SUMMING THRESHOLD LOGS WITH PARTON SHOWER

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Parton Showers and Resummation 2015

Some Conclusions from 2007

ZN, DIS 2007 April 16, 2007 Plenary Talk

Instead of having defined LO, NLO and shower calculation separately and patching the gap between them by matching schemes

Born Level calculations

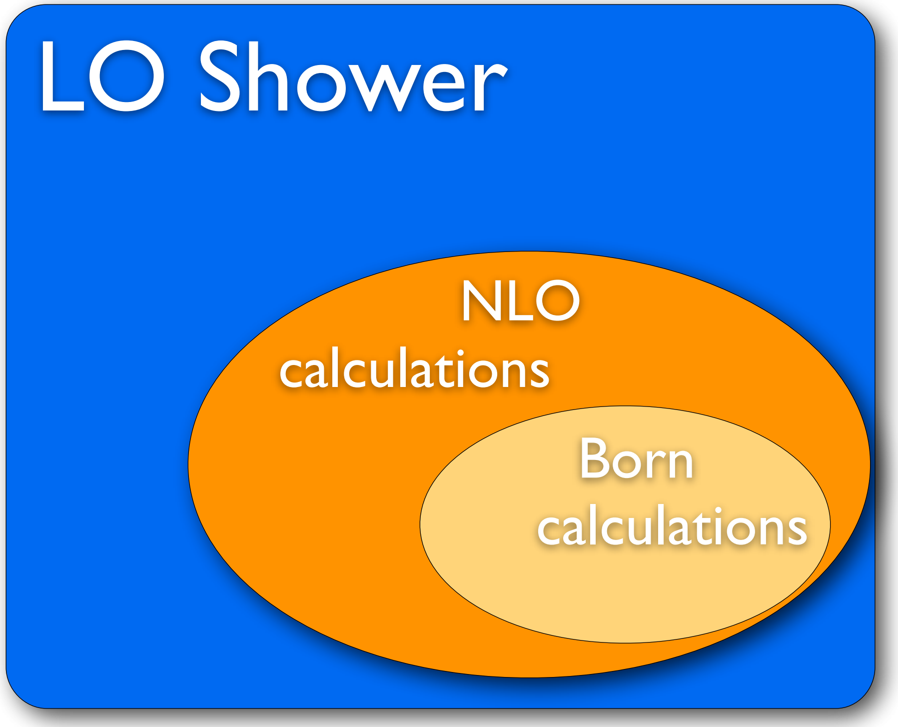
LO Matching Schemes

LO Parton Shower

NLO Matching Schemes

NLO Level calculations

we should define a new shower concept that can naturally cooperate with NLO calculations



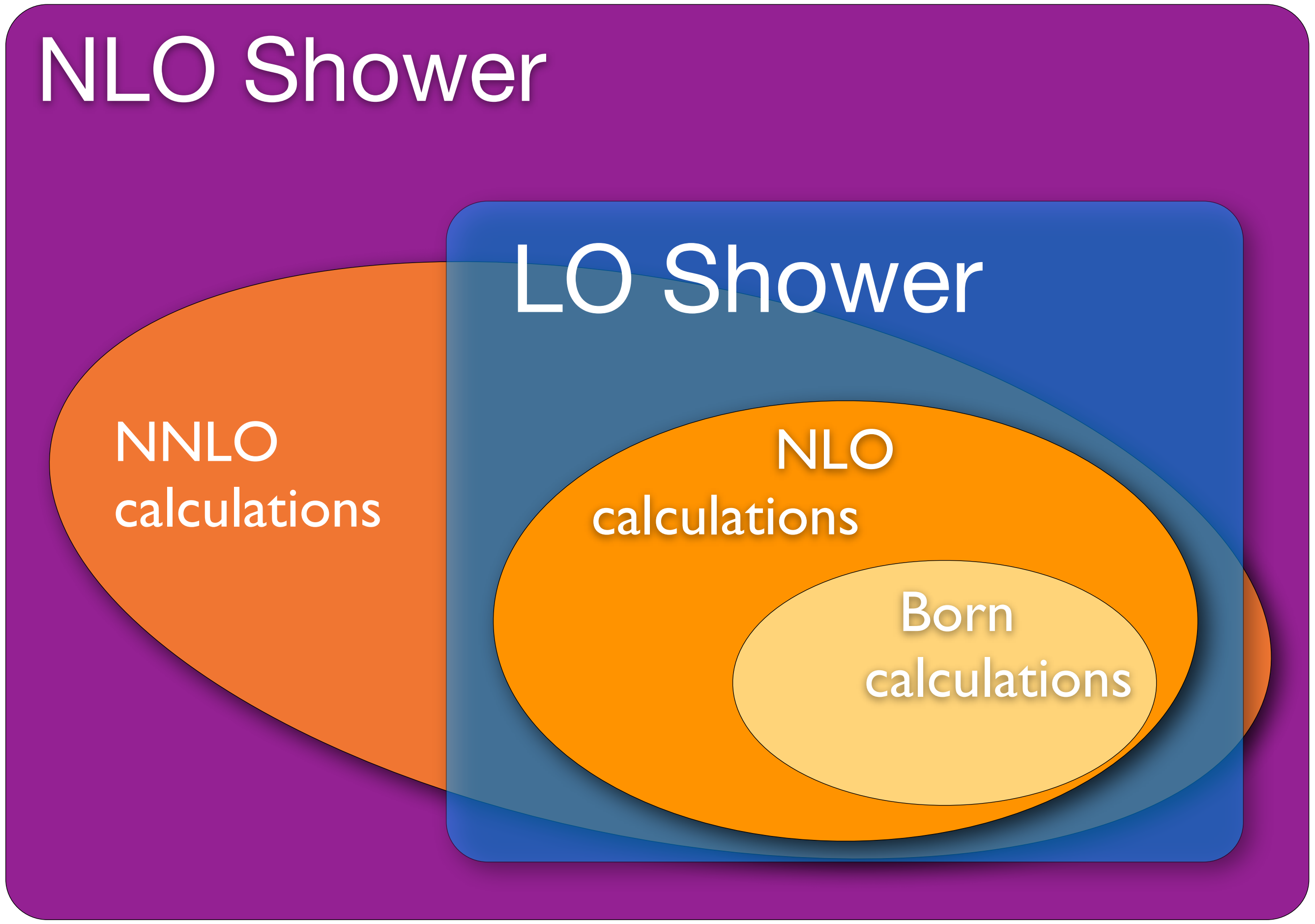
LO Shower

NLO calculations

Born calculations

Some Conclusions from 2007

Or, one can be more ambitious and define this framework at NLO level.



Some Conclusions from 2007

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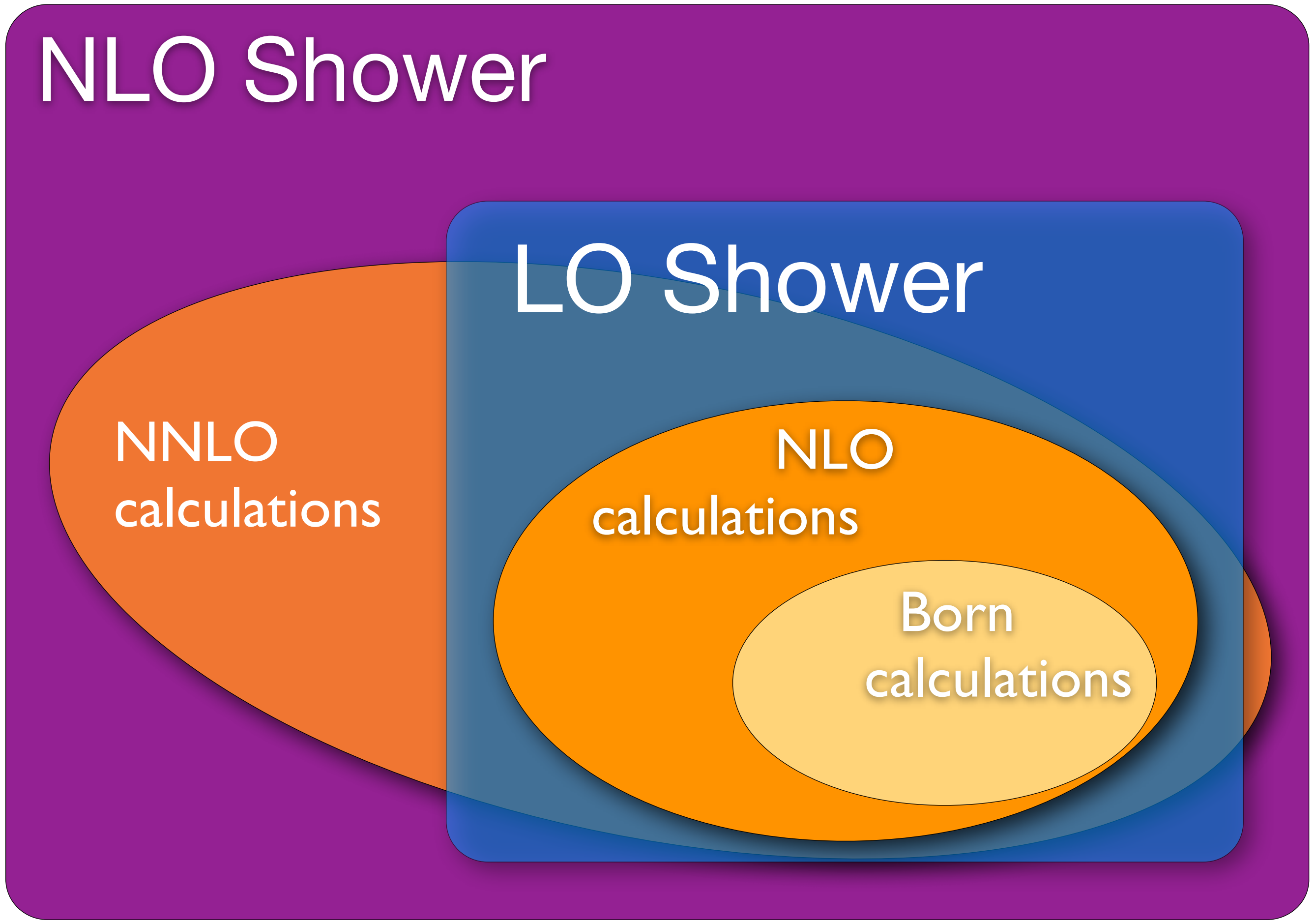


NLO Shower

LO Shower

NNLO calculations

NLO calculations

Born calculations

Back to 2015: *Actually we need an all order pQCD definition of the parton shower.*

Factorisation

- Let us consider an infrared safe observable and it has a typical resolution scale μ_J^2. This means every radiation under this scale is unresolvable and not visible by the observable.
- The all order cross section can be written in a factorised form. The soft and hard part is separated by the factorisation (or shower) scale μ^2.

"Soft part"

"Hard part"

$$\sigma[O] = \Big(1\Big|\mathcal{O}(\mu_J^2)\mathcal{D}_{\text{NP}}(\mu^2)\Big[\mathcal{F}_{\text{F.S.}}(\mu^2)\circ\mathcal{X}_{\text{S.S.}}(\mu^2)\Big]\Big|\rho_H(\mu^2)\Big)$$

This is the soft part and every radiation with $k_\perp^2, q^2, \cdots < \mu^2$ are considered here.

This is the hard part and every radiation with $k_\perp^2, q^2, \cdots > \mu^2$ are considered here.

- It is important that we factorise out the parton emissions (real and virtual) instead of some kind of jet, soft and hard function.
- We work with states and operators in the statistical space.
- We don't have an all order proof for this factorisation, yet. (But we are working on it…)
- We know the QCD amplitudes factorise in the singular limits and that what we use here.

Singular Operator

Renormalized PDF

Collinear counter-terms with explicit poles

Defines the shower scheme

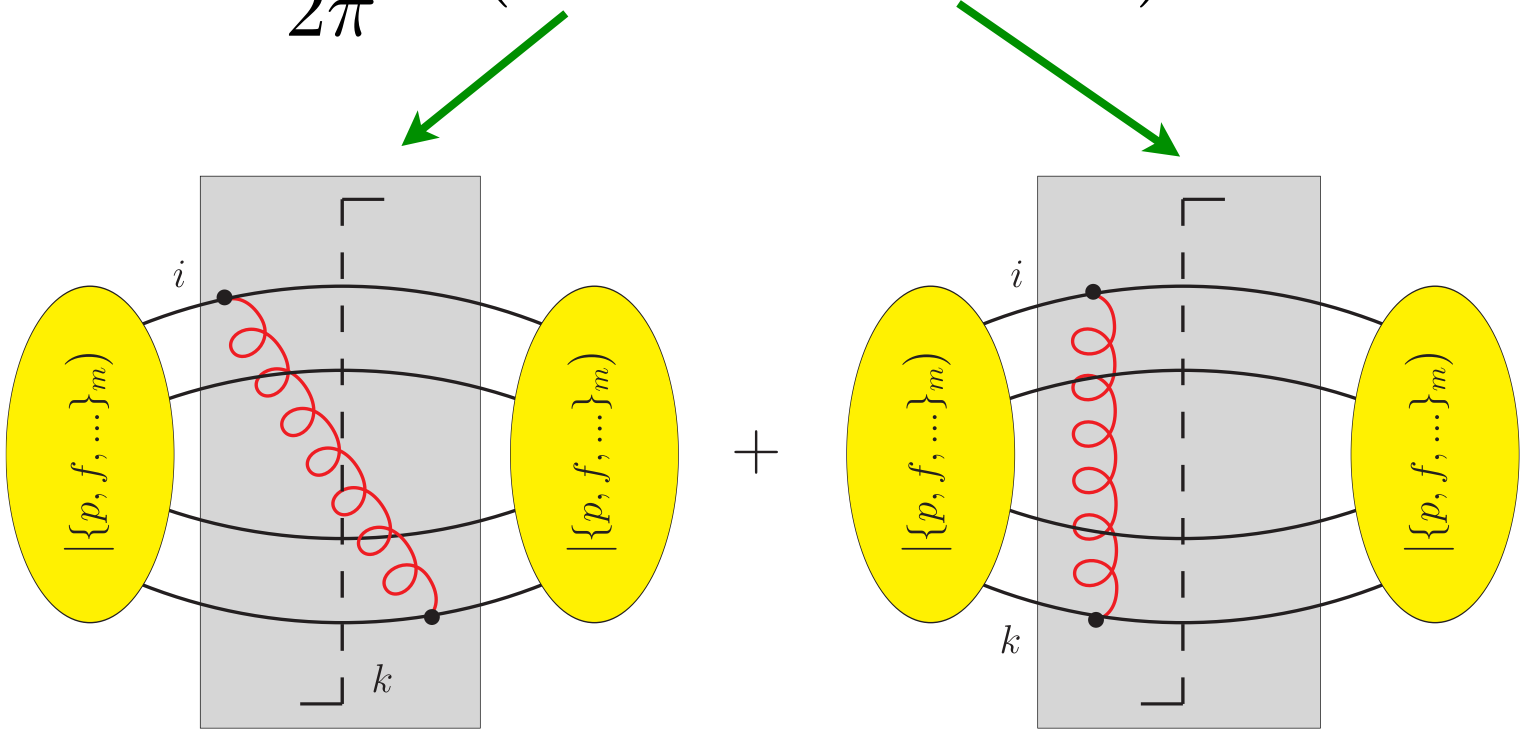
$$\mathcal{D}_{\rm NP}(\mu^2) = \left[\mathcal{F}_{\rm F.S.}(\mu^2) \circ \mathcal{K}_{\rm F.S.}(\mu^2) \circ \mathcal{Z}_F(\mu^2)\right] \mathcal{D}(\mu^2) \left[\mathcal{F}_{\rm F.S.}(\mu^2) \circ \mathcal{X}_{\rm S.S.}\right]^{-1}$$

Defines the factorization scheme

Partonic splitting operator with explicit and implicit singularities

The inverse operator of *D* is analogous to that is usually called to NLO subtraction term

$$\mathcal{D}(\mu^2) = 1 + \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\left(\mathcal{H}_{\rm R}(\mu^2) + \mathcal{H}_{\rm V}(\mu^2)\right) + \mathcal{O}(\alpha_{\rm s}^2)$$



Visible Logs

Now, what happens when the factorization scale is much bigger than the jet resolution scale

$$\mu^2 \gg \mu_J^2$$

In this case the soft part suffers on large logarithms

$$\big(1\big|\mathcal{O}(\mu_J^2)\mathcal{D}_{\rm NS}(\mu^2) = \big(1\big|\left[1 + \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\mathcal{O}\left(\log^2\frac{\mu^2}{\mu_J^2}\right) + \cdots\right]$$

- This indicates that we have to choose the factorization scale to be small, something like

$$\mu^2 = \mu_{\rm f}^2 \sim 1{\rm GeV}$$

- Now the operator $\mathcal{D}(\mu^2)$ describes only soft or collinear emissions and the observable is insensitive to them

$$\big(1\big|\mathcal{O}(\mu_J^2)\mathcal{D}(\mu_{\rm f}^2) \approx \big(1\big|\mathcal{D}(\mu_{\rm f}^2)\mathcal{O}(\mu_J^2)$$

- We want to keep $\big(1\big|\mathcal{D}(\mu_{\rm f}^2)$ free from large perturbative correction to be able to replace it with hadronization.

Meanwhile on the hard side…

Visible Logs

When the factorization scale is small then the hard part suffers on large logarithms

$$\mathcal{F}(\mu^2)\big|\rho_{\mathrm{H}}(\mu^2)\big) = \left[1 + \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\mathcal{O}\left(\log^2 \frac{\mu^2}{M^2}\right)\right] \underbrace{\mathcal{F}(M^2)\big|\rho_{\mathrm{H}}^{(0)}(M^2)\big)}_{\textit{Born level hard part}}$$

➠ This demands large factorization scale otherwise the hard part will be completely unreliable. It is clear that the 1GeV scale choice would be a disaster here.

Visible Logs

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$$\sigma[O] = \big(1\big|\mathcal{O}(\mu_J^2)\mathcal{D}_{\rm NP}(\mu_{\rm f}^2) \qquad \big[\mathcal{F}_{\rm F.S.}(M^2)\circ\mathcal{X}_{\rm S.S.}(M^2)\big]\big|\rho_H(M^2)\big)$$

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$$\sigma[O] = \big(1\big|\mathcal{O}(\mu_J^2)\mathcal{D}_{\rm NP}(\mu_{\rm f}^2)\quad\textit{The gap between the hard and soft parts has to be bridged by partons shower.}\quad\big[\mathcal{F}_{\rm F.S.}(M^2)\circ\mathcal{X}_{\rm S.S.}(M^2)\big]\big|\rho_H(M^2)\big)$$

Visible Logs

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The gap between the hard and soft parts has to be bridged by partons shower.

Renormalization Group Equation

"Soft part" *"Hard part"*

$$\sigma[O] = \big(1\big|\mathcal{O}(\mu_J^2)\mathcal{D}_{\rm NP}(\mu_{\rm f}^2)\underbrace{\mathbb{T}\exp\left\{\int_{\mu_{\rm f}^2}^{M^2}\mathcal{D}_{\rm NP}^{-1}(\mu^2)d\mathcal{D}_{\rm NP}(\mu^2)\right\}}_{\mathcal{U}(\mu_{\rm f}^2,M^2)}\big[\mathcal{F}_{\rm F.S.}(M^2)\circ\mathcal{X}_{\rm S.S.}(M^2)\big]\big|\rho_H(M^2)\big)$$

Standard Shower

As a first approximation: If we define the NLO (NNLO,…) subtraction scheme then we can have a reasonable parton shower algorithm. According to the previous slides we have to make sure that

$$\big(1\big|\mathcal{D}_{\rm NS}(\mu_{\rm f}^2) = \big(1\big|\left[1 + \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\mathcal{O}(1) + \cdots\right]$$

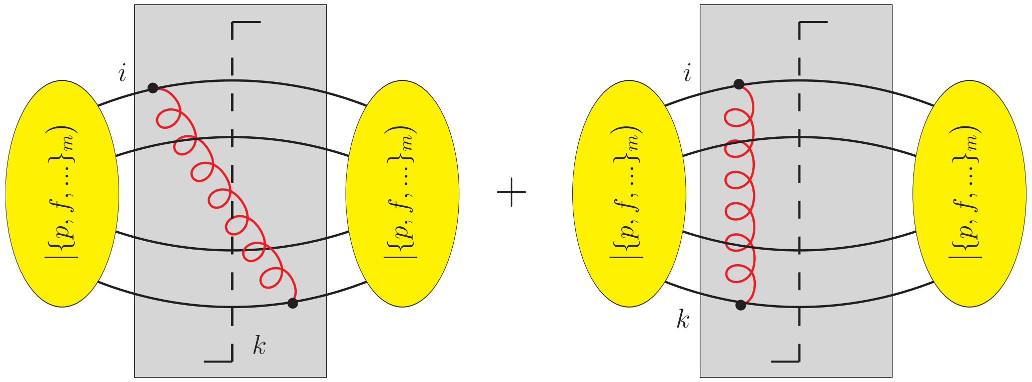
This can be easily done… ⇨ *Unitarity!*

Unitarity condition: Do NOT define subtraction term directly for the 1-loop graphs, use the inclusive version of the real subtraction term via

$$\boxed{-\big(1\big|\big[\mathcal{F}(\mu^2)\circ\mathcal{H}_{\rm V}(\mu^2)\big] = \big(1\big|\big[\mathcal{F}(\mu^2)\circ\mathcal{V}(\mu^2)\big] \equiv \big(1\big|\mathcal{F}(\mu^2)\mathcal{H}_{\rm R}(\mu^2)}$$

This is the definitions of the inclusive splitting operator.

$$\mathcal{D}(\mu^2) = 1 + \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\big(\mathcal{H}_{\rm R}(\mu^2) + \mathcal{H}_{\rm V}(\mu^2)\big) + \mathcal{O}(\alpha_{\rm s}^2)$$



- ✓ This certainly fulfils our requirement

  $$\big(1\big|\mathcal{D}_{\rm NS}(\mu^2) = \big(1\big| \qquad \text{for every } \mu^2$$

- ✓ This leads to a good NLO subtraction scheme.
- ✓ The meaning of the factorization scale is still debatable (kT, virtuality, angle or something else). See Bryan's talk!
- ✗ Is that all? Unfortunately not!

Invisible Logs

Let us consider the total cross section of the Drell-Yan process. It is fully inclusive quantity, $\mu_J^2 = M^2$.
This can be calculated analytically and the partonic cross section is

$$\frac{d^2\hat{\sigma}}{dY\,dM^2} \sim \delta(1-z) + \frac{\alpha_{\mathrm{s}}(M^2)}{2\pi} C_{\mathrm{F}} \left((1+z^2) \left[\frac{1}{1-z} \log \frac{(1-z)^2}{z} \right]_+ + \cdots \right) + \mathcal{O}(\alpha_{\mathrm{s}}^2)$$

Threshold logs

$$\frac{d}{dz} \frac{f_{a/A}\left(\frac{\eta}{z}, \mu^2\right)}{f_{a/A}(\eta, \mu^2)} \gg 1$$

- We have these large contributions in the hard state at large scale at $\mu^2 = M^2$.
- Obviously these logs have to summed up, but can the standard shower deal with it?
- Obviously, standard parton shower cannot deal with it at all.
  It is easy to see from the unitarity condition

$$\big(1\big|\mathcal{D}_{\mathrm{NS}}(\mu^2)\big) = \big(1\big|\mathcal{U}(\mu^2, M^2)\big) = \big(1\big| \qquad \text{for every } \mu^2 < M^2$$

$$\sigma[1] = \big(1\big|\mathcal{D}_{\mathrm{NP}}(\mu_{\mathrm{f}}^2)\,\mathcal{U}(\mu_{\mathrm{f}}^2, M^2)\,\mathcal{F}_{\mathrm{F.S.}}(M^2)\big|\rho_H(M^2)\big) = \big(1\big|\mathcal{F}_{\mathrm{F.S.}}(M^2)\big|\rho_H(M^2)\big) = \sigma_{\mathrm{tot}}$$

All the threshold contributions remain in the hard part. The shower doesn't sum them up.

Shower Compatibility

After applying the subtraction scheme the NLO cross section is

$$\sigma[1] = \big(1\big|\mathcal{D}_{\rm NP}(\mu^2)\mathcal{F}_{\rm F.S.}(\mu^2)\big|\rho_H^{(0)}(\mu^2)\big) + \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\big(1\big|\mathcal{F}_{\rm F.S.}(\mu^2)\big|\rho_H^{(1)}(\mu^2)\big) + \mathcal{O}(\alpha_{\rm s}^2)$$

This is the soft part.

- It has to be free from direct logs at low factorization scale.
- At low scale it has to be free from threshold logs avoiding to tune large perturbative contributions into the hadronization.

This is the NLO correction of the hard part.

- At large shower scale it is free from direct logarithms of the factorization scale.
- At large scale it has to be free from threshold logarithms and other potentially large contributions.

$$\lim_{\mu^2\to 0}\big(1\big|\mathcal{D}_{\rm NS}(\mu^2) = \big(1\big|\left[1 + \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\mathcal{O}(\mu^2\log\mu^2) + \cdots\right]$$

$$\big(1\big|\mathcal{F}_{\rm F.S.}(M^2)\big|\rho_H(M^2)\big) = \left(1 + \frac{\alpha_{\rm s}(M^2)}{2\pi}\mathcal{O}(1) + \cdots\right)\big(1\big|\mathcal{F}_{\rm F.S.}(M^2)\big|\rho_H^{(0)}(M^2)\big)$$

Fixed Order & Parton Shower

The fixed order cross section (at all order level) is

$$\sigma[O] = \big(1\big|\mathcal{O}(\mu_J^2)\mathcal{D}_{\mathrm{NP}}(\mu^2)\,\big[\mathcal{F}_{\mathrm{F.S.}}(\mu^2)\circ\mathcal{X}_{\mathrm{S.S.}}(\mu^2)\big]\big|\rho_H(\mu^2)\big)$$

All the elements of this expression here is well defined in 4 dimension

Since we cannot calculate all order, these series are always truncated, so we have to do resummation

Solving the renormalization group equation

This leads to the parton shower cross section:

$$\sigma[O] = \big(1\big|\mathcal{O}(\mu_J^2)\mathcal{D}_{\mathrm{NP}}(\mu_{\mathrm{f}}^2)\,\underbrace{\mathbb{T}\exp\left\{\int_{\mu_{\mathrm{f}}^2}^{M^2}\mathcal{D}_{\mathrm{NP}}^{-1}(\mu^2)\,d\mathcal{D}_{\mathrm{NP}}(\mu^2)\right\}}_{\mathcal{U}(\mu_{\mathrm{f}}^2,\,M^2)}\big[\mathcal{F}_{\mathrm{F.S.}}(M^2)\circ\mathcal{X}_{\mathrm{S.S.}}(M^2)\big]\big|\rho_H(M^2)\big)$$

IMPORTANT: The primary goal of the parton showers is to sum up parton emissions (both real and virtual).

NLO level

Let us start with the singular operator. This operator also defines the subtraction terms.

Collinear counter-terms
with explicit poles

Renormalized PDF

$$\mathcal{D}_{\text{NP}}(\mu^2) = \left[\mathcal{F}_{\text{F.S.}}(\mu^2) \circ \mathcal{K}_{\text{F.S.}}(\mu^2) \circ \mathcal{Z}_F(\mu^2)\right] \mathcal{D}(\mu^2) \left[\mathcal{F}_{\text{F.S.}}(\mu^2) \circ \mathcal{X}_{\text{S.S.}}\right]^{-1}$$

Defines the
factorization scheme

Partonic splitting operator
with explicit and implicit
singularities

NLO level

Let us start with the singular operator. This operator also defines the subtraction terms.

$$\mathcal{D}_{\mathrm{NP}}(\mu^2) = 1 + \frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\Big\{\mathcal{F}_{\mathrm{F.S.}}(\mu^2)\mathcal{H}_{\mathrm{R}}(\mu^2)\mathcal{F}^{-1}_{\mathrm{F.S.}}(\mu^2)$$

$$+\,\mathcal{H}_{\mathrm{V}}(\mu^2) + \mathcal{R}^{(1)}_{\mathrm{LSZ}}(\mu^2)$$

$$+\left[\mathcal{F}_{\mathrm{F.S.}}(\mu^2)\circ\left(\mathcal{K}^{(1)}_{\mathrm{F.S.}}(\mu^2) + \mathcal{Z}^{(1)}_{F}(\mu^2) - \mathcal{X}^{(1)}_{\mathrm{S.S.}}(\mu^2)\right)\right]\mathcal{F}^{-1}_{\mathrm{F.S.}}(\mu^2)\Big\} + \mathcal{O}(\alpha_{\mathrm{s}}^2)\quad.$$

Real subtraction term

(implicit singularities)

Virtual subtraction term

(explicit 1/ϵ singularities)

Collinear counter-term

(explicit 1/ϵ singularities)

NLO level

Let us start with the singular operator. This operator also defines the subtraction terms.

$$\begin{aligned}\mathcal{D}_{\rm NP}(\mu^2) = 1 + \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\Big\{&\mathcal{F}_{\rm F.S.}(\mu^2)\mathcal{H}_{\rm R}(\mu^2)\mathcal{F}^{-1}_{\rm F.S.}(\mu^2)\\ &+\mathcal{H}_{\rm V}(\mu^2)+\mathcal{R}^{(1)}_{\rm LSZ}(\mu^2)\\ &+\Big[\mathcal{F}_{\rm F.S.}(\mu^2)\circ\Big(\mathcal{K}^{(1)}_{\rm F.S.}(\mu^2)+\mathcal{Z}^{(1)}_{F}(\mu^2)-\mathcal{X}^{(1)}_{\rm S.S.}(\mu^2)\Big)\Big]\mathcal{F}^{-1}_{\rm F.S.}(\mu^2)\Big\}+\mathcal{O}(\alpha_{\rm s}^2)\quad .\end{aligned}$$

Real subtraction term

(implicit singularities)

Virtual subtraction term

(explicit 1/ϵ singularities)

Collinear counter-term

(explicit 1/ϵ singularities)

It is still useful to introduce the inclusive splitting operator and its approximation as

$$\big(1\big|\big[\mathcal{F}(\mu^2)\circ\mathcal{V}(\mu^2)\big] = \big(1\big|\Big[\mathcal{F}(\mu^2)\circ\Big(\widetilde{\mathcal{V}}(\mu^2)+\mathcal{X}_{\rm S.S}(\mu^2)\Big)\Big] = \big(1\big|\mathcal{F}(\mu^2)\mathcal{H}_{\rm R}(\mu^2)$$

Defines the shower scheme

(only power suppressed terms)

NLO level

Let us start with the singular operator. This operator also defines the subtraction terms.

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Defines the shower scheme

(only power suppressed terms)

Unitary part, this leads to the standard shower

$$\mathcal{D}_{\rm NP}(\mu^2) = 1 + \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\Big\{\Big(\mathcal{F}_{\rm F.S.}(\mu^2)\mathcal{H}_{\rm R}(\mu^2) - \big[\mathcal{F}_{\rm F.S.}(\mu^2)\circ\mathcal{V}(\mu^2)\big]\Big)\mathcal{F}^{-1}_{\rm F.S.}(\mu^2) \boxed{\begin{aligned}&+ \mathcal{H}_{\rm V}(\mu^2) + \mathcal{R}^{(1)}_{\rm LSZ}(\mu^2)\\ &+ \Big[\mathcal{F}_{\rm F.S.}(\mu^2)\circ\Big(\mathcal{K}^{(1)}_{\rm F.S.}(\mu^2) + \mathcal{Z}^{(1)}_{F}(\mu^2) + \widetilde{\mathcal{V}}(\mu^2)\Big)\Big]\,\mathcal{F}^{-1}_{\rm F.S.}(\mu^2)\end{aligned}}\Big\} + \mathcal{O}(\alpha_{\rm s}^2) \;.$$

Finite in d=4 dimension

Completely neglected in the standard showers. The threshold contributions are here.

Inclusive Splitting Operator

We have to study the approximated inclusive splitting operator and we are interested in the contribution that describes initial state splittings

$$
\begin{aligned}
\widetilde{\mathcal{V}}(\mu^2)\big|\{p,f,s',c',s,c\}_m\big) = {} & \sum_{k\neq \mathrm{a}} \widetilde{\mathcal{V}}_{\mathrm{a},k}(\mu^2)\big|\{p,f,s',c',s,c\}_m\big) + \sum_{k\neq \mathrm{b}} \widetilde{\mathcal{V}}_{\mathrm{b},k}(\mu^2)\big|\{p,f,s',c',s,c\}_m\big) \\
& + \sum_{l=1}^{m} \sum_{k\neq l} \widetilde{\mathcal{V}}_{l,k}(\mu^2)\big|\{p,f,s',c',s,c\}_m\big)
\end{aligned}
$$

The initial state operator is

$$
\widetilde{\mathcal{V}}_{\mathrm{a},k}(\mu^2)\big|\{p,f,s',c',s,c\}_m\big) = \lambda_{\mathrm{a}k}(\mu^2;z)\,\frac{1}{2}\,\underbrace{\big[(\boldsymbol{T}_{\mathrm{a}}\cdot\boldsymbol{T}_k)\otimes 1 + 1\otimes(\boldsymbol{T}_{\mathrm{a}}\cdot\boldsymbol{T}_k)\big]}\,\big|\{p,f,s',c',s,c\}_m\big)
$$

The color structure has to be matched
to the 1-loop color structure.

$\mu^2 > m_\perp^2 \approx 1\,\mathrm{GeV}$ fixed

$$
\begin{aligned}
[\lambda_{\mathrm{a}k}(\mu^2;z)]_{\hat{a}a} = {} & \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\int_0^{\frac{1-z}{z}}\frac{dy}{y}\,y^{-\epsilon}(1-z(1+y))^{-\epsilon} \\
& \times \Big[1-\theta\big((1-z)^{\beta}yQ^2>\mu^2\big)\,\theta\big((1-z)yQ^2>m_\perp^2\big)\Big] \\
& \times \left[\theta(k=\mathrm{a})\,\frac{\hat{P}_{a\hat{a}}(z,\epsilon)}{C_a} - \theta(k\neq \mathrm{a})\,\delta_{a\hat{a}}\,\frac{2z}{1-z}\,w\Big(\xi_{\mathrm{a}k},\frac{zy}{1-z}\Big)\right]\ ,
\end{aligned}
$$

Defines the singular region

NOTE: This is nothing else but the integral of the initial state NLO real subtraction term over a limited phase space region.

Inclusive Splitting Operator

We have to study the approximated inclusive splitting operator and we are interested in the contribution that describes initial state splittings

$$\widetilde{\mathcal{V}}(\mu^2)\big|\{p \qquad ', c', s, c\}_m\big)$$

$$\beta = \begin{cases} -1 & \text{angular ordering, } yQ^2/(1-z) > \mu^2 \\ 0 & \text{virtuality ordering, } yQ^2 > \mu^2 \\ 1 & \text{transverse momentum ordering, } (1-z)yQ^2 > \mu^2 \end{cases}$$

The initial state operator is

$$\widetilde{\mathcal{V}}_{\mathrm{a},k}(\mu^2)\big|\{p, f, s', c', s, c\}_m\big) = \lambda_{\mathrm{a}k}(\mu^2; z)\frac{1}{2}\underbrace{\left[(\boldsymbol{T}_{\mathrm{a}} \cdot \boldsymbol{T}_k) \otimes 1 + 1 \otimes (\boldsymbol{T}_{\mathrm{a}} \cdot \boldsymbol{T}_k)\right]}\big|\{p, f, s', c', s, c\}_m\big)$$

The color structure has to be matched to the 1-loop color structure.

$\mu^2 > m_\perp^2 \approx 1\,\mathrm{GeV}$ fixed

$$\begin{aligned}[\lambda_{\mathrm{a}k}(\mu^2; z)]_{\hat{a}a} = {} & \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{Q^2}\right)^\epsilon \int_0^{\frac{1-z}{z}} \frac{dy}{y}\, y^{-\epsilon}(1 - z(1+y))^{-\epsilon} \\ & \times \left[1 - \theta\big((1-z)^\beta yQ^2 > \mu^2\big)\, \theta\big((1-z)yQ^2 > m_\perp^2\big)\right] \\ & \times \left[\theta(k = \mathrm{a})\, \frac{\hat{P}_{a\hat{a}}(z,\epsilon)}{C_a} - \theta(k \neq \mathrm{a})\, \delta_{a\hat{a}}\, \frac{2z}{1-z}\, w\Big(\xi_{\mathrm{a}k}, \frac{zy}{1-z}\Big)\right] ,\end{aligned}$$

Defines the singular region

NOTE: This is nothing else but the integral of the initial state NLO real subtraction term over a limited phase space region.

Inclusive Splitting Operator

After performing the y integral, we have

$$[\lambda_{\mathrm{a}k}(\mu^2;z)]_{\hat{a}a} = -\underbrace{\frac{1}{C_a}D_{a\hat{a}}(\mu^2/Q^2,z)}_{\text{Soft x Collinear + Collinear}} -\delta_{a\hat{a}}\overbrace{C(\xi_{\mathrm{a}k},\mu^2/Q^2,z)}^{\text{Wide angle soft}}$$

$$\begin{aligned}
C(\xi,y_{\mathrm{S}},z) = {} & \left[\frac{2z}{1-z}\,\theta\big((1-z)^{\beta+1} < z\,y_{\mathrm{S}}\,r_\perp(z)\big)\right]_+ \log\xi \\
& - \left[\frac{2z}{1-z}\,\theta\big((1-z)^{\beta+1} > z\,y_{\mathrm{S}}\;r_\perp(z)\big)\log\frac{\xi+\sqrt{\xi^2+(1-\xi)\frac{4z^2y_{\mathrm{S}}^2r_\perp(z)^2}{(1-z)^{2\beta+2}}}}{2\xi}\right]_+ \\
& - \delta(1-z)\left\{\frac{(4\pi\,y_{\mathrm{S}})^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon}+2\right)\log\xi - \int_0^1\frac{dx}{x}\,[\log x+\log(1-x)]\,w(\xi,x)\right\} \\
& - \delta(1-z)\int_0^1 du\,\frac{2u}{1-u}\,\theta\big((1-u)^{\beta+1} > u\,y_{\mathrm{S}}\,r_\perp(u)\big) \\
& \qquad \times\log\frac{\xi+\sqrt{\xi^2+(1-\xi)\frac{4u^2y_{\mathrm{S}}^2r_\perp(u)^2}{(1-u)^{2\beta+2}}}}{2}\quad .
\end{aligned}$$

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$$\begin{aligned}
C(\xi,y_{\mathrm{s}},z) = {} & \left[\frac{2z}{1-z}\,\theta\big((1-z)^{\beta+1} < z\,y_{\mathrm{s}}\,r_\perp(z)\big)\right]_+ \log\xi \\
& - \left[\frac{2z}{1-z}\,\theta\big((1-z)^{\beta+1} > z\,y_{\mathrm{s}}\,r_\perp(z)\big)\log\frac{\xi+\sqrt{\xi^2+(1-\xi)\frac{4z^2y_{\mathrm{s}}^2r_\perp(z)^2}{(1-z)^{2\beta+2}}}}{2\xi}\right]_+
\end{aligned}$$

$$\boxed{\begin{aligned}
& -\delta(1-z)\left\{\frac{(4\pi\,y_{\mathrm{s}})^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon}+2\right)\log\xi - \int_0^1\frac{dx}{x}\,[\log x+\log(1-x)]\,w(\xi,x)\right\} \\
& -\delta(1-z)\int_0^1 du\,\frac{2u}{1-u}\,\theta\big((1-u)^{\beta+1} > u\,y_{\mathrm{s}}\,r_\perp(u)\big) \\
& \qquad\times\log\frac{\xi+\sqrt{\xi^2+(1-\xi)\frac{4u^2y_{\mathrm{s}}^2r_\perp(u)^2}{(1-u)^{2\beta+2}}}}{2}\quad.
\end{aligned}}$$

Wide angle soft singularity
Cancelled by the 1-loop graphs

Inclusive Splitting Operator

After performing the y integral, we have

$$[\lambda_{\mathrm{a}k}(\mu^2; z)]_{\hat{a}a} = \underbrace{-\frac{1}{C_a} D_{a\hat{a}}(\mu^2/Q^2, z)}_{\textit{Soft} \text{ x } \textit{Collinear} + \textit{Collinear}} - \delta_{a\hat{a}}\, \overbrace{C(\xi_{\mathrm{a}k}, \mu^2/Q^2, z)}^{\textit{Wide angle soft}}$$

Threshold logs with the right behaviour. They disappear in the $\mu^2 \to 0$ limit, thus they are summed up in the shower evolution.

$$\begin{aligned}
C(\xi, y_{\mathrm{S}}, z) = {} & \boxed{\begin{aligned} & \left[\frac{2z}{1-z}\,\theta\big((1-z)^{\beta+1} < z\, y_{\mathrm{S}}\, r_\perp(z)\big)\right]_+ \log\xi \\ & - \left[\frac{2z}{1-z}\,\theta\big((1-z)^{\beta+1} > z\, y_{\mathrm{S}}\, r_\perp(z)\big) \log\frac{\xi + \sqrt{\xi^2 + (1-\xi)\frac{4z^2 y_{\mathrm{S}}^2 r_\perp(z)^2}{(1-z)^{2\beta+2}}}}{2\xi}\right]_+ \end{aligned}} \\
& - \delta(1-z)\left\{\frac{(4\pi\, y_{\mathrm{S}})^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon} + 2\right)\log\xi - \int_0^1 \frac{dx}{x}\,[\log x + \log(1-x)]\, w(\xi, x)\right\} \\
& - \delta(1-z)\int_0^1 du\,\frac{2u}{1-u}\,\theta\big((1-u)^{\beta+1} > u\, y_{\mathrm{S}}\, r_\perp(u)\big) \\
& \qquad \times \log\frac{\xi + \sqrt{\xi^2 + (1-\xi)\frac{4u^2 y_{\mathrm{S}}^2 r_\perp(u)^2}{(1-u)^{2\beta+2}}}}{2}\ .
\end{aligned}$$

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$$\begin{aligned} D_{\hat{a}a}(y_{\mathrm{s}},z) = &-\frac{1}{\epsilon}\frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)}P_{a\hat{a}}(z) \\ &+ P^{(\epsilon)}_{a\hat{a}}(z) - P^{\mathrm{reg}}_{a\hat{a}}(z)\log\frac{(1-z)^{\beta-1}}{r_\perp(z)} - 2C_a\delta_{a\hat{a}}\left[\frac{1}{1-z}\log\frac{(1-z)^{\beta-1}}{r_\perp(z)}\right]_+ \\ &+ 2C_a\delta_{a\hat{a}}\left[\theta\big((1-z)^{\beta+1} < z\,y_{\mathrm{s}}\,r_\perp(z)\big)\frac{1}{1-z}\log\frac{(1-z)^{\beta+1}}{z\,y_{\mathrm{s}}\,r_\perp(z)}\right]_+ \\ &+ P^{\mathrm{reg}}_{a\hat{a}}(z)\,\theta\big((1-z)^{\beta+1} < z\,y_{\mathrm{s}}\,r_\perp(z)\big)\log\frac{(1-z)^{\beta+1}}{z\,y_{\mathrm{s}}\,r_\perp(z)} \\ &+ \delta(1-z)\,\delta_{a\hat{a}}\frac{(4\pi\,y_{\mathrm{s}})^{\epsilon}}{\Gamma(1-\epsilon)}\left\{\frac{1}{\epsilon^2}C_a + \frac{1}{\epsilon}\gamma_a + \frac{\pi^2}{6}C_a\right\} \\ &- \delta(1-z)\,\delta_{a\hat{a}}\,2C_a\int_0^1\frac{du}{1-u}\log\frac{(1-u)^{\beta+1}}{u\,y_{\mathrm{s}}\,r_\perp(u)}\,\theta\big((1-u)^{\beta+1} > uy_{\mathrm{s}}\,r_\perp(u)\big) \end{aligned}$$

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$$\begin{aligned}
D_{\hat{a}a}(y_{\mathrm{s}},z) = &\boxed{-\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}P_{a\hat{a}}(z) \qquad \begin{array}{r}\text{Collinear singularity,}\\ \text{taken by the PDF renormalization}\end{array}} \\
&+ P^{(\epsilon)}_{a\hat{a}}(z) - P^{\mathrm{reg}}_{a\hat{a}}(z)\log\frac{(1-z)^{\beta-1}}{r_\perp(z)} - 2C_a\delta_{a\hat{a}}\left[\frac{1}{1-z}\log\frac{(1-z)^{\beta-1}}{r_\perp(z)}\right]_+ \\
&+ 2C_a\delta_{a\hat{a}}\left[\theta\big((1-z)^{\beta+1} < z\,y_{\mathrm{s}}\,r_\perp(z)\big)\frac{1}{1-z}\log\frac{(1-z)^{\beta+1}}{z\,y_{\mathrm{s}}\,r_\perp(z)}\right]_+ \\
&+ P^{\mathrm{reg}}_{a\hat{a}}(z)\,\theta\big((1-z)^{\beta+1} < z\,y_{\mathrm{s}}\,r_\perp(z)\big)\log\frac{(1-z)^{\beta+1}}{z\,y_{\mathrm{s}}\,r_\perp(z)} \\
&+ \delta(1-z)\,\delta_{a\hat{a}}\frac{(4\pi\,y_{\mathrm{s}})^\epsilon}{\Gamma(1-\epsilon)}\left\{\frac{1}{\epsilon^2}C_a + \frac{1}{\epsilon}\gamma_a + \frac{\pi^2}{6}C_a\right\} \\
&- \delta(1-z)\,\delta_{a\hat{a}}\,2C_a\int_0^1\frac{du}{1-u}\log\frac{(1-u)^{\beta+1}}{u\,y_{\mathrm{s}}\,r_\perp(u)}\,\theta\big((1-u)^{\beta+1} > uy_{\mathrm{s}}\,r_\perp(u)\big)
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After performing the y integral, we have

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$$\begin{aligned}
D_{\hat{a}a}(y_{\mathrm{s}}, z) = & -\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} P_{a\hat{a}}(z) \\
& + P^{(\epsilon)}_{a\hat{a}}(z) - P^{\mathrm{reg}}_{a\hat{a}}(z) \log\frac{(1-z)^{\beta-1}}{r_\perp(z)} - 2C_a\delta_{a\hat{a}}\left[\frac{1}{1-z}\log\frac{(1-z)^{\beta-1}}{r_\perp(z)}\right]_+ \\
& + 2C_a\delta_{a\hat{a}}\left[\theta\big((1-z)^{\beta+1} < z\, y_{\mathrm{s}}\, r_\perp(z)\big)\frac{1}{1-z}\log\frac{(1-z)^{\beta+1}}{z\, y_{\mathrm{s}}\, r_\perp(z)}\right]_+ \\
& + P^{\mathrm{reg}}_{a\hat{a}}(z)\,\theta\big((1-z)^{\beta+1} < z\, y_{\mathrm{s}}\, r_\perp(z)\big)\log\frac{(1-z)^{\beta+1}}{z\, y_{\mathrm{s}}\, r_\perp(z)}
\end{aligned}$$

Soft and collinear singularities and logs. They are cancelled by the 1-loop graphs.

$$\begin{aligned}
& + \delta(1-z)\,\delta_{a\hat{a}}\frac{(4\pi\, y_{\mathrm{s}})^\epsilon}{\Gamma(1-\epsilon)}\left\{\frac{1}{\epsilon^2}C_a + \frac{1}{\epsilon}\gamma_a + \frac{\pi^2}{6}C_a\right\} \\
& - \delta(1-z)\,\delta_{a\hat{a}}\, 2C_a\int_0^1 \frac{du}{1-u}\log\frac{(1-u)^{\beta+1}}{u\, y_{\mathrm{s}}\, r_\perp(u)}\,\theta\big((1-u)^{\beta+1} > u y_{\mathrm{s}}\, r_\perp(u)\big)
\end{aligned}$$

Inclusive Splitting Operator

After performing the y integral, we have

$$[\lambda_{\mathrm{a}k}(\mu^2;z)]_{\hat{a}a} = -\underbrace{\frac{1}{C_a} D_{a\hat{a}}(\mu^2/Q^2,z)}_{\text{Soft x Collinear + Collinear}} -\delta_{a\hat{a}}\overbrace{C(\xi_{\mathrm{a}k},\mu^2/Q^2,z)}^{\text{Wide angle soft}}$$

Threshold logs with the right behaviour. They disappear in the $\mu^2 \to 0$ limit, thus they are summed up in the shower evolution.

$$\begin{aligned}
D_{\hat{a}a}(y_{\mathrm{s}},z) = {} & -\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} P_{a\hat{a}}(z) \\
& + P^{(\epsilon)}_{a\hat{a}}(z) - P^{\mathrm{reg}}_{a\hat{a}}(z)\log\frac{(1-z)^{\beta-1}}{r_\perp(z)} - 2C_a\delta_{a\hat{a}}\left[\frac{1}{1-z}\log\frac{(1-z)^{\beta-1}}{r_\perp(z)}\right]_+ \\
& + 2C_a\delta_{a\hat{a}}\left[\theta\big((1-z)^{\beta+1} < z\,y_{\mathrm{s}}\,r_\perp(z)\big)\frac{1}{1-z}\log\frac{(1-z)^{\beta+1}}{z\,y_{\mathrm{s}}\,r_\perp(z)}\right]_+ \\
& + P^{\mathrm{reg}}_{a\hat{a}}(z)\,\theta\big((1-z)^{\beta+1} < z\,y_{\mathrm{s}}\,r_\perp(z)\big)\log\frac{(1-z)^{\beta+1}}{z\,y_{\mathrm{s}}\,r_\perp(z)} \\
& + \delta(1-z)\,\delta_{a\hat{a}}\frac{(4\pi\,y_{\mathrm{s}})^\epsilon}{\Gamma(1-\epsilon)}\left\{\frac{1}{\epsilon^2}C_a + \frac{1}{\epsilon}\gamma_a + \frac{\pi^2}{6}C_a\right\} \\
& - \delta(1-z)\,\delta_{a\hat{a}}\,2C_a\int_0^1 \frac{du}{1-u}\log\frac{(1-u)^{\beta+1}}{u\,y_{\mathrm{s}}\,r_\perp(u)}\,\theta\big((1-u)^{\beta+1} > uy_{\mathrm{s}}\,r_\perp(u)\big)
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After performing the y integral, we have

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Threshold logs, they DON'T disappear in the $\mu^2 \to 0$ limit.

$$\begin{aligned}
D_{\hat{a}a}(y_{\mathrm{s}}, z) = & -\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} P_{a\hat{a}}(z) \\
& \boxed{+ P^{(\epsilon)}_{a\hat{a}}(z) - P^{\mathrm{reg}}_{a\hat{a}}(z) \log\frac{(1-z)^{\beta-1}}{r_\perp(z)} - 2C_a\delta_{a\hat{a}}\left[\frac{1}{1-z}\log\frac{(1-z)^{\beta-1}}{r_\perp(z)}\right]_+} \\
& + 2C_a\delta_{a\hat{a}}\left[\theta\big((1-z)^{\beta+1} < z\, y_{\mathrm{s}}\, r_\perp(z)\big)\frac{1}{1-z}\log\frac{(1-z)^{\beta+1}}{z\, y_{\mathrm{s}}\, r_\perp(z)}\right]_+ \\
& + P^{\mathrm{reg}}_{a\hat{a}}(z)\, \theta\big((1-z)^{\beta+1} < z\, y_{\mathrm{s}}\, r_\perp(z)\big) \log\frac{(1-z)^{\beta+1}}{z\, y_{\mathrm{s}}\, r_\perp(z)} \\
& + \delta(1-z)\, \delta_{a\hat{a}} \frac{(4\pi\, y_{\mathrm{s}})^\epsilon}{\Gamma(1-\epsilon)}\left\{\frac{1}{\epsilon^2}C_a + \frac{1}{\epsilon}\gamma_a + \frac{\pi^2}{6}C_a\right\} \\
& - \delta(1-z)\, \delta_{a\hat{a}}\, 2C_a \int_0^1 \frac{du}{1-u}\log\frac{(1-u)^{\beta+1}}{u\, y_{\mathrm{s}}\, r_\perp(u)}\, \theta\big((1-u)^{\beta+1} > u y_{\mathrm{s}}\, r_\perp(u)\big)
\end{aligned}$$

PDF Factorisation Scheme

Some of the threshold logarithms has to be summed up by the PDF functions by choosing factorisation scheme appropriately. The first order kernel of the factorisation scheme is

$$[K^{(1)}_{\text{F.S.}}(z,\mu^2)]_{a\hat{a}} = -P^{(\epsilon)}_{a\hat{a}}(z) + P^{\text{reg}}_{a\hat{a}}(z)\log\frac{(1-z)^{\beta-1}}{r_\perp(z)} + 2C_a\delta_{a\hat{a}}\left[\frac{1}{1-z}\log\frac{(1-z)^{\beta-1}}{r_\perp(z)}\right]_+$$

$$r_\perp(z) = \max\left\{1, (1-z)^{\beta-1}\frac{m_\perp^2}{\mu^2}\right\}$$

✓ Transverse momentum ordered shower $\beta = 1$

$$[K^{(1)}_{\text{F.S.}}(z,\mu^2)]_{a\hat{a}} = -P_{a\hat{a}}(z)\log\big(\max\{1, m_\perp^2/\mu^2\}\big) - P^{(\epsilon)}_{a\hat{a}}(z)$$

- For $m_\perp^2 < \mu^2$ we don't have to change the factorisation scheme. MSbar works perfectly.
- For $m_\perp^2 > \mu^2$ the PDFs get frozen.

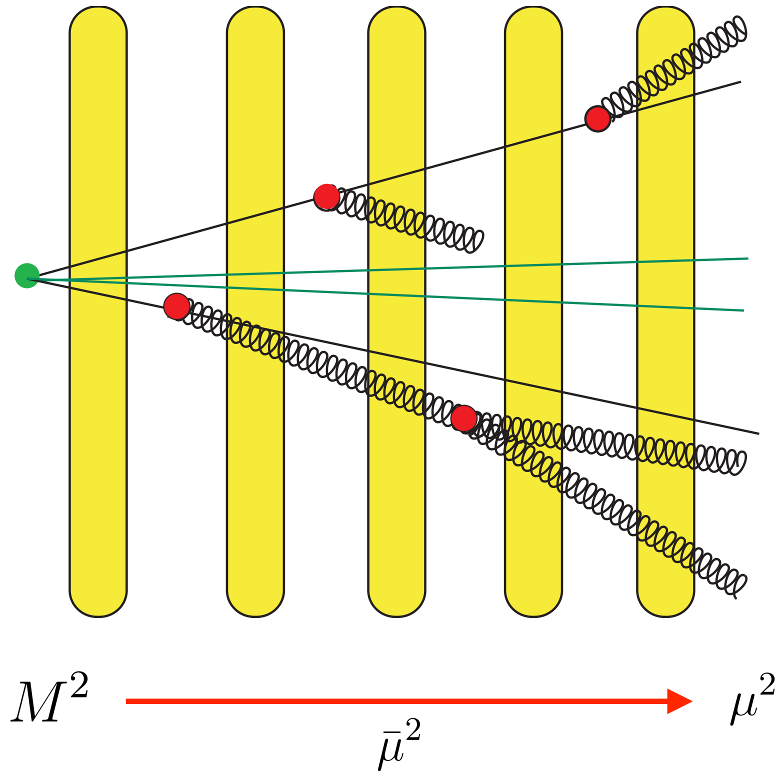
✓ For other orderings (virtuality and angular) $\beta < 1$

$$\begin{aligned}[K^{(1)}_{\text{F.S.}}(z,\mu^2)]_{a\hat{a}} = & -P^{(\epsilon)}_{a\hat{a}}(z) - P^{\text{reg}}_{a\hat{a}}(z)\log\left(\max\left\{(1-z)^{1-\beta}, \frac{m_\perp^2}{\mu^2}\right\}\right) \\ & - 2C_a\delta_{a\hat{a}}\left[\frac{1}{1-z}\log\left(\max\left\{(1-z)^{1-\beta}, \frac{m_\perp^2}{\mu^2}\right\}\right)\right]_+\end{aligned}$$

Shower Evolution

Now the shower evolution operator is

$$\exp\left\{\int_{\mu_2^2}^{\mu_1^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left(\underbrace{\mathcal{S}_{\rm uni}(\bar{\mu}^2)}_{\textit{Unitary part}}+\overbrace{\left[\mathcal{F}_{\rm F.S.}(\bar{\mu}^2)\circ\mathcal{S}_{\rm thr}(\bar{\mu}^2)\right]\mathcal{F}^{-1}_{\rm F.S.}(\bar{\mu}^2)}^{\textit{Threshold contribution}}\right)\right\}$$

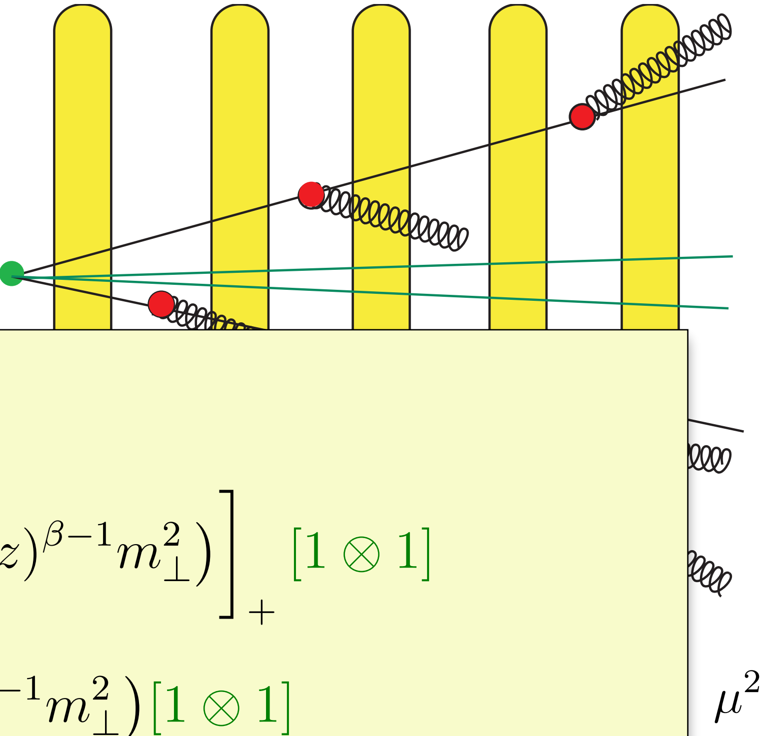


- ✓ This leads to a non-unitary shower.
- ✓ The threshold splitting operator doesn't change the number of the partons and their momenta. It operates in the colour and flavour space only.
- ✓ In LC+ approximation it leads to an extra factor that we have to insert after every step of the shower evolution.

$$\exp\left\{\int_{\mu_2^2}^{\mu_1^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\mathcal{F}_{\rm F.S.}(\mu^2)\circ\mathcal{S}^{\rm LC+}_{\rm thr}(\bar{\mu}^2)\right]\mathcal{F}^{-1}_{\rm F.S.}(\mu^2)\right\}$$

Shower Evolution

Now the shower evolution operator is



Threshold contribution

Direct term with diagonal color

$$
\begin{aligned}
\mathcal{S}_{\rm thr}(\mu^2) = &-\frac{\alpha_{\rm s}(\mu^2)}{2\pi}\left[\delta_{a\hat{a}}\frac{2Ca}{1-z}\,\theta\big((1-z)^{\beta+1} < z\,y_{\rm s}\,r_\perp(z)\big)\,\theta\big(\mu^2 > (1-z)^{\beta-1}m_\perp^2\big)\right]_+ [1\otimes 1]\\
&-\frac{\alpha_{\rm s}(\mu^2)}{2\pi}P^{\rm reg}_{a\hat{a}}(z)\,\theta\big((1-z)^{\beta+1} < z\,y_{\rm s}\,r_\perp(z)\big)\,\theta\big(\mu^2 > (1-z)^{\beta-1}m_\perp^2\big)[1\otimes 1]\\
&-\frac{\alpha_{\rm s}(\mu^2)}{2\pi}\sum_{k\neq {\rm a}}\left[w\left(\xi_{{\rm a}k},\frac{y_{\rm s}r_\perp(z)}{(1-z)^{\beta+1}}\right)\theta\big((1-z)^{\beta+1} > z\,y_{\rm s}\,r_\perp(z)\big)\,\theta\big(\mu^2 > (1-z)^{\beta-1}m_\perp^2\big)\right]_+\\
&\qquad\times\frac{1}{2}\left[(\boldsymbol{T}_{\rm a}\cdot\boldsymbol{T}_k)\otimes 1 + 1\otimes(\boldsymbol{T}_{\rm a}\cdot\boldsymbol{T}_k)\right] + \cdots
\end{aligned}
$$

Colour interference part of the threshold logs

$$
\exp\left\{\int_{\mu_2^2}^{\mu_1^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\mathcal{F}_{\rm F.S.}(\mu^2)\circ\mathcal{S}^{\rm LC+}_{\rm thr}(\bar{\mu}^2)\right]\mathcal{F}^{-1}_{\rm F.S.}(\mu^2)\right\}
$$

Conclusions

✓ We have defined parton shower.

- ➠ We defined parton shower based on pQCD and factorisation of QCD density matrices. The aim is the gain as much control as possible on the approximations (like unitarity condition)…
- ➠ We still need the all order proof of the factorisation of the physical states. We want a constructive proof. Splitting operators (with many loops), momentum mapping, shower scale definition, …
- ➠ At higher order it is not possible to turn every subtraction scheme to parton shower.

✓ It works at NLO level.

- ➠ We recovered what is called "Standard Shower".
- ➠ We obtained threshold resummation basically for free. *Shower is not unitary!*
- ➠ If you want unitary shower, you need process dependent PDFs.
- ➠ Some threshold logs get resummed in the PDFs. MSbar PDFs only for transverse momentum ordered showers. In other shower schemes the PDF factorisation scheme has to be adjusted.
- ➠ There is a plan to implement the new factorisation schemes in HERAFITTER.

✗ I didn't discuss in the talk.

- ➠ We obtained NLO matching for free, it is just part of the scheme.
- ➠ Genuine loop effects like $\mathrm{i}\pi/\epsilon$ terms.
- ➠ Final state heavy flavour threshold logs

Where is the Code?

- DEDUCTOR is designed to do a better job with color, spin and resummation of large logarithms compared to other shower generators.
    - Lambda ordering with and without initial state massive quarks
    - LC+ color treatment. It allows us to do color evolution at amplitude level
    - Spin correlations are not yet computed
- Next version is available soon…
    - The shower equation is implemented at very abstract level. It allows us to use other ordering variables like kT or angle (massless or massive initial state partons).
    - Initial state threshold log resummation.
    - Subleading (wide angle subleading colour, Coulomb gluon,…) contribution perturbative.
- It is available at

`http://www.desy.de/~znagy/deductor`
`http://pages.uoregon.edu/soper/deductor`

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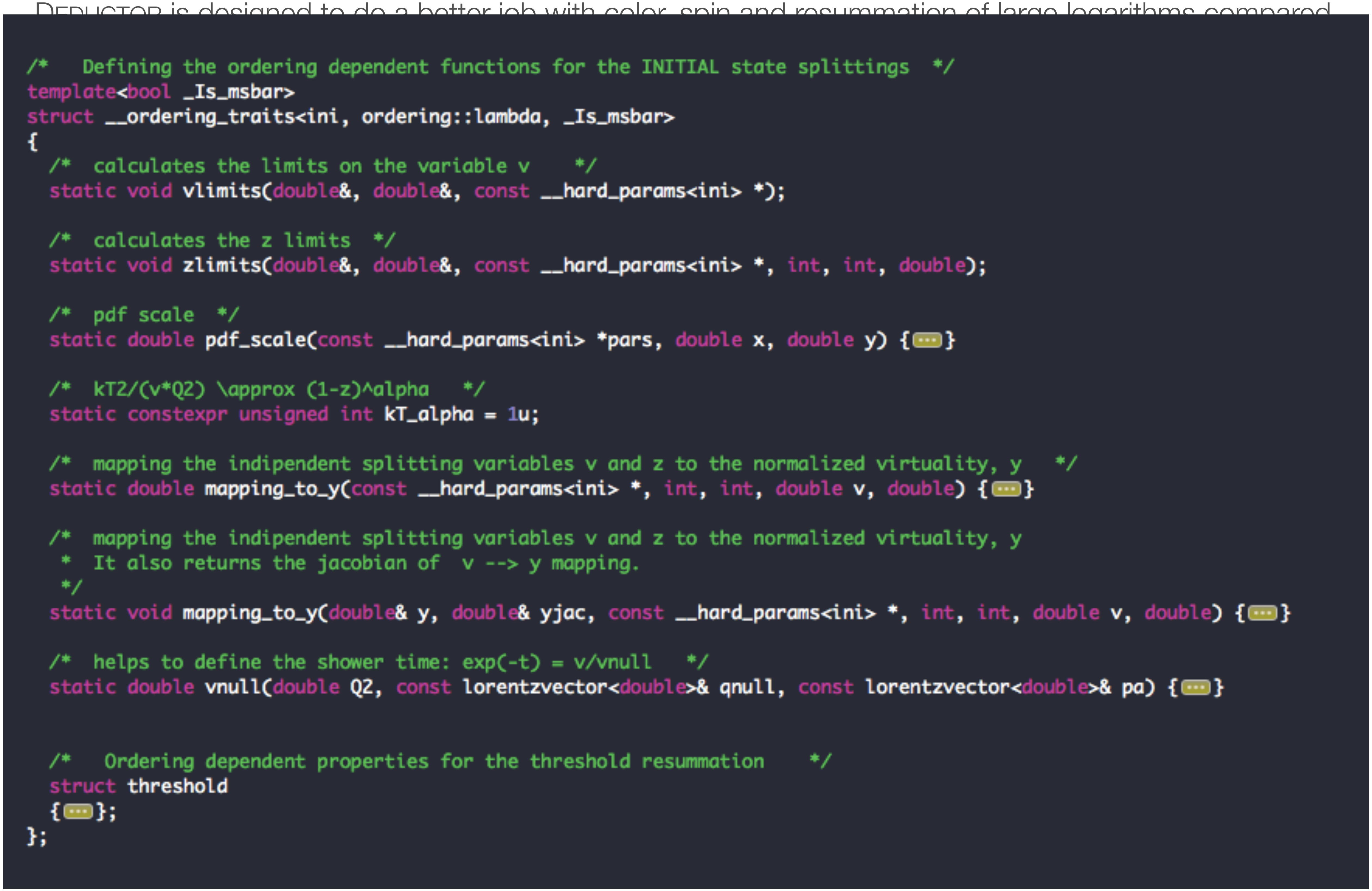
- DEDUCTOR is designed to do a better job with color, spin and resummation of large logarithms compared



```
/*   Defining the ordering dependent functions for the INITIAL state splittings  */
template<bool _Is_msbar>
struct __ordering_traits<ini, ordering::lambda, _Is_msbar>
{
  /*  calculates the limits on the variable v    */
  static void vlimits(double&, double&, const __hard_params<ini> *);

  /*  calculates the z limits  */
  static void zlimits(double&, double&, const __hard_params<ini> *, int, int, double);

  /*  pdf scale  */
  static double pdf_scale(const __hard_params<ini> *pars, double x, double y) {⋯}

  /*  kT2/(v*Q2) \approx (1-z)^alpha   */
  static constexpr unsigned int kT_alpha = 1u;

  /*  mapping the indipendent splitting variables v and z to the normalized virtuality, y   */
  static double mapping_to_y(const __hard_params<ini> *, int, int, double v, double) {⋯}

  /*  mapping the indipendent splitting variables v and z to the normalized virtuality, y
   *  It also returns the jacobian of  v --> y mapping.
   */
  static void mapping_to_y(double& y, double& yjac, const __hard_params<ini> *, int, int, double v, double) {⋯}

  /*  helps to define the shower time: exp(-t) = v/vnull   */
  static double vnull(double Q2, const lorentzvector<double>& qnull, const lorentzvector<double>& pa) {⋯}


  /*   Ordering dependent properties for the threshold resummation    */
  struct threshold
  {⋯};
};
```

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