

Automated Calculation of Dijet Soft Functions in Soft Collinear Effective Theory

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Outline

1. *A very brief introduction to SCET*

- (a) SCET_I and SCET_{II}
- (b) SCET dijet factorisation
- (c) Renormalisation group resummation (for thrust)

2. *Universal soft functions at NLO*

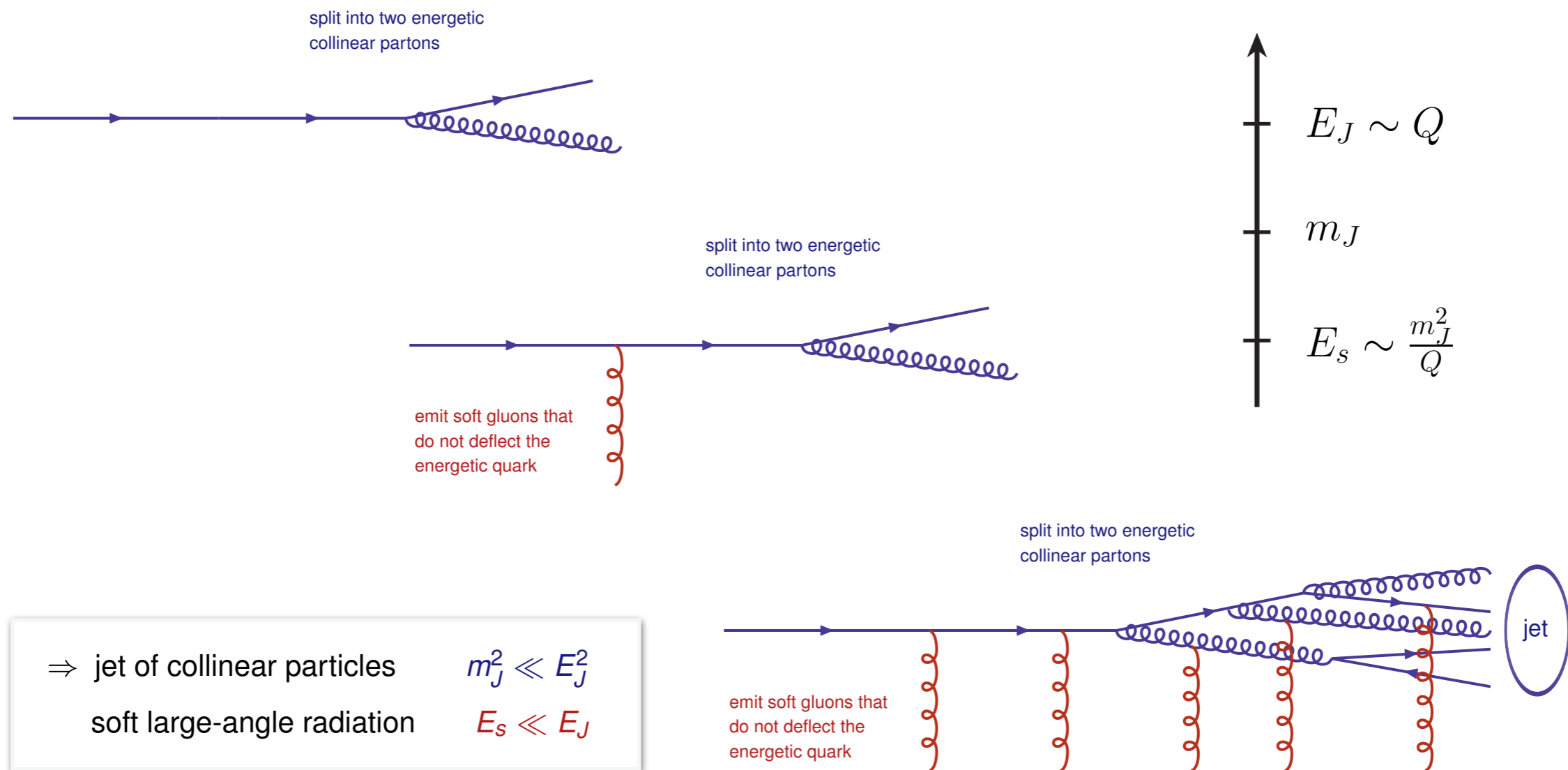
- (d) Divergence structures, measurement functions, and Laplace space
- (e) naive subtraction

3. *Soft automation to NNLO*

- (f) NLO vs. NNLO: the need for automation
- (g) Sector Decomposition, parameterising phase space, and *SecDec*
- (h) Analytic regulator at NNLO
- (i) Results

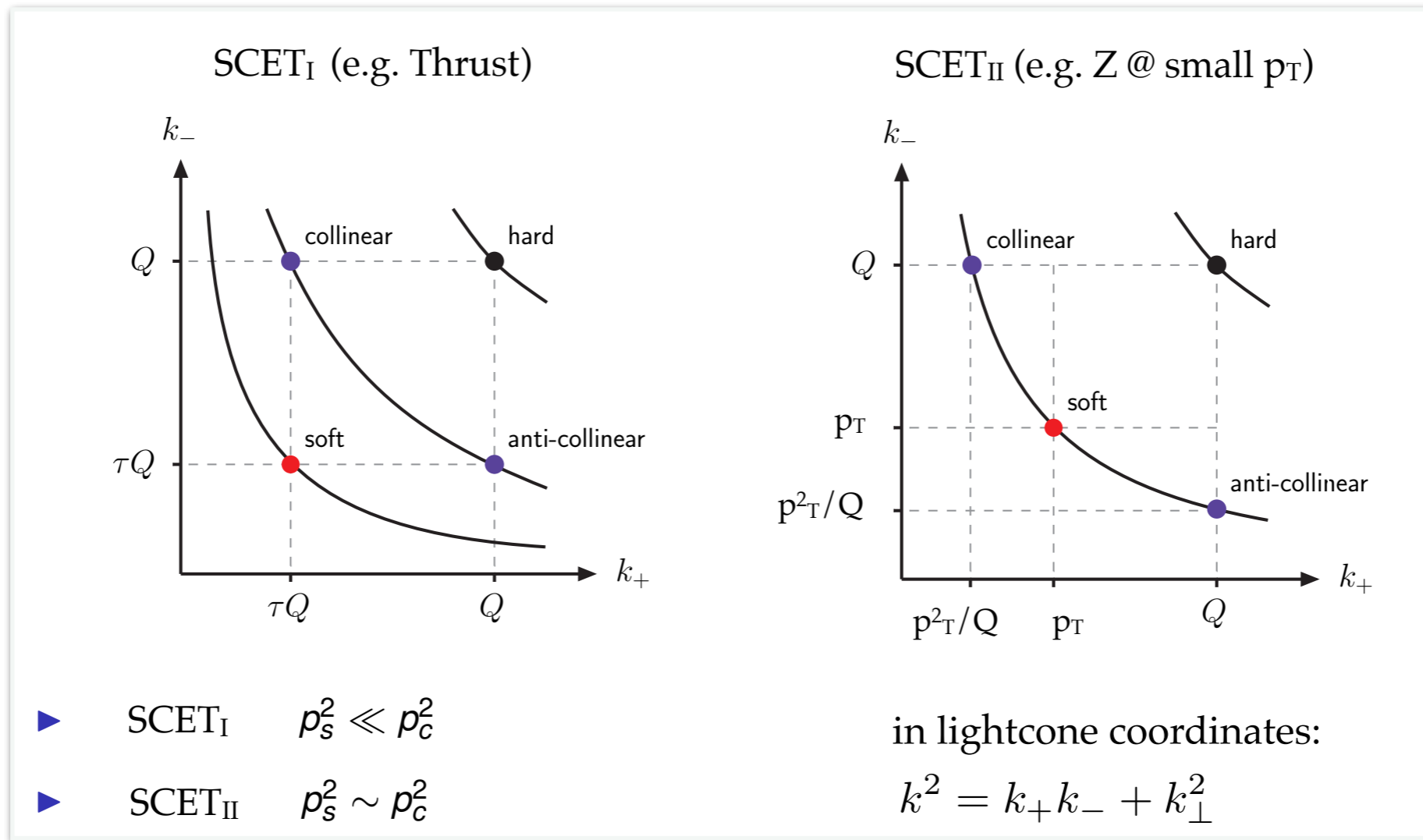
Introducing SCET: intuition

- SCET is an effective theory whose degrees of freedom are soft and collinear partons



Introducing SCET: SCET_I vs. SCET_{II}

- There are two types of SCET, depending on the relative scaling of soft and collinear modes:



- In SCET_{II} scaling alone does not suffice to distinguish—introduce rapidity regulator:

$$\mathcal{R}_\alpha(\nu, k_i) = \prod_{i=1}^n \left(\frac{\nu}{k_i^+} \right)^{\frac{\alpha}{n}}$$

Factorisation in SCET

$$\text{thrust } T = \max_n \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|}$$

- The SCET Lagrangian contains distinct fields for soft and collinear modes
- Field redefinition allows decoupling \rightarrow all order factorisation

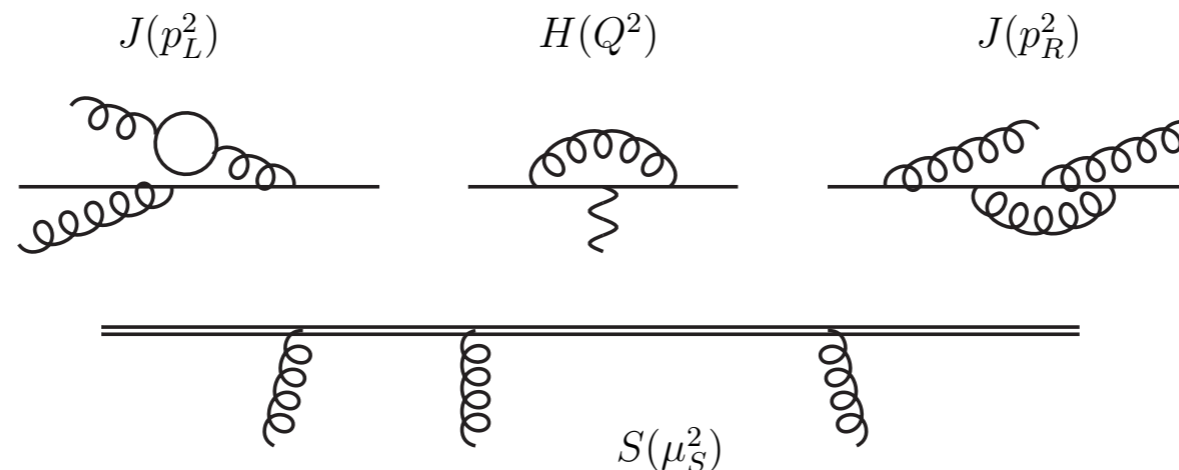
$$|C_V|^2 \sum_X |\langle 0 | \mathcal{O}_{n\bar{n}} | X \rangle|^2$$

$$= |C_V|^2 \langle 0 | [\bar{\zeta}_{\bar{n}}^0 W_{\bar{n}}^{0,\dagger}] [\bar{\zeta}_{\bar{n}}^0 W_{\bar{n}}^{0,\dagger}]^\dagger | 0 \rangle \langle 0 | [W_n^0 \zeta_n^0] [W_n^0 \zeta_n^0]^\dagger | 0 \rangle \langle 0 | [S_{\bar{n}}^\dagger S_n] [S_{\bar{n}}^\dagger S_n]^\dagger | 0 \rangle$$



$$\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} = H(Q^2, \mu) \int dp_L^2 \int dp_R^2 J(p_L^2, \mu) J(p_R^2, \mu) S(\tau Q - \frac{p_L^2 + p_R^2}{Q}, \mu)$$

(for dijet thrust)
($\tau = 1 - T$)



Thrust w/ SCET: resummation

- H , J^i , and S contain logs of the form (respectively):

$$\ln \frac{\mu^2}{Q^2}, \quad \ln \frac{\mu^2}{\tau Q^2}, \quad \ln \frac{\mu^2}{\tau^2 Q^2}$$

- We evaluated H , J^i , and S at a common scale. Yet there are ‘natural’ scales at which the logarithms are no longer large:

$$\mu_h \sim Q \quad \mu_j \sim Q\sqrt{\tau} \quad \mu_s \sim Q\tau$$

- We thus wish to RG run our functions up to their natural scales. Take H as a simple example:

$$H(Q^2, \mu) = H(Q^2, \mu_h) U_h(\mu_h, \mu)$$

- Where the function U is a solution to the RG equation for the hard function:

$$\frac{dH(Q^2, \mu)}{d \ln \mu} = \left[2\Gamma_{cusp} \ln \left(\frac{Q^2}{\mu^2} \right) + 4\gamma_H(\alpha_s) \right] H(Q^2, \mu)$$

- Which, at LL approximation, has the following form:

$$H(Q^2, \mu) = \exp \left[\frac{4\pi\Gamma_0}{\beta_0^2} \frac{1}{\alpha_s(Q)} \left(1 - \frac{1}{r} - \ln r \right) \right] = 1 - \frac{\Gamma_0}{2} \frac{\alpha_s(Q)}{4\pi} \ln^2 \left(\frac{Q^2}{\mu^2} \right) + \mathcal{O}(\alpha_s^2), \quad r = \frac{\alpha_s(\mu)}{\alpha_s(Q)}$$

- Similar (in Laplace space) for jet and soft functions...

Resummation: Technicalities

Logarithmic Accuracy	Γ_{Cusp}	$\gamma_H, \gamma_J, \gamma_S$	C_H, C_J, C_S
LL	1-loop	tree	tree
NLL	2-loop	1-loop	tree
NNLL	3-loop	2-loop	1-loop
N ³ LL	4-loop	3-loop	2-loop

- **SCET Observables @ NNLL:** broadening, Z/W/H @ small p_T , jet-veto ...
- **SCET Observables @ N³LL:** thrust, C-parameter, Z/W/H @ large p_T , ...
- Automated code for QCD resummation @NLL: *CAESAR* (Banfi, Salam, Zanderighi)
- Recently extended to NNLL (Banfi, McAslan, Monni, Zanderighi)
- We want to automate soft functions in SCET...

Universal dijet soft functions

- We can write down a universal dijet soft function as the vacuum matrix element of a product of Wilson lines along the direction of energetic quarks.

$$S(\omega, \mu) = \sum_{X, reg.} \mathcal{M}(\omega, \{k_i\}) |\langle X | S_n^\dagger(0) S_{\bar{n}}(0) | 0 \rangle|^2 \quad S_n(x) = P \exp(i g_s \int_{-\infty}^0 n \cdot A_s(x + sn) ds)$$

- The **matrix element** of soft wilson lines is *independent of the observable*. It contains the universal (implicit) UV/IR-divergences of the function.
- The **measurement function** (M) encodes all of the information of the particular observable at hand. It is *independent of the singularity structure*. Take thrust as an example:

$$\mathcal{M}_{thrust}(\omega, \{k_i\}) = \delta(\omega - \sum_{i \in L} k_i^+ - \sum_{i \in R} k_i^-)$$

- Idea:** isolate singularities at each order and calculate the associated coefficient numerically:

$$\bar{\mathcal{S}}(\tau) \sim 1 + \alpha_s \left\{ \frac{c_2}{\epsilon^2} + \frac{c_1}{\epsilon^1} + c_0 \right\} + \mathcal{O}(\alpha_s^2)$$

- The coefficients depend on the observable, we typically work in **Laplace space**, to avoid distributions.

Universal soft functions: NLO

- At 1-loop the virtual corrections are scaleless in DR and we can write the NLO soft function as:

$$\bar{S}^{(1)}(\tau, \mu) = \frac{\mu^{2\epsilon}}{(2\pi)^{d-1}} \int \delta(k^2) \theta(k^0) \mathcal{R}_\alpha(\nu; k_+, k_-) \left(\frac{16\pi\alpha_s C_F}{k_+ k_-} \right) \bar{\mathcal{M}}(\tau, k) d^d k$$

- Where we use a symmetric version of the analytic SCET_{II} regulator (*Becher, Bell / 1112.3907*):

$$\mathcal{R}_\alpha(\nu, k_+, k_-) = \theta(k_- - k_+) (\nu/k_-)^\alpha + \theta(k_+ - k_-) (\nu/k_+)^\alpha$$

- We want to disentangle all of the UV and IR divergences. We thus split the integration region into two hemispheres and make the following physical substitutions:

$$k_- \rightarrow \frac{k_T}{\sqrt{y}} \quad k_+ \rightarrow k_T \sqrt{y}$$

- We can now specify the measurement function M . We assume it can be written in terms of two dimensionless functions f & g :

$$\bar{\mathcal{M}}(\tau, k) = g(\tau k_T, y, \theta) \exp(-\tau k_T f(y, \theta))$$

Universal soft functions: examples

$$\bar{\mathcal{M}}(\tau, k) = g(\tau k_T, y, \theta) \exp(-\tau k_T f(y, \theta))$$

Obs.	$g(\tau k_T, y, \theta)$	$f(y, \theta)$
Thrust	1	\sqrt{y}
Angularities	1	$y^{(1-A)/2}$
C-Parameter	1	$\sqrt{y}/(1+y)$
Broadening	$\Gamma(1-\epsilon) \left(\frac{z\tau k_T}{4}\right)^\epsilon \mathcal{J}_{-\epsilon}\left(\frac{z\tau k_T}{2}\right)$	1/2
W/H @ large p_T	1	$\frac{1+y-2\sqrt{y}\cos\theta}{\sqrt{y}}$
Transverse Thrust	1	$\frac{1}{ s } \left\{ \sqrt{1 + \frac{1}{4} \left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)^2 s^2 + \left(\frac{1}{\sqrt{y}} - \sqrt{y}\right) cs \cos\theta - s^2 \cos^2\theta} - c \cos\theta + \frac{1}{2} \left(\frac{1}{\sqrt{y}} - \sqrt{y}\right) s \right\}$

Universal soft functions: NLO master formula

- We switch to a dimensionless variable (x) and extract the scaling of the observables in the collinear limit $y \Rightarrow 0$:

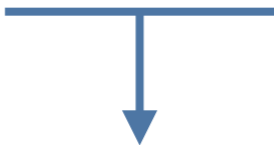
$$\tau k_T f(y, \theta) \rightarrow x \quad f(y, \theta) \rightarrow y^{\frac{n}{2}} \hat{f}(y, \theta)$$

- We are now in a position to write a master formula for the calculation of NLO dijet soft functions:

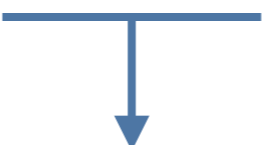
$$\bar{S}^{(1)}(\tau, \mu) \sim \int_{-1}^1 \sin^{-1-2\epsilon} \theta \, d \cos \theta \int_0^\infty dx \int_0^1 dy \, x^{-1-2\epsilon-\alpha} y^{-1+n\epsilon+(n-1)\alpha/2} \hat{g}(x, y, \theta) [\hat{f}(y, \theta)]^{2\epsilon+\alpha} e^{-x}$$

- Note that $n=0$ corresponds to a SCET_{II} observable.
- We are in a position to apply a subtraction technique to extract the singularities. Consider a simple 1-D example:

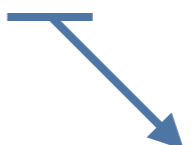
$$\int_0^1 dx \, x^{-1-n\epsilon} f(x) = \int_0^1 dx \, x^{-1-n\epsilon} \{f(x) - f(0) + f(0)\}$$



divergent

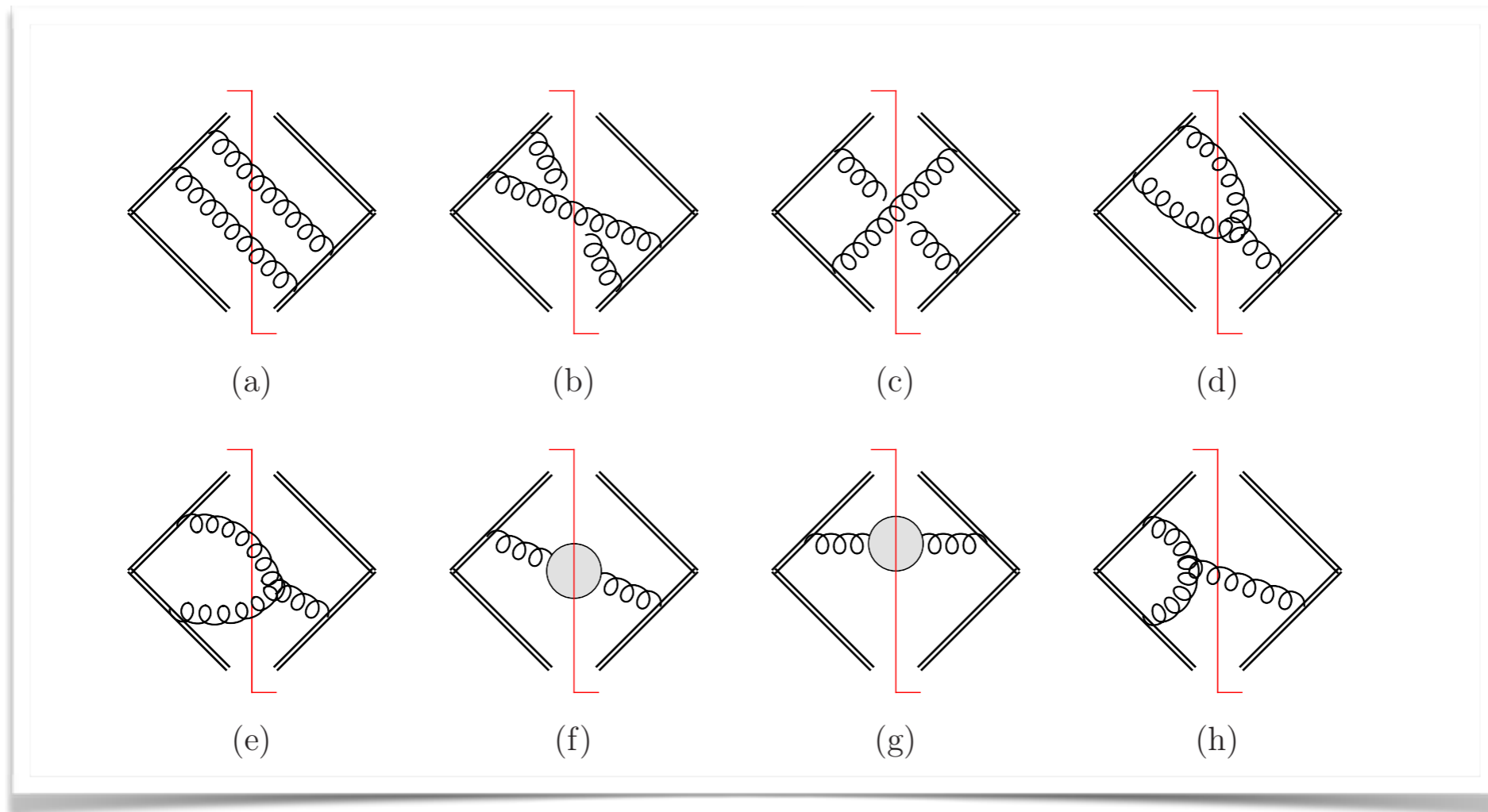


- finite / $O(x)$
- expand in ϵ
- integrate numerically


 $\sim -\frac{1}{n\epsilon}$

- singularity isolated

NNLO diagrams



- Three color structures are present: C_F^2 , $C_F C_A$, $C_F T_F n_f$
- We use analytic results for the C_F^2 terms (for now).

**** All results presented are preliminary! ****

Automation: NLO vs. NNLO

- Consider the double real emission (and drop additional regulator):

$$\bar{S}_{RR}^2(\tau) = \frac{\mu^{4\epsilon}}{(2\pi)^{2d-2}} \int d^d k \delta(k^2) \theta(k^0) \int d^d l \delta(l^2) \theta(l^0) |\mathcal{A}(k, l)|^2 \bar{\mathcal{M}}(\tau, k, l)$$

- Decompose into light-cone coordinates and perform trivial integrations:

$$\begin{aligned} \bar{S}_{RR}^{(2)}(\tau) &\sim \Omega_{d-3} \Omega_{d-4} \int_0^\infty dk_+ \int_0^\infty dk_- \int_0^\infty dl_+ \int_0^\infty dl_- \int_{-1}^1 d \cos \theta_k \sin^{d-5} \theta_k \\ &\times \int_{-1}^1 d \cos \theta_l \sin^{d-5} \theta_l \int_{-1}^1 d \cos \theta_1 \sin^{d-6} \theta_1 (k_+ k_- l_+ l_-)^{-\epsilon} |\mathcal{A}(k, l)|^2 \bar{\mathcal{M}}(\tau, k, l) \end{aligned}$$

- Consider, e.g., the $C_F T_{Ff}$ color structure:

$$|\mathcal{A}(k, l)|^2 = 128\pi^2 \alpha_s^2 C_F T_{Ff} n_f \frac{2k \cdot l (k_- + l_-)(k_+ + l_+) - (k_- l_+ - k_+ l_-)^2}{(k_- + l_-)^2 (k_+ + l_+)^2 (2k \cdot l)^2}$$

- It is clear the singularity structure is non-trivial, and that the singularities are overlapping...

Automation: sector decomposition

- Consider a simple integral over a unit hypercube with 'overlapping singularities' (singular as x, y simultaneously tend to 0):

$$I = \int_0^1 dx \int_0^1 dy (x + y)^{-2+\epsilon}$$

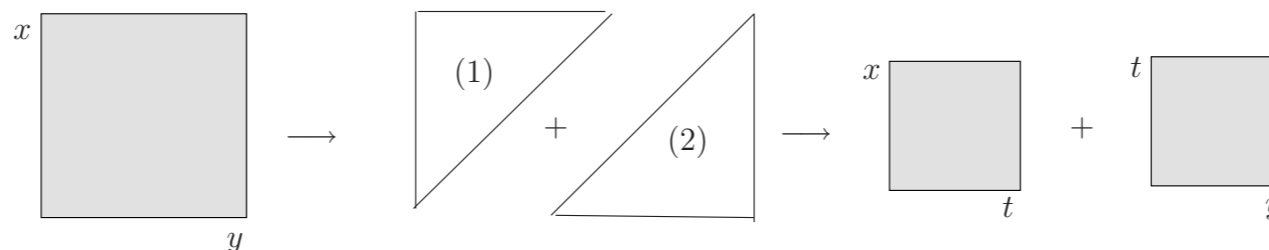
- We want to factorise such singularities. Split the hypercube with two sectors ($x > y$) and ($y > x$):

$$I = I_1 + I_2 = \int_0^1 dx \int_0^x dy (x + y)^{-2+\epsilon} + \int_0^1 dy \int_0^y dx (x + y)^{-2+\epsilon}$$

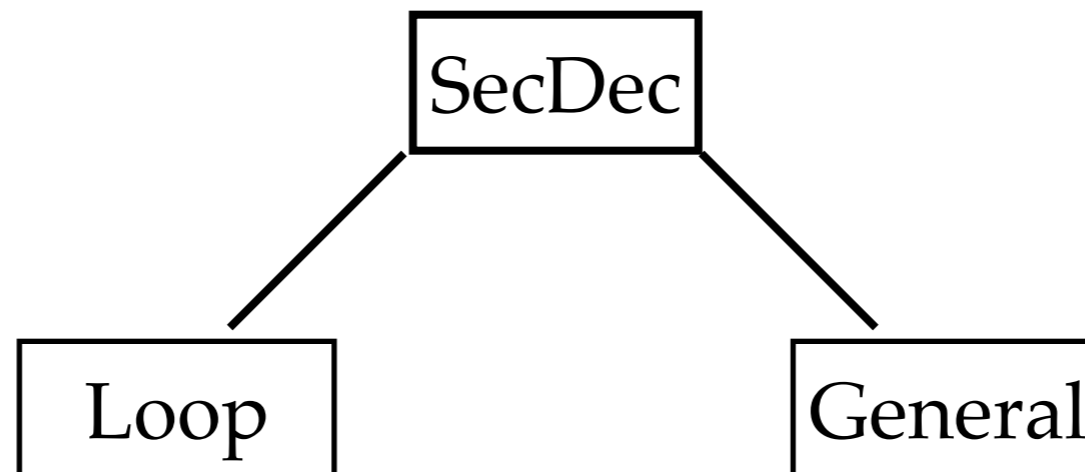
- Now substitute $y = xt$ in first sector and $x = yt$ in second:

$$I_1 = \int_0^1 dx \int_0^1 dt x^{-1+\epsilon} (1+t)^{-2+\epsilon}$$

$$I_2 = \int_0^1 dy \int_0^1 dt y^{-1+\epsilon} (1+t)^{-2+\epsilon}$$



- A tool is already on the market that exploits the sector decomposition algorithm: *SecDec*
- “A program to evaluate dimensionally regularised parameter integrals numerically”



- We use *SecDec*'s 'general' mode, as it allows the definition of *dummy functions*.
- *SecDec* provides: Simple interface to our NLO and NNLO master formulas (✓), numerical code output (✓), multiple numerical integrators for crosschecks (✓)
- Currently limited to SCET_I observables, though additional rapidity regulator in development.

Automation: NNLO parameterisation

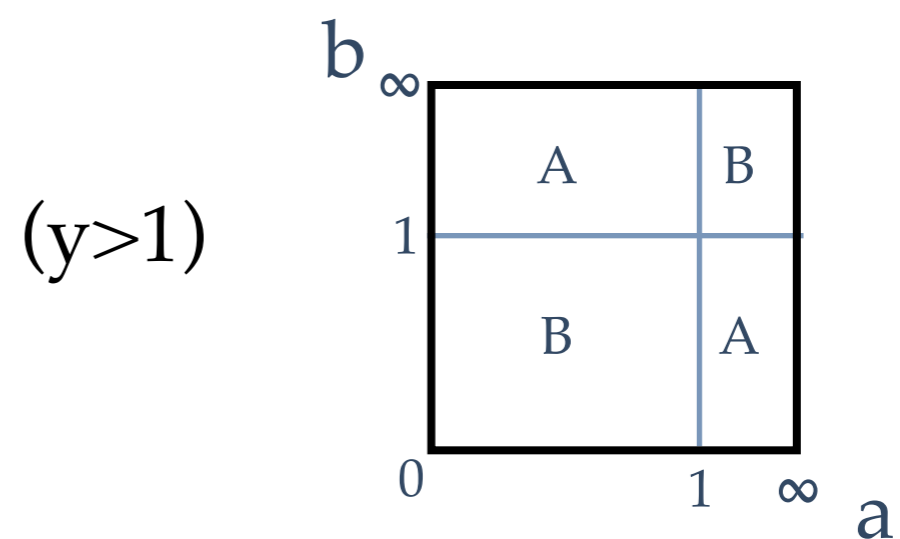
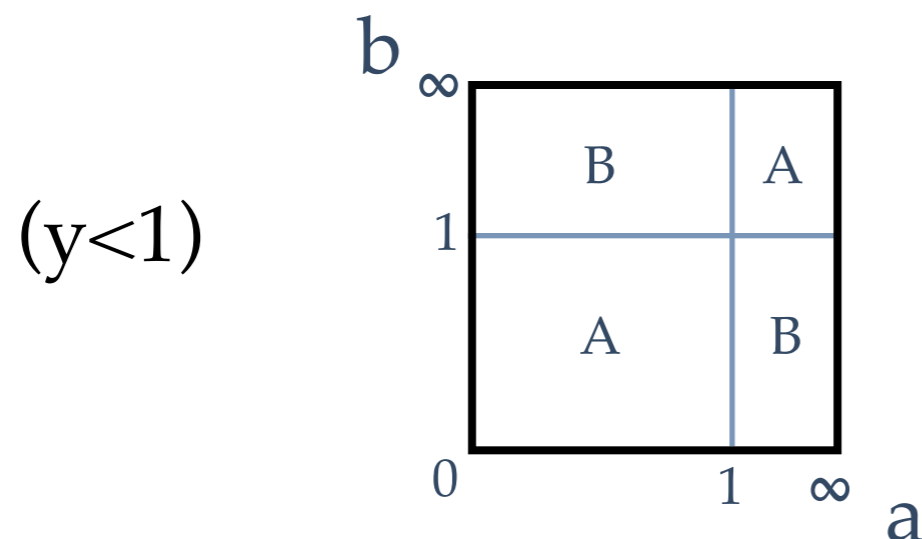
- We thus need to find an appropriate phase space parameterisation that exposes the divergence structure and is amenable to sector decomposition (*SecDec*):

$$\begin{aligned}
 p_- &= k_- + l_- & a &= \sqrt{\frac{k_- l_+}{k_+ l_-}} = e^{-(\eta_k - \eta_l)} \\
 p_+ &= k_+ + l_+ & b &= \sqrt{\frac{k_- k_+}{l_- l_+}} = \frac{k_T}{l_T}
 \end{aligned}$$

- We further write the total momentum components in terms of p_T and y (as in NLO case):

$$p_- \rightarrow \frac{p_T}{\sqrt{y}} \quad p_+ \rightarrow p_T \sqrt{y}$$

- Finally, we map onto the unit hyper-cube:



Automation: NNLO master formula

- We again assume an exponentiated form for the (Laplace space) measurement function:

$$\tilde{\mathcal{M}}(\tau, k, l) = \exp\{-\tau p_T F(a, b, y, t_i(\theta_i))\}$$

- We again factorise the y -dependence in $F(a, b, y)$:

$$F(a, b, y, t_i) = y^{\frac{N}{2}} \hat{F}(a, b, y, t_i)$$

- We arrive at an NNLO master formula (here shown for $C_F T_{FN_f}$ amplitude):

$$\begin{aligned} \bar{\mathcal{S}}_{RR}^{(2)}(\tau) &\propto \tau^{4\epsilon} \int_0^1 da \int_0^1 db \int_0^1 dy \int_0^1 dt y^{-1+2N\epsilon} a^{2-2\epsilon} b^{-2\epsilon} (a+b)^{-2+2\epsilon} (1+ab)^{-2+2\epsilon} (4t\bar{t})^{-\frac{1}{2}-\epsilon} \\ &\times \int_0^1 dt_1 \int_0^1 dt_2 2^{-1-4\epsilon} \frac{\epsilon}{\pi} (t_1\bar{t}_1)^{-\frac{1}{2}-\epsilon} (t_2\bar{t}_2)^{-1-\epsilon} \\ &\times \frac{(1-a)^2(1-b)^2 + 4t(a+b)(1+ab)}{[(1-a)^2 + 4at]^2} \left\{ \left[\hat{F}(a, b, y, t_i) \right]^{4\epsilon} + \left[\hat{F}(1/a, b, y, t_i) \right]^{4\epsilon} \right\} \end{aligned}$$

$$[\bar{t}_{(i)} = (1 - t_{(i)})]$$

NNLO Measurement functions: examples

Observable	$F(y, a, b, t_i(\theta_i))$
Inclusive Drell-Yan	$\frac{1}{\sqrt{y}} + \sqrt{y}$
C-Parameter	$a\sqrt{y} \left(\frac{b}{a(a+b) + y(1+ab)} + \frac{1}{a+b + ay(1+ab)} \right)$
Thrust	$\sqrt{y} + \Theta \left(y - a \frac{1+ab}{a+b} \right) \left(\frac{a}{\sqrt{y}(a+b)} - \frac{\sqrt{y}}{1+ab} \right)$
Angularities	$\sqrt{y}^{1-A} \left[b \left(\frac{a}{1+ab} \right)^{1-\frac{A}{2}} \left(\frac{1}{a+b} \right)^{\frac{A}{2}} + \left(\frac{1}{1+ab} \right)^{1-\frac{A}{2}} \left(\frac{a}{a+b} \right)^{\frac{A}{2}} \right. \\ \left. + \Theta \left(y - a \frac{1+ab}{a+b} \right) \left(y^{A-1} \left(\frac{a}{a+b} \right)^{1-\frac{A}{2}} \left(\frac{1}{1+ab} \right)^{\frac{A}{2}} - \left(\frac{1}{1+ab} \right)^{1-\frac{A}{2}} \left(\frac{a}{a+b} \right)^{\frac{A}{2}} \right) \right]$
W/H at high p_T	$\frac{1}{\sqrt{y}} + \sqrt{y} - 2\sqrt{\frac{a}{(a+b)(1+ab)}} \left[1 - 2t_2 + b \left(1 - 2t + t_2 - 2tt_2 - 2(1-2t_1)\sqrt{tt_2t_2} \right) \right]$
Recoil-free broadening	$\sqrt{\frac{a}{(a+b)(1+ab)}}(1+b)$

Results: *Thrust*

$$\mathcal{M}_{thrust}(\omega, \{k_i\}) = \delta(\omega - \sum_{i \in L} k_i^+ - \sum_{i \in R} k_i^-)$$

- We use *SecDec* to calculate the double emission contribution. To obtain the renormalized soft function we have to add the counterterms, which are known analytically at the required order.
- We show the cancellation of the divergences for thrust, setting $\ln(\mu\bar{\tau}) \rightarrow 0$

$$\begin{aligned} \tilde{S}_{ren}^{(2)} = & \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left\{ C_A C_F \left(\frac{0}{\epsilon^4} - \frac{5.07333 \times 10^{-9}}{\epsilon^3} + \frac{1.07523 \times 10^{-6}}{\epsilon^2} + \frac{.0000102661}{\epsilon} \right) \right. \\ & \left. + C_F T_F n_f \left(-\frac{1.40667 \times 10^{-8}}{\epsilon^3} + \frac{6.83778 \times 10^{-8}}{\epsilon^2} - \frac{1.44697 \times 10^{-8}}{\epsilon} \right) \right\} + \tilde{S}_0^{(2)} \end{aligned}$$

- We thus also have an indication of our numerical precision...
- For the finite portion, we find (setting again $\ln(\mu\bar{\tau}) \rightarrow 0$):

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2} (48.7045 C_F^2 - 56.4992 C_A C_F + 43.3902 C_F T_F n_f)$$

- Versus the analytic expression calculated by *Kelley, Schabinger, Schwartz, Zhu* / 1105.3676 (see also *Monni, Gehrmann, Luisoni* / 1105.4560):

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2} (48.7045 C_F^2 - 56.4990 C_A C_F + 43.3905 C_F T_F n_f)$$

Results: *C*-parameter

- *C*-parameter measurement function:

$$\mathcal{M}_C(\omega, \{k_i\}) = \delta(\omega - \sum_i \frac{k_+^i k_-^i}{k_+^i + k_-^i})$$

- For *C*-parameter, we obtain:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2} (5.41162 C_F^2 - 57.9754 C_A C_F + 43.8179 C_F T_F n_f)$$

- Where *Hoang, Kolodrubetz, Mateu, Stewart* / 1411.6633 extracted (using EVENT2) the following:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2} ((5.41162) C_F^2 - (58.16 \pm .26) C_F C_A + (43.74 \pm .06) C_F T_F n_f)$$

- We find similar numerical precision in the subtractions.

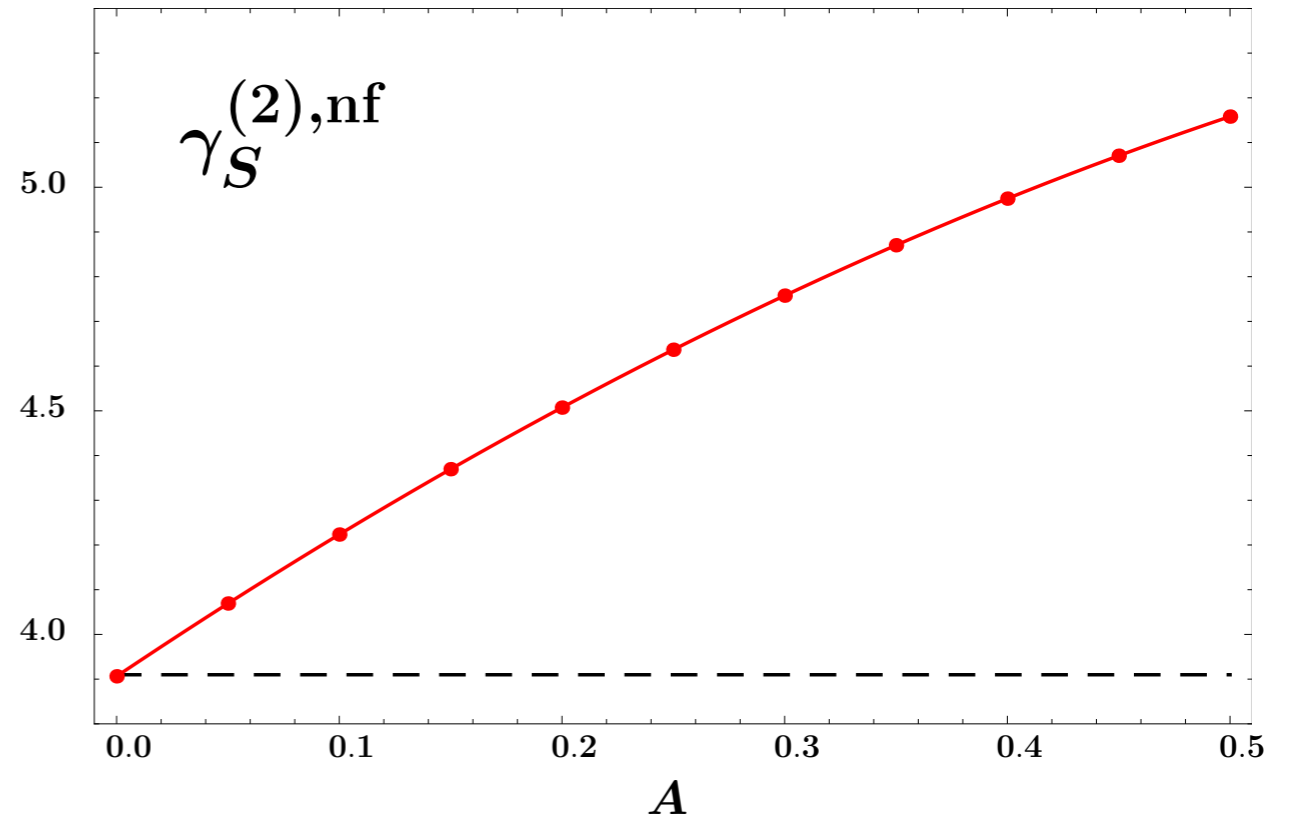
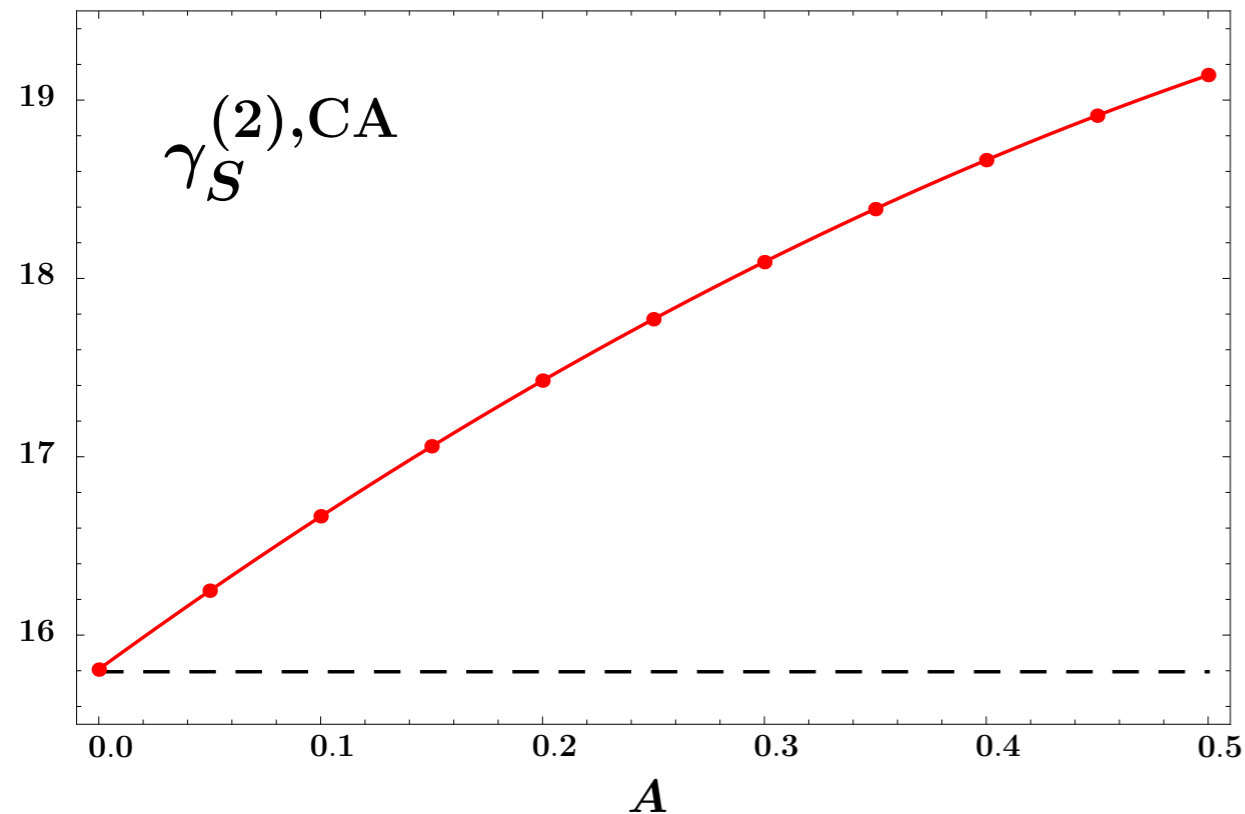
Results: *Angularities*

- *Angularities* measurement function:

$$\mathcal{M}_{Ang}(\omega, \{k_i\}) = \delta\left(\omega - \sum_{i \in L} k_{i,+}^{1-A/2} k_{i,-}^{A/2} - \sum_{i \in R} k_{i,+}^{A/2} k_{i,-}^{1-A/2}\right)$$

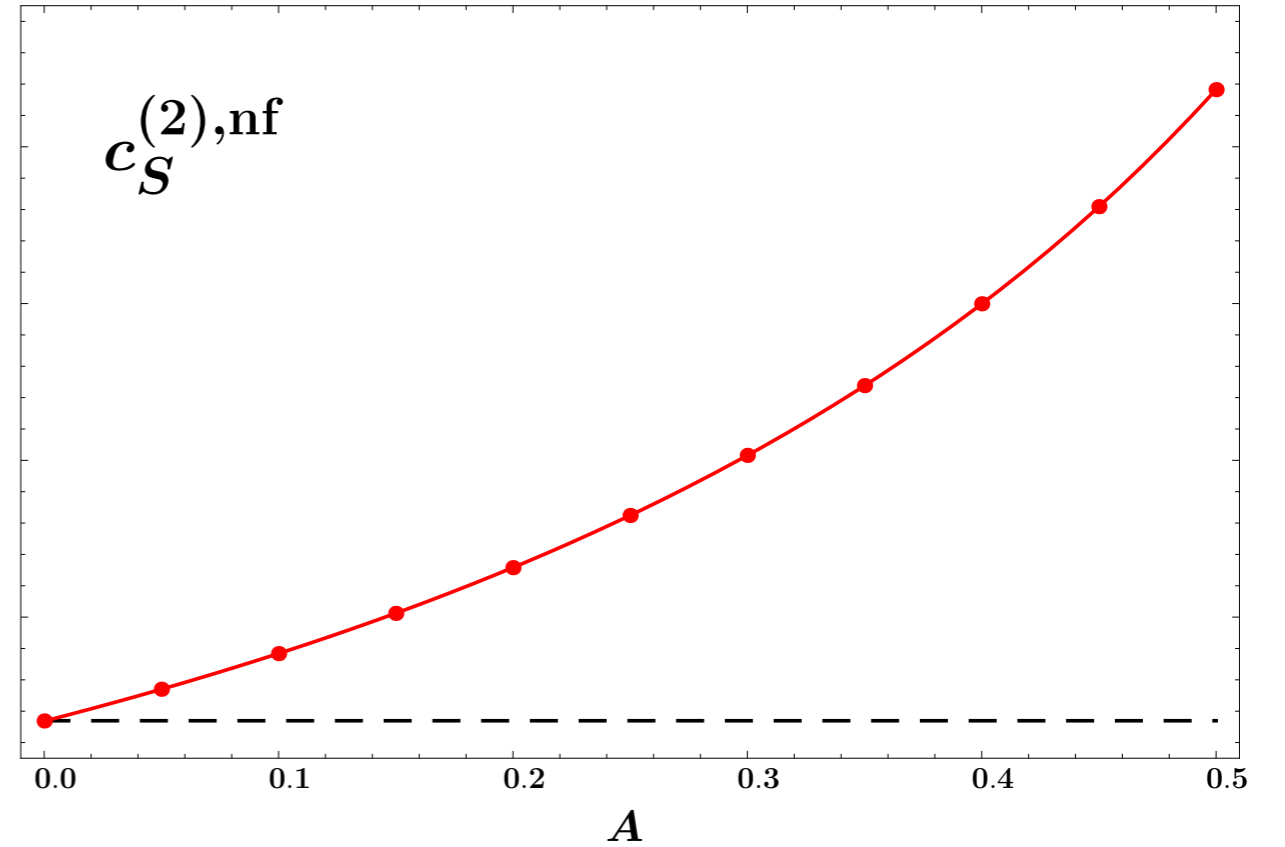
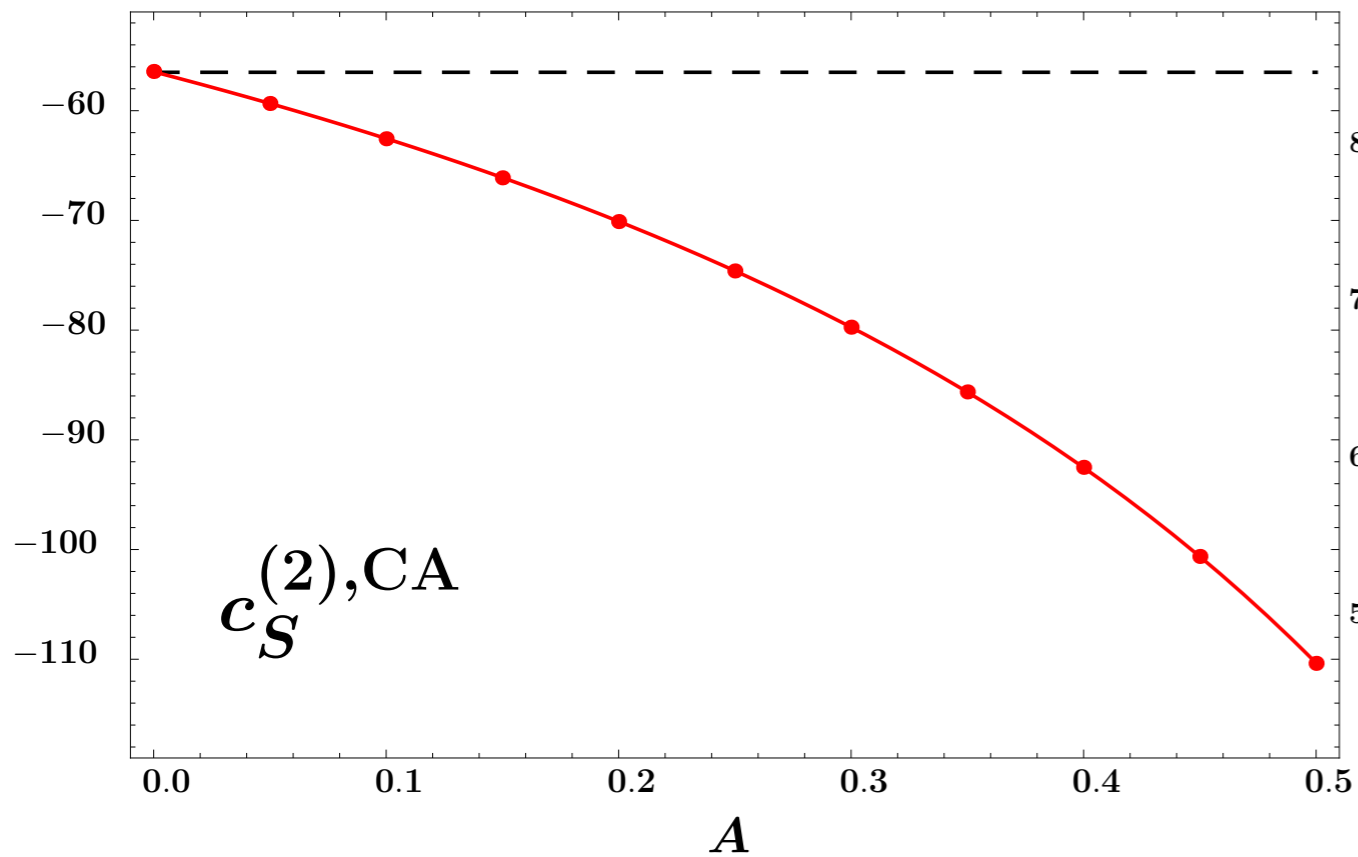
- The two-loop soft anomalous dimension is not known. We define in Laplace space:

$$\frac{d\tilde{S}(\tau)}{d \ln \mu} = -\frac{1}{(1-A)} [4\Gamma_{cusp} \ln(\mu\bar{\tau}) - 2\gamma_S] \tilde{S}(\tau)$$



Results: *Angularities*

$$\mathcal{M}_{Ang}(\omega, \{k_i\}) = \delta(\omega - \sum_{i \in L} k_{i,+}^{1-A/2} k_{i,-}^{A/2} - \sum_{i \in R} k_{i,+}^{A/2} k_{i,-}^{1-A/2})$$



Results: W/H @ large p_T

- With two beams and one recoiling jet the soft function depends on two initial state (1,2) and one final state (J) Wilson lines:

$$S(\omega) = \sum_X \delta(\omega - n_J \cdot p_X) |\langle X | S_1 S_2 S_J | 0 \rangle|^2$$

- However, due to rescaling invariance of light-cone vectors and colour conservation, the diagrams that contribute @ NNLO only involve attachments to the initial state Wilson Lines S_1 and S_2 .
- Hence, up to NNLO, we encounter the same dijet matrix element as before.
- However, there is also now an angular dependence in the measurement function, giving six-dimensional integrals...

Results: W/H @ large p_T

- W/H production @ large p_T :

$$\mathcal{M}_{W/H}(\omega, \{k_i\}) = \delta(\omega - \sum_i (k_i^+ + k_i^- - 2k_i^T \cos \theta_i))$$

- We have similar color structures with the following definitions:

$$C_s = \begin{cases} C_F - C_A/2 & q\bar{q} \rightarrow g \\ C_A/2 & qg \rightarrow q \text{ and } gg \rightarrow g \end{cases}$$

- For W/H production @ large p_T , we obtain:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2} \left(48.7045 C_s^2 + \left(\underbrace{108.62}_{bare} - \underbrace{111.40}_{ren} = -2.78 \right) C_A C_s - 25.2824 C_s n_f T_F \right)$$

- Whereas *Becher, Bell, Marti* / 1201.5572 calculate:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2} (48.7045 C_s^2 - 2.6501 C_A C_s - 25.3073 C_s n_f T_F)$$

Conclusions and future work

- SCET provides an efficient, analytic approach to high-order resummations necessary for precision collider physics via renormalisation group methods.
 - We have presented an algorithm for automated computation of dijet soft functions for a wide class of observables in SCET
 - Our master formulas coupled with *SecDec* can quickly and easily produce predictions for a wide class of SCET_I soft functions at one and two-loops.
 - This is an important ingredient for NNLL resummations in SCET...
 - Next steps: Better understanding of the numerics, SCET_{II} observables, n-jet soft functions and a public code...
-

Thanks!