightarrow$ the color space may be decomposed into irreducible representations
- Orthogonal basis vectors corresponding to irreducible representations may be constructed
- The construction of the corresponding basis vectors is non-trivial, and a general strategy was presented relatively recently
  (S. Keppeler & M.S. JHEP09(2012)124, arXiv:1207.0609)
- With general, I mean general: general number of quarks and gluons, general order in α_s and general N_c
- In this presentation I will – for comparison – talk about processes with gluons only, however, processes with quarks can be treated similarly

- The gluon basis vectors are of form



and can thus be characterized by a chain of representations $\alpha_1, \alpha_2, ...$ (In principle we have to differentiate between different vertices as well)

- These vectors are orthogonal ($\rightarrow$ minimal) by construction

For many partons the size of the vector space is much smaller for $N_c = 3$ (exponential), compared to for $N_c \to \infty$ (factorial)

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$ trace bases	LO Vectors $N_c \to \infty$ LO trace bases
4	8	9	3!=6
5	32	44	4!=24
6	145	265	120
7	702	1 854	720
8	3 598	14 833	5 040
9	19 280	133 496	40 320
10	107 160	1 334 961	362 880

Number of basis vectors for N_g gluons *without* imposing vectors to appear in charge conjugation invariant combinations

... but the real advantage comes when squaring as the multiplet bases are orthogonal and the trace bases are not

N_g	Vectors $N_c = 3$	Vectors $N_c \to \infty$ trace bases	LO Vectors $N_c \to \infty$ LO trace bases
4	8	$(9)^2$	$(6)^2$
5	32	$(44)^2$	$(24)^2$
6	145	$(265)^2$	$(120)^2$
7	702	$(1\ 854)^2$	$(720)^2$
8	3 598	$(14\ 833)^2$	$(5\ 040)^2$
9	19 280	$(133\ 496)^2 \sim 10^{10}$	$(40\ 320)^2 \sim 10^9$
10	107 160	$(1\ 334\ 961)^2 \sim 10^{12}$	$(362\ 880)^2 \sim 10^{11}$

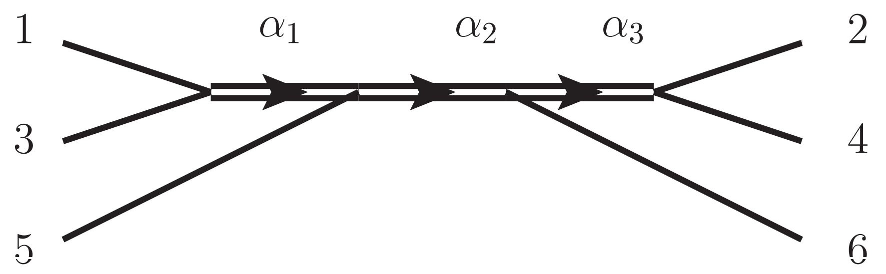
Number of terms from color when squaring for N_g gluons *without* imposing charge conjugation invariant combinations

- Multiplet bases can potentially speed up exact calculations in color space very significantly, as squaring amplitudes becomes much quicker
- Before squaring, amplitudes must be decomposed in color bases
- How quickly can amplitudes be constructed in multiplet bases?
- ... using parton showers?
- ... for resummation?

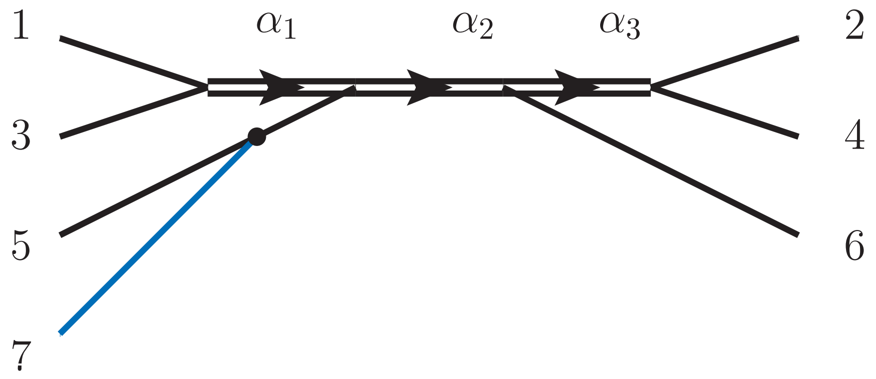
A parton shower perspective

- In a parton shower we start with some amplitude which we can assume that we have decomposed in the multiplet basis

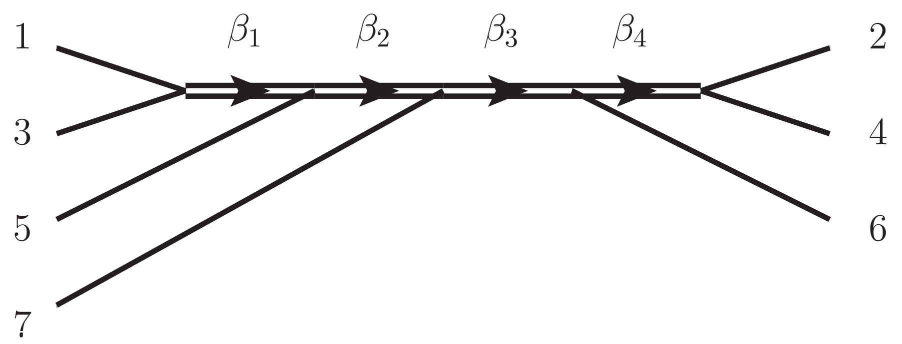
$$\text{Amp} = \sum_{\alpha_1,\alpha_2,\alpha_3} c_{\alpha_1,\alpha_2,\alpha_3}$$



- Knowing the decomposition for $N_g - 1$ gluons, how can we decompose the N_g gluon amplitude?

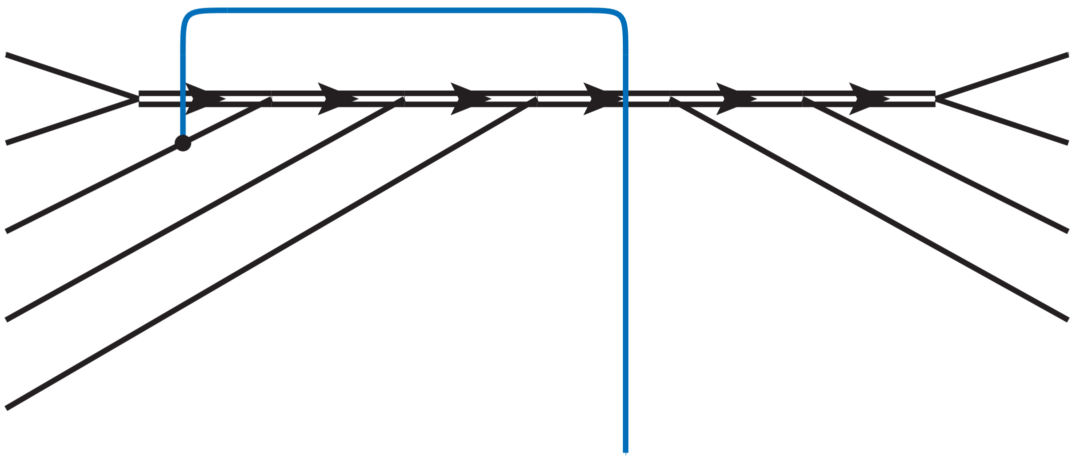


$$= \sum_{\beta_1,\beta_2,\ldots} \tilde{c}_{\beta_1,\beta_2,\ldots}$$



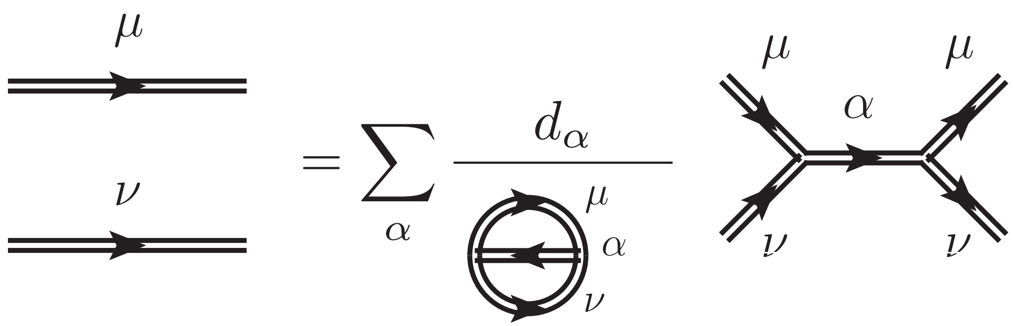
- Scalar products? Too slow!

Let one of the gluons emit a new gluon:

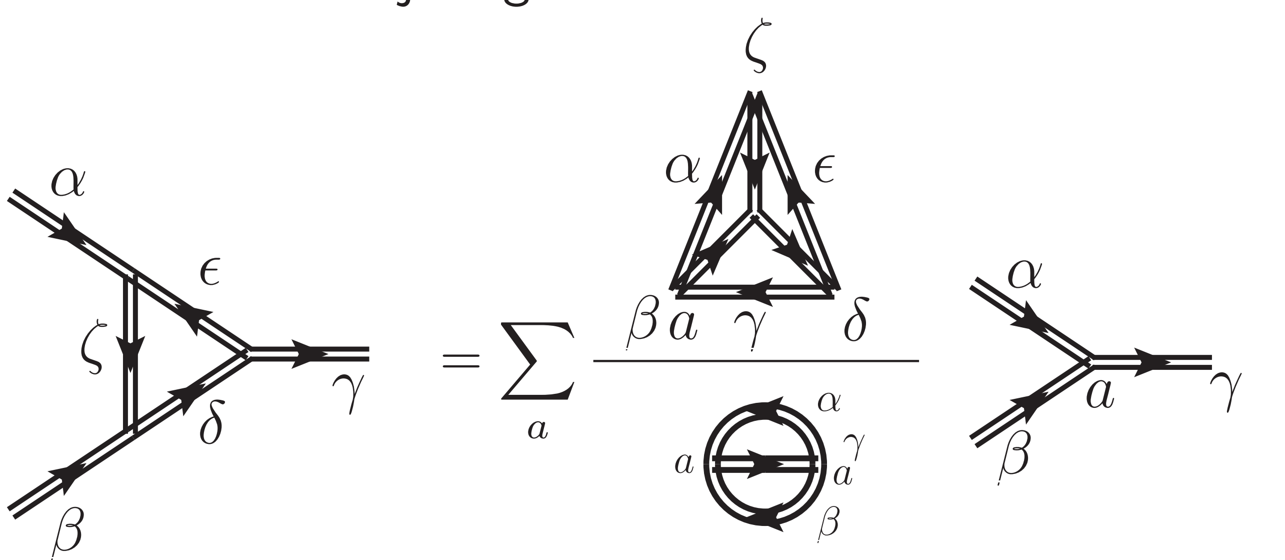


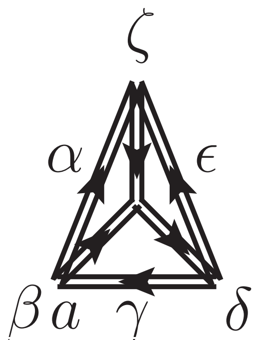
To express the color structure in the new basis we need a few tricks

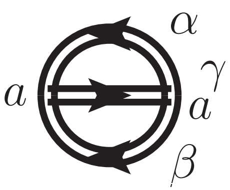
- The completeness relation



- A vertex correction just gives a constant



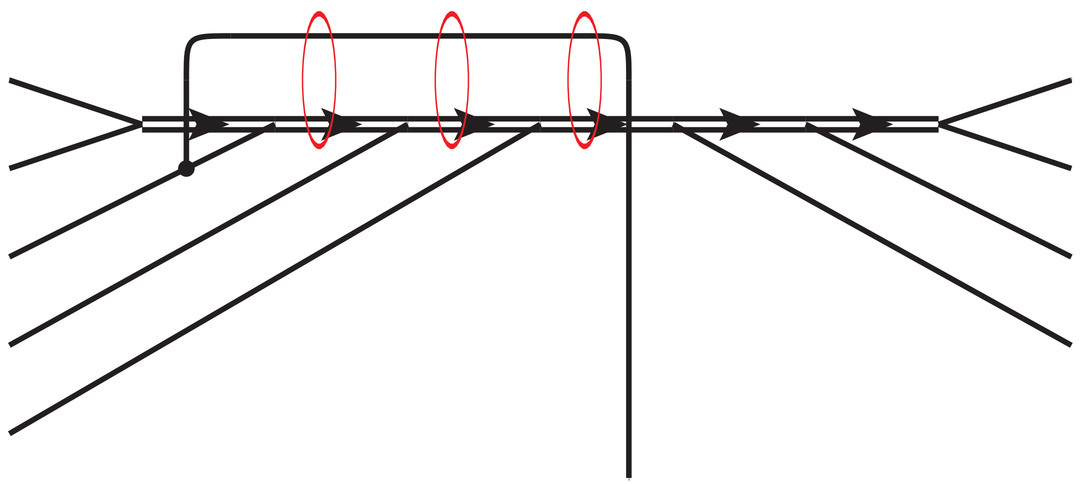
The symbols 

and 

are Wigner 6j and 3j coefficients and their values can be calculated once and for all

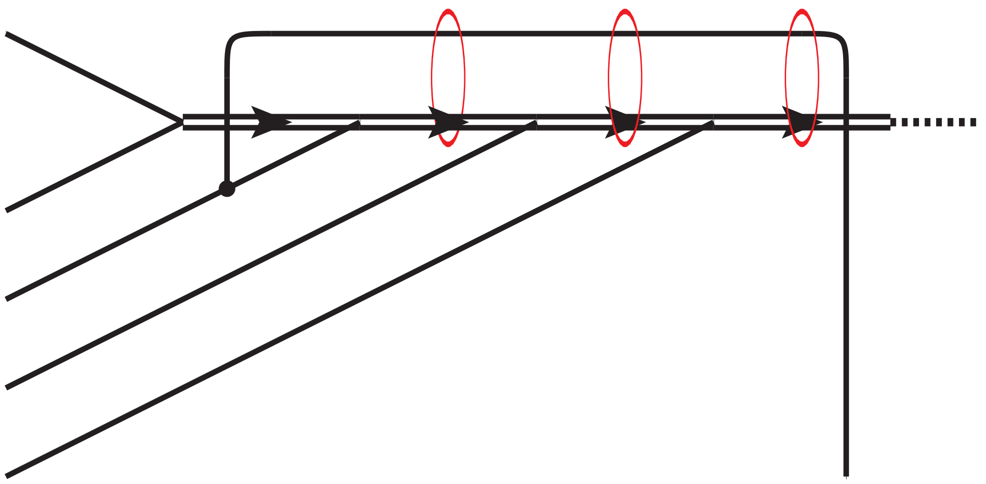
(M.S. & J. Thorén, work in preparation)

To decompose the affected side, we may insert the completeness relation, repeatedly:

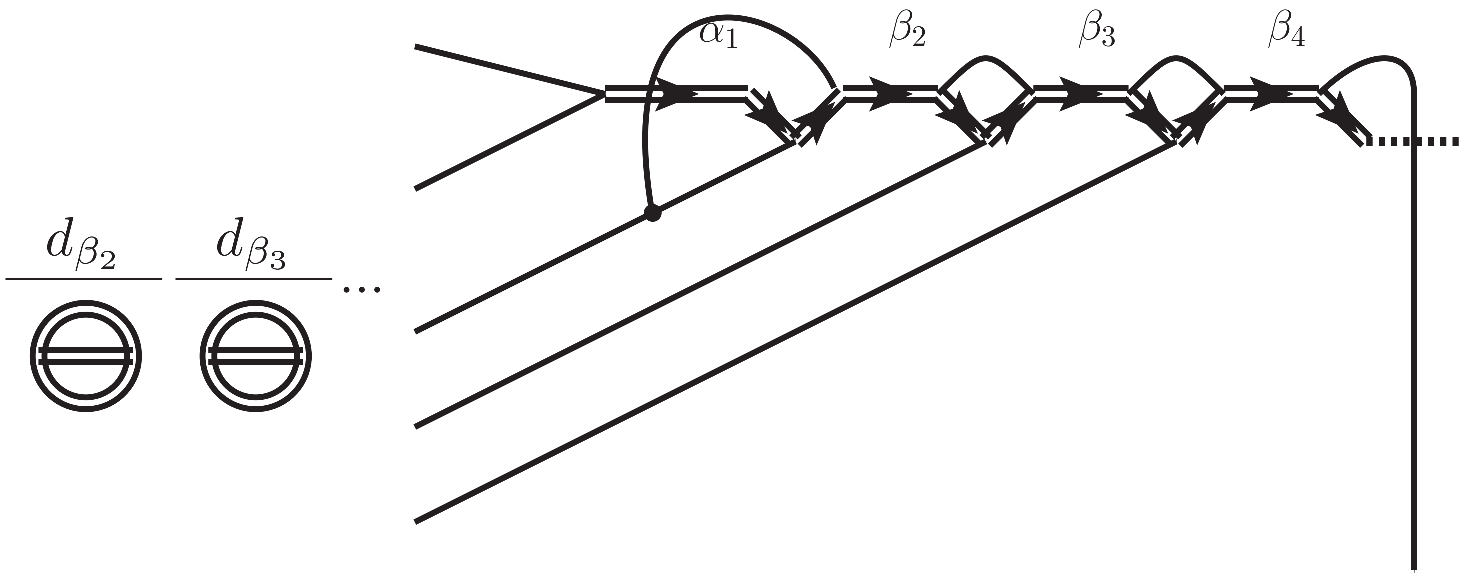


The representations on the other side (here right) don't change

Consider the affected side:

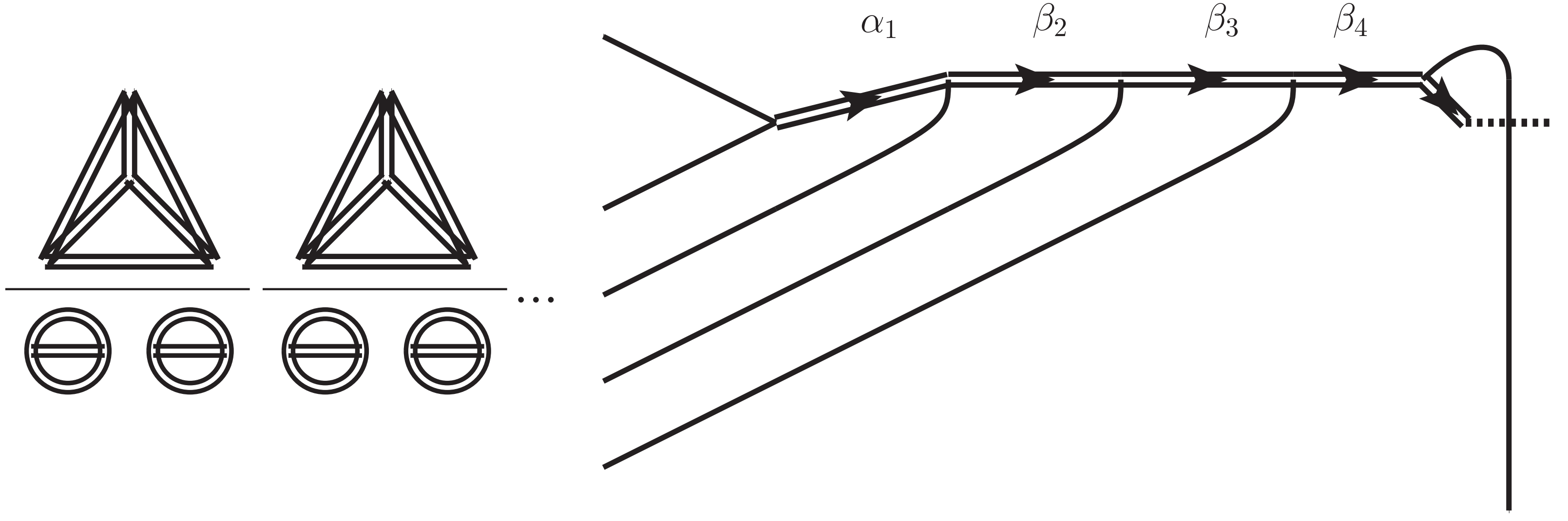


Inserting completeness relations we get a sum of terms of form:



What we have here are just vertex corrections which can be rewritten in terms of 3j and 6j coefficients

Giving us a sum of terms of form:



i.e., knowing the 3j and 6j symbols we can write down the resulting vectors

- By inserting the new gluon "in the middle" in the basis we guarantee that the emitted gluon need never "be transported" across more than $\sim$ half of the reps
- Typically we get only a small fraction of all basis vectors in the larger basis:

N_g	5→6	6→7	7→8	8→9	9→10
$N_c = 3$	0.094	0.027	0.012	0.0032	0.0014
$N_c \geq N_g$	0.071	0.014	0.0054	0.00092	0.00032

(Yi-Jian Du, M.S. & Johan Thorén, arXiv:1503.00530, accepted by JHEP)

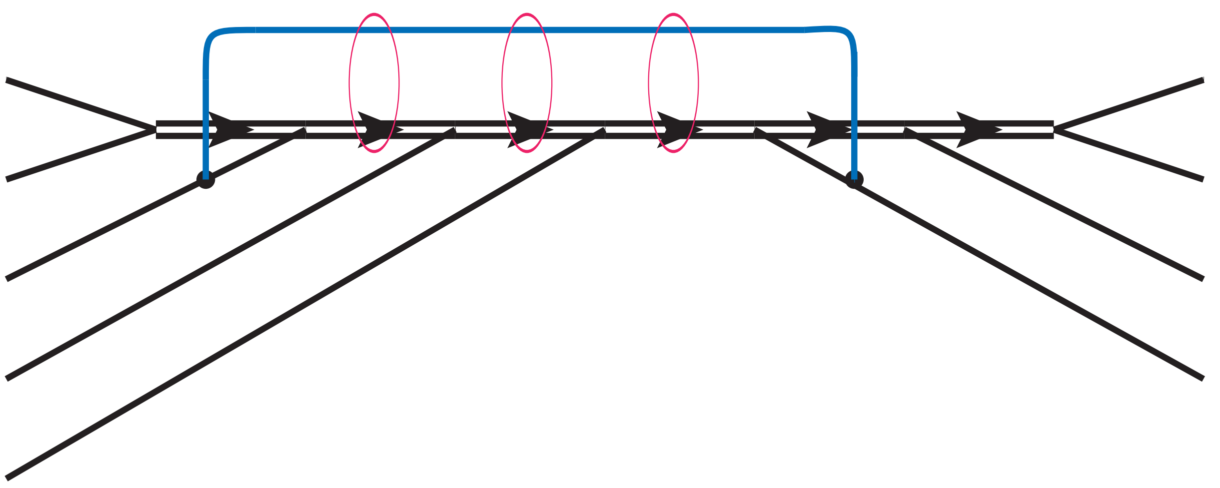
Consider the sum of all terms from all emissions (all emitters and all vectors) and compare to the number encountered when squaring a tree-level amplitude

N_g	Fraction ($N_c = 3$)	All terms ($N_c = 3$)	(# tree vectors)^2 (any N_c)
5→6	0.094	2 184	$(120)^2$
6→7	0.027	16 372	$(720)^2$
7→8	0.012	212 914	$(5\,040)^2$
8→9	0.0032	1 758 620	$(40\,320)^2 \sim 10^9$
9→10	0.0014	25 407 328	$(362\,880)^2 \sim 10^{11}$

Numbers will be somewhat reduced by clever vertex choices, and non-general linear combinations

Gluon exchange

- For higher order calculations or for resummation we need to describe the effect of gluon exchange on the color structure
- Gluon exchange may be treated similar to emission



- Here we get a linear combination of basis vectors where only the intermediate representations can have changed

- For each starting color vector we may immediately write down the linear combination of basis vectors after the gluon exchange in terms of 3j and 6j coefficients
- As only the in-between multiplets can be affected, the result is typically a linear combination of a small fraction of all basis vectors
- $\rightarrow$ The soft anomalous dimension matrices may be written down directly, and they are relatively sparse...
- $\rightarrow$ fewer terms for writing down matrices for all exchanges in multiplet bases than terms in squaring trace bases (for more than $\sim$ 8 gluons)

- But this is actually not the relevant comparison for soft gluon resummation since exponentiation scales as (matrix size)^3

N_g	(QCD/large N_c)^3	QCD $N_c = 3$	Trace bases
6	0.163	145^3	$(265)^3$
7	0.054	$(702)^3$	$(1\ 854)^3$
8	0.0143	$(3\ 598)^3$	$(14\ 833)^3$
9	0.0030	$(19\ 280)^3$	$(133\ 496)^3$
10	0.00052	$(107\ 160)^3$	$(1\ 334\ 961)^3$

- Furthermore, as the multiplet bases are minimal one does not need tricks to exponentiate singular matrices

Summary

So, what basis is best? according to me...

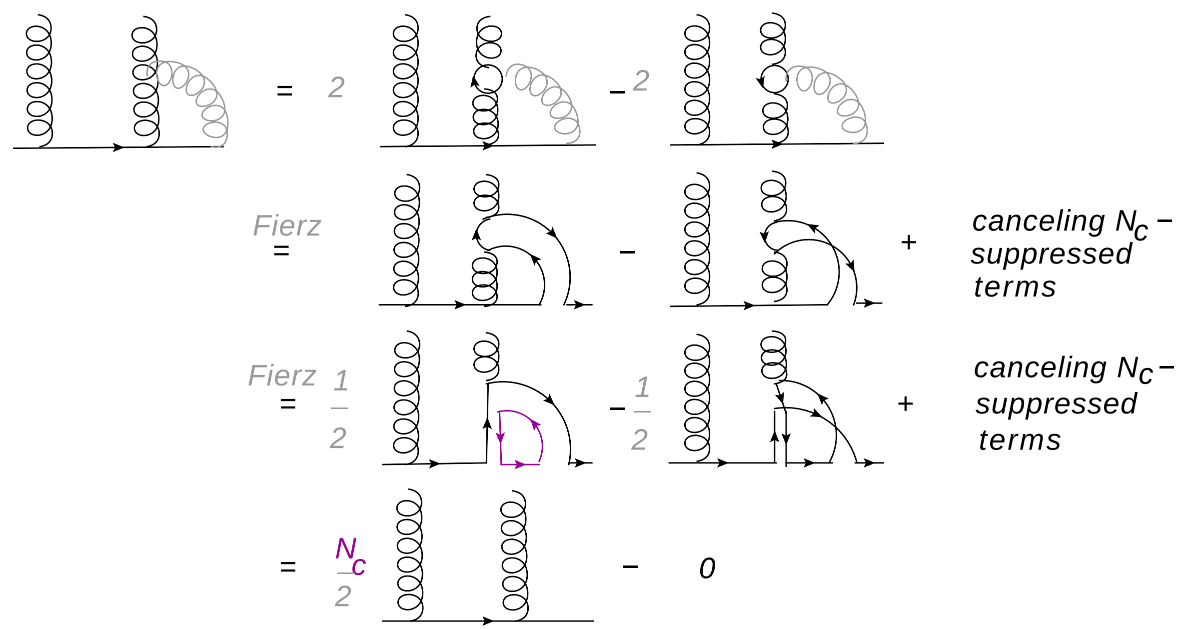
- A moderate number of partons, up to $\sim$ 6 gluons plus $q\bar{q}$-pairs: It doesn't really matter $\rightarrow$ trace bases, easy to deal with and implemented in several codes such as ColorFull
  (Later on possibly multiplet bases)
- Monte Carlo over colors: Color flow bases
  (Multiplet bases in the future?)
- Tree-level gluon amplitudes: DDM

- Parton showers (exact color): So far trace bases, they are implemented and have a trivial $N_c \to \infty$ limit, but it's hard to go beyond ~ 8 gluons plus $q\bar{q}$-pairs while keeping full color structure. In the future, multiplet bases since it will be possible to treat more partons with full color structure, (the issues of gluon exchange and gluon emission have been proved not to scale as badly as squaring in trace bases)
- Resummation: Here the bottle neck is not the amplitude squaring, scaling like N^2_{basis}, but the exponentiation, scaling like N^3_{basis} $\to$ the smaller basis is best $\to$ multiplet bases will be better (once efficiently implemented...)

Thank you for your attention

Backup: Gluon exchange

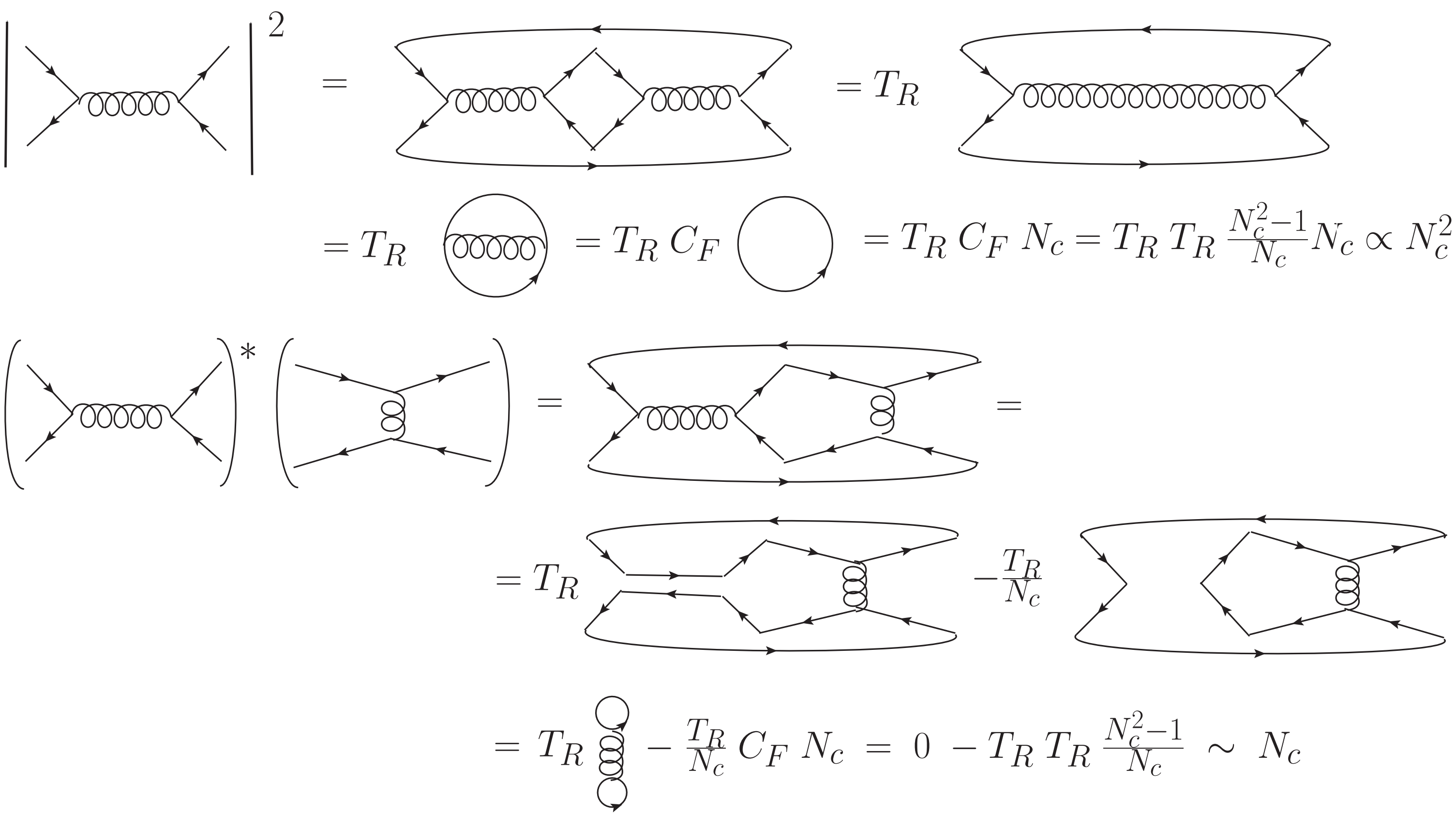
A gluon exchange in this basis “directly” i.e. without using scalar products gives back a linear combination of (at most 4) basis tensors



- N_c-enhancement possible only for near by partons
  → only “color neighbors” radiate in the $N_c \to \infty$ limit

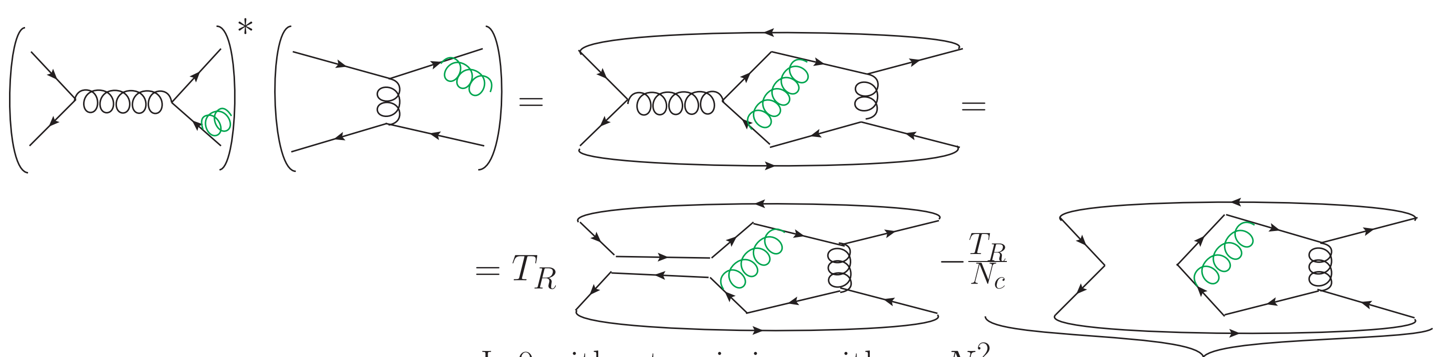
Backup: N_c-suppressed terms

That non-leading color terms are suppressed by $1/N_c^2$, is guaranteed only for same order α_s diagrams with only gluons ('t Hooft 1973)



Backup: N_c-suppressed terms

For a parton shower there may also be terms which only are suppressed by one power of N_c



Is 0 without emission, with $\sim N_c^2$ did not enter in any form, genuine "shower" contribution

Is $\sim N_c$ without emission, with $\sim N_c^2$ "included" in shower, contribution from hard process

The leading N_c contribution scales as N_c^2 before emission and N_c^3 after