Real radiation at next-to-next-to-leading order in QCD

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Schemes

First attempts

Sector decomposition applied to phase spaces

Idea by Anastasiou, Melnikov and Petriello ’03
Also discussed by original authors of sector decomposition Binoth and Heinrich ’04

First applied to Higgs production by Anastasiou, Melnikov and Petriello ’04

Highly process dependent and difficult to apply
Besides Higgs production with decay only applied to Drell-Yan Melnikov, Petriello ‘06

Schemes

Antenna subtraction Gehrmann-De Ridder, Gerhmann, Glover '05

First applied to e^+e^- -> 3jets Gerhmann-De Ridder, Gehrmann, Glover, Heinrich '07

Subsequent successes:

- Dijet production
  - Leading color, only gluons Gehrmann-De Ridder, Gehrmann, Glover, Pires '13
  - Full color, only gluons Currie, Gehrmann-De Ridder, Gerhmann, Glover, Pires '13
  - Quark-anti-quark at leading color Currie, Glover, Wells '13

- Higgs + jet (gluons) Chen, Gehrmann, Glover, Jacquier '14

- Top-pair production (only quark scattering) Abelof, Gerhmann-De Ridder '14

Schemes

q_T subtraction Catani, Grazzini '07

Originally introduced for Higgs boson production
Based on transverse momentum resummation

Recentlly computed processes:

- $W\gamma$, $Z\gamma$ Grazzini, Kallweit, Rathlev '15
- W^+W^- Gehrmann, Grazzini, Kallweit, Maierhoeffer,von Manteuffel,Pozzorini, Rathlev, Tancredi '14
- ZH Ferrara, Grazzini, Tramontano '14
- ZZ Cascioli, Gehrmann, Grazzini, Kallweit, Maierhoeffer,von Manteuffel,Pozzorini, Rathlev, Tancredi, Weihs '14
- $Z\gamma$ Grazzini, Kallweit, Rathlev, Torre '13
- $\gamma\gamma$ Catani, Cieri, de Florian, Ferrara, Grazzini '11
- WH Ferrara, Grazzini, Tramontano '11 '13

Schemes

"Colorful subtraction" del Duca, Somogyi, Trocsanyi '05

Applied to Higgs boson decay to b-quarks

Del Duca, Duhr, Somogyi, Tramontano, Trocsanyi '15

Schemes

SecToR ImProved Phase spacE for real Radiation (STRIPPER)

Also known as Sector Improved Residue Subtraction Scheme

Invented: MC ’10 First applied: MC ’11 (top-quarks)

Subsequently applied by others to several non-trivial problems:

- $Z \to e^+e^-$ (as a warmup) Boughezal, Melnikov, Petriello ’11
- top quark decay Brucherseifer, Caola, Melnikov ’13
- $b \to X_u e\nu$ Brucherseifer, Caola, Melnikov ‘13
- H + jet Boughezal, Caola, Melnikov, Petriello, Schulze ’13
- Muon decay spin asymmetry Caola, Czarnecki, Liang, Melnikov, Szafron ’14
- Single top-quark production Brucherseifer, Caola, Melnikov ’14
- Higgs + jet Boughezal, Caola, Melnikov, Petriello, Schulze ‘15

And by the inventor

- Total cross sections for top pair production Baernreuther, MC, Fiedler, Mitov 12’, 13’
- Differential top pair production MC, Fiedler, Mitov ‘14

Schemes

Most recent developments

N-jettiness subtraction

Introduced in Boughezal, Focke, Liu, Petriello '15 (W + jet)
Also discussed in Gaunt, Stahlhofen, Tackmann, Walsh '15

Applied to W + jet and Higgs + jet Boughezal, Focke, Giele, Liu, Petriello '15

Goals?

1) Be completely general
2) Use all information on factorization

Seems that currently only STRIPPER satisfies both criteria
"Colorful subtraction" will probably reach these same goals

	analytic	FS colour	IS colour	local
antenna subtraction	✓	✓	✓	✓ (ish)
STRIPPER	✗	✓	✓	✓
q subtraction	✓	✓	✗	✗
"colorful subtraction"	✓	✓	✗	✓
inverse unitarity	✓	✗	✓	N/A

Currie '15

Stripper Documentation

Implementation of the

Sector-improved residue subtraction scheme

STRIPPER (SecToR Improved Phase sPacE for Real radiation)

following (CH)

Four-dimensional formulation of the sector-improved residue subtraction scheme
M. Czakon and D. Heymes
Nucl. Phys. B890 (2015) 152

The original idea of the subtraction scheme has been described in

A novel subtraction scheme for double-real radiation at NNLO
M. Czakon
Phys. Lett. B693 (2010) 259

while the first implementation may be found in

Double-real radiation in hadronic top quark pair production as a proof of a certain concept
M. Czakon
Nucl. Phys. B849 (2011) 250

Inspiration

I. Frixione, Kunszt, Signer ‘96 (FKS)

BUT:

1) No analytic integration over subtraction terms
2) Solution to overlapping singularities not present at NLO

II. Binoth, Heinrich ‘00 (Sector decomposition)
Anastasiou, Melnikov, Petriell ‘04 (in the context of phase spaces)

BUT:

1) Process independent
2) Decomposition driven by physical singularities

What's there to compute?

$$\sigma_{h_1 h_2}(P_1, P_2) = \sum_{ab} \iint_0^1 \mathrm{d}x_1 \mathrm{d}x_2\, f_{a/h_1}(x_1, \mu_F^2)\, f_{b/h_2}(x_2, \mu_F^2)\, \hat{\sigma}_{ab}(x_1 P_1, x_2 P_2; \alpha_s(\mu_R^2), \mu_R^2, \mu_F^2)$$

$$\hat{\sigma}_{ab} = \hat{\sigma}_{ab}^{(0)} + \hat{\sigma}_{ab}^{(1)} + \hat{\sigma}_{ab}^{(2)}$$

$$\hat{\sigma}_{ab}^{(0)} = \hat{\sigma}_{ab}^{\mathrm{B}} = \frac{1}{2\hat{s}} \frac{1}{N_{ab}} \int \mathrm{d}\boldsymbol{\Phi}_n \langle \mathcal{M}_n^{(0)} | \mathcal{M}_n^{(0)} \rangle \mathrm{F}_n$$

$$\hat{\sigma}_{ab}^{(1)} = \hat{\sigma}_{ab}^{\mathrm{R}} + \hat{\sigma}_{ab}^{\mathrm{V}} + \hat{\sigma}_{ab}^{\mathrm{C}}$$

$$\hat{\sigma}_{ab}^{\mathrm{R}} = \frac{1}{2\hat{s}} \frac{1}{N_{ab}} \int \mathrm{d}\boldsymbol{\Phi}_{n+1} \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(0)} \rangle \mathrm{F}_{n+1},$$

$$\hat{\sigma}_{ab}^{\mathrm{V}} = \frac{1}{2\hat{s}} \frac{1}{N_{ab}} \int \mathrm{d}\boldsymbol{\Phi}_n\, 2\,\mathrm{Re} \langle \mathcal{M}_n^{(0)} | \mathcal{M}_n^{(1)} \rangle \mathrm{F}_n,$$

$$\hat{\sigma}_{ab}^{\mathrm{C}}(p_1, p_2) = \frac{\alpha_s}{2\pi} \frac{1}{\epsilon} \left(\frac{\mu_R^2}{\mu_F^2} \right)^{\epsilon} \sum_c \int_0^1 \mathrm{d}z \left[P_{ca}^{(0)}(z)\, \hat{\sigma}_{cb}^{\mathrm{B}}(z p_1, p_2) + P_{cb}^{(0)}(z)\, \hat{\sigma}_{ac}^{\mathrm{B}}(p_1, z p_2) \right]$$

What's there to compute?

$$\hat{\sigma}_{ab}^{(2)} = \hat{\sigma}_{ab}^{\mathrm{RR}} + \hat{\sigma}_{ab}^{\mathrm{RV}} + \hat{\sigma}_{ab}^{\mathrm{VV}} + \hat{\sigma}_{ab}^{\mathrm{C1}} + \hat{\sigma}_{ab}^{\mathrm{C2}}$$

$$\hat{\sigma}_{ab}^{\mathrm{RR}} = \frac{1}{2\hat{s}}\frac{1}{N_{ab}}\int \mathrm{d}\boldsymbol{\Phi}_{n+2}\langle\mathcal{M}_{n+2}^{(0)}|\mathcal{M}_{n+2}^{(0)}\rangle \mathrm{F}_{n+2},$$

$$\hat{\sigma}_{ab}^{\mathrm{RV}} = \frac{1}{2\hat{s}}\frac{1}{N_{ab}}\int \mathrm{d}\boldsymbol{\Phi}_{n+1}2\,\mathrm{Re}\langle\mathcal{M}_{n+1}^{(0)}|\mathcal{M}_{n+1}^{(1)}\rangle \mathrm{F}_{n+1},$$

$$\hat{\sigma}_{ab}^{\mathrm{VV}} = \frac{1}{2\hat{s}}\frac{1}{N_{ab}}\int \mathrm{d}\boldsymbol{\Phi}_{n}\big(2\,\mathrm{Re}\langle\mathcal{M}_{n}^{(0)}|\mathcal{M}_{n}^{(2)}\rangle + \langle\mathcal{M}_{n}^{(1)}|\mathcal{M}_{n}^{(1)}\rangle\big)\mathrm{F}_{n}$$

$$\hat{\sigma}_{ab}^{\mathrm{C1}}(p_1,p_2) = \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu_R^2}{\mu_F^2}\right)^{\epsilon}\sum_c\int_0^1 \mathrm{d}z\big[P_{ca}^{(0)}(z)\,\hat{\sigma}_{cb}^{\mathrm{R}}(zp_1,p_2) + P_{cb}^{(0)}(z)\,\hat{\sigma}_{ac}^{\mathrm{R}}(p_1,zp_2)\big]$$

$$\begin{aligned}
&\hat{\sigma}_{ab}^{\mathrm{C2}}(p_1,p_2) \\
&= \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu_R^2}{\mu_F^2}\right)^{\epsilon}\sum_c\int_0^1 \mathrm{d}z\big[P_{ca}^{(0)}(z)\,\hat{\sigma}_{cb}^{\mathrm{V}}(zp_1,p_2) + P_{cb}^{(0)}(z)\,\hat{\sigma}_{ac}^{\mathrm{V}}(p_1,zp_2)\big] \\
&\quad + \left(\frac{\alpha_s}{2\pi}\right)^2\frac{1}{2\epsilon}\left(\frac{\mu_R^2}{\mu_F^2}\right)^{2\epsilon}\sum_c\int_0^1 \mathrm{d}z\big[P_{ca}^{(1)}(z)\,\hat{\sigma}_{cb}^{\mathrm{B}}(zp_1,p_2) + P_{cb}^{(1)}(z)\,\hat{\sigma}_{ac}^{\mathrm{B}}(p_1,zp_2)\big] \\
&\quad + \left(\frac{\alpha_s}{2\pi}\right)^2\frac{\beta_0}{4\epsilon^2}\left[\left(\frac{\mu_R^2}{\mu_F^2}\right)^{2\epsilon} - 2\left(\frac{\mu_R^2}{\mu_F^2}\right)^{\epsilon}\right] \\
&\quad \times\sum_c\int_0^1 \mathrm{d}z\big[P_{ca}^{(0)}(z)\,\hat{\sigma}_{cb}^{\mathrm{B}}(zp_1,p_2) + P_{cb}^{(0)}(z)\,\hat{\sigma}_{ac}^{\mathrm{B}}(p_1,zp_2)\big] \; + \left(\frac{\alpha_s}{2\pi}\right)^2\frac{1}{2\epsilon^2}\left(\frac{\mu_R^2}{\mu_F^2}\right)^{2\epsilon} \\
&\quad \times\sum_{cd}\int_0^1 \mathrm{d}z\big[\big(P_{cd}^{(0)}\otimes P_{da}^{(0)}\big)(z)\,\hat{\sigma}_{cb}^{\mathrm{B}}(zp_1,p_2) + \big(P_{cd}^{(0)}\otimes P_{db}^{(0)}\big)(z)\,\hat{\sigma}_{ac}^{\mathrm{B}}(p_1,zp_2)\big] \\
&\quad + \left(\frac{\alpha_s}{2\pi}\right)^2\frac{1}{\epsilon^2}\left(\frac{\mu_R^2}{\mu_F^2}\right)^{2\epsilon}\sum_{cd}\iint_0^1 \mathrm{d}z\,\mathrm{d}\bar{z}\big[P_{ca}^{(0)}(z)\,P_{db}^{(0)}(\bar{z})\,\hat{\sigma}_{cd}^{\mathrm{B}}(zp_1,\bar{z}p_2)\big],
\end{aligned}$$

Outline of the scheme

I. D-dimensional formulation

1) Phase space decomposition
2) Phase space parameterization
3) Generation of subtraction and integrated subtraction terms

II. 4-dimensional formulation

1) Average over azimuthal angles
2) Separation of finite contributions
3) ‘t Hooft-Veltman regularization of separately finite contributions

Phase space decomposition

Goal: split the phase space into sectors with a controllable number of singularities

NLO

$$\sum_{ik} \mathcal{S}_{i,k} = 1 \qquad \mathcal{S}_{i,k} = \frac{1}{D_1\, d_{i,k}}, \qquad D_1 = \sum_{ik} \frac{1}{d_{i,k}} \qquad d_{i,k} = \left(\frac{E_i}{\sqrt{\hat{s}}}\right)^{\alpha} (1-\cos\theta_{ik})^{\beta}$$

Patterns:

$(g,g),\ (g,q),\ (g,\bar{q}),\ (q,g),\ (\bar{q},g),\ (q,q),\ (\bar{q},\bar{q})$ Initial state reference

$(g,g),\ (g,q),\ (g,\bar{q}),\ (q,\bar{q})$ final state reference

NNLO

$$\sum_{ij}\left[\sum_k \mathcal{S}_{ij,k} + \sum_{kl} \mathcal{S}_{i,k;j,l}\right] = 1$$

$$\mathcal{S}_{ij,k} = \frac{1}{D_2\, d_{ij,k}}, \qquad \mathcal{S}_{i,k;j,l} = \frac{1}{D_2\, d_{i,k} d_{j,l}}$$

$$D_2 = \sum_{ii}\left[\sum_k \frac{1}{d_{ij,k}} + \sum_{kl} \frac{1}{d_{i,k} d_{j,l}}\right].$$

$$d_{ij,k} = \left(\frac{E_i}{\sqrt{\hat{s}}}\right)^{\alpha_i} \left(\frac{E_j}{\sqrt{\hat{s}}}\right)^{\alpha_j} \left[(1-\cos\theta_{ij})(1-\cos\theta_{ik})(1-\cos\theta_{jk})\right]^{\beta}$$

$(g,g,g),\ (g,g,q),\ (g,g,\bar{q}),\ (g,q,g),\ (g,\bar{q},g),\ (g,q,q),\ (g,\bar{q},\bar{q}),\ (q,\bar{q},g),$
$(q,\bar{q},q'),\ (q',q,q),\ (q',\bar{q},\bar{q}).$ Initial state reference

$(g,g,g),\ (g,g,q),\ (g,g,\bar{q}),\ (g,q,\bar{q}),\ (q,\bar{q},g),\ (q,\bar{q},q')$ final state reference

Phase space decomposition

Example: gg -> gggg

NLO

nr.	(i,k)	flavors	count
0	(2,0)	(g,g)	4
1	(2,1)	(g,g)	4
2	(2,3)	(g,g)	12

NNLO

nr.	(i,j,k)	flavors	sector	count
0	(3,2,0)	(g,g,g)	1	12
1	(3,2,0)	(g,g,g)	2	12
2	(3,2,0)	(g,g,g)	3	12
3	(3,2,0)	(g,g,g)	4	12
4	(3,2,0)	(g,g,g)	5	12
5	(3,2,1)	(g,g,g)	1	12
6	(3,2,1)	(g,g,g)	2	12
7	(3,2,1)	(g,g,g)	3	12
8	(3,2,1)	(g,g,g)	4	12
9	(3,2,1)	(g,g,g)	5	12
10	(2,4,3)	(g,g,g)	1	24
11	(2,4,3)	(g,g,g)	2	24
12	(2,4,3)	(g,g,g)	3	24
13	(2,4,3)	(g,g,g)	4	24
14	(2,4,3)	(g,g,g)	5	24

nr.	(i,k)(j,l)	flavors	sector	count
15	(2,0)(3,1)	(g,g)(g,g)	1	12
16	(3,1)(2,0)	(g,g)(g,g)	1	12
17	(2,0)(3,4)	(g,g)(g,g)	1	24
18	(3,4)(2,0)	(g,g)(g,g)	1	24
19	(2,1)(3,4)	(g,g)(g,g)	1	24
20	(3,4)(2,1)	(g,g)(g,g)	1	24
21	(2,3)(4,5)	(g,g)(g,g)	1	24
22	(2,3)(4,5)	(g,g)(g,g)	2	12

Phase space decomposition

Example: gg -> gguu~

NLO

nr.	(i,k)	flavors	count
0	(2,0)	(g,g)	2
1	(4,0)	(u,g)	1
2	(5,0)	(u~,g)	1
3	(2,1)	(g,g)	2
4	(4,1)	(u,g)	1
5	(5,1)	(u~,g)	1
6	(2,3)	(g,g)	2
7	(2,4)	(g,u)	2
8	(2,5)	(g,u~)	2
9	(4,5)	(u,u~)	1

NNLO

nr.	(i,j,k)	flavors	sector	count
0	(3,2,0)	(g,g,g)	1	2
1	(3,2,0)	(g,g,g)	2	2
2	(3,2,0)	(g,g,g)	3	2
3	(3,2,0)	(g,g,g)	4	2
4	(3,2,0)	(g,g,g)	5	2
5	(2,4,0)	(g,u,g)	1	2
6	(4,2,0)	(u,g,g)	1	2
7	(2,4,0)	(g,u,g)	2	2
8	(4,2,0)	(u,g,g)	2	2

nr.	(i,k)(j,l)	flavors	sector	count
100	(2,0)(3,1)	(g,g)(g,g)	1	2
101	(3,1)(2,0)	(g,g)(g,g)	1	2
102	(2,0)(4,1)	(g,g)(u,g)	1	2
103	(4,1)(2,0)	(u,g)(g,g)	1	2
104	(2,0)(5,1)	(g,g)(u~,g)	1	2
105	(5,1)(2,0)	(u~,g)(g,g)	1	2
143	(4,5)(2,3)	(u,u~)(g,g)	1	2
144	(2,3)(4,5)	(g,g)(u,u~)	2	2
145	(2,4)(3,5)	(g,u)(g,u~)	1	2
146	(3,5)(2,4)	(g,u~)(g,u)	1	2
147	(2,4)(3,5)	(g,u)(g,u~)	2	2

Phase space parameterization

Goal: parameterize the phase space in terms of reference and unresolved momenta

$$\int \mathrm{d}\boldsymbol{\Phi}_{n+n_u} = \int \mathrm{d}\boldsymbol{\Phi}_{\substack{\text{reference} \\ \text{unresolved}}} \int \mathrm{d}\boldsymbol{\Phi}_{n-n_{fr}}(Q), \quad n \geq 2,\ n_u \in \{1,2\},\ 0 \leq n_{fr} \leq n_u$$

Goal: Parameterize unresolved momenta through energy and angle variables

NLO-type or rather single-unresolved

$$\iint_0^1 \mathrm{d}\eta\, \mathrm{d}\xi\, \eta^{a_1 - b_1 \epsilon}\, \xi^{a_2 - b_2 \epsilon}$$

NNLO-type, i.e.
triple- and double-unresolved

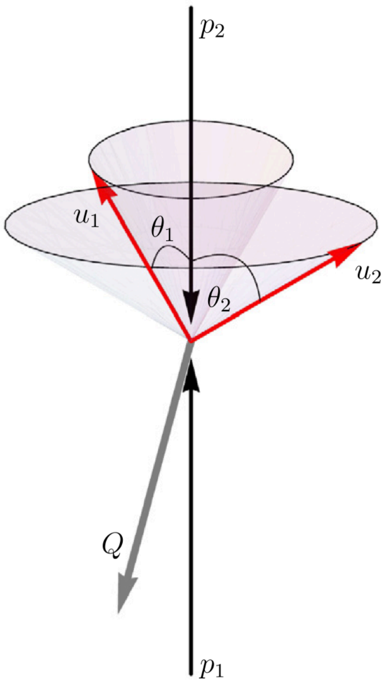
$$\iiiint_0^1 \mathrm{d}\eta_1 \mathrm{d}\eta_2 \mathrm{d}\xi_1 \mathrm{d}\xi_2\, \eta_1^{a_1 - b_1 \epsilon}\, \eta_2^{a_2 - b_2 \epsilon}\, \xi_1^{a_3 - b_3 \epsilon}\, \xi_2^{a_4 - b_4 \epsilon}$$

Phase space parameterization

Example: triple-collinear sector parameterization

$$r^\mu = r^0\,\hat{r}^\mu = r^0 \begin{pmatrix} 1 \\ \hat{\boldsymbol{r}} \end{pmatrix}, \qquad u_1^\mu = u_1^0\,\hat{u}_1^\mu = u_1^0 \begin{pmatrix} 1 \\ \hat{\boldsymbol{u}}_1 \end{pmatrix}, \qquad u_2^\mu = u_2^0\,\hat{u}_2^\mu = u_2^0 \begin{pmatrix} 1 \\ \hat{\boldsymbol{u}}_2 \end{pmatrix}$$

$$\begin{aligned} \hat{\boldsymbol{r}} &= \hat{\boldsymbol{n}}^{(3-2\epsilon)}(\alpha_1, \alpha_2, \ldots), \\ \hat{\boldsymbol{u}}_1 &= \boldsymbol{R}_1^{(3-2\epsilon)}(\alpha_1, \alpha_2, \ldots)\,\hat{\boldsymbol{n}}^{(3-2\epsilon)}(\theta_1, \phi_1, \rho_1, \rho_2, \ldots), \\ \hat{\boldsymbol{u}}_2 &= \boldsymbol{R}_1^{(3-2\epsilon)}(\alpha_1, \alpha_2, \ldots)\,\boldsymbol{R}_2^{(3-2\epsilon)}(\phi_1, \rho_1, \rho_2, \ldots)\,\hat{\boldsymbol{n}}^{(3-2\epsilon)}(\theta_2, \phi_2, \sigma_1, \sigma_2, \ldots) \end{aligned}$$



$$\int \mathrm{d}\boldsymbol{\Phi}_{n+2} = \int \mathrm{d}\boldsymbol{\Phi}_{\text{unresolved}} \int \mathrm{d}\boldsymbol{\Phi}_n(p_1 + p_2 - u_1 - u_2)$$

$$\int \mathrm{d}\boldsymbol{\Phi}_{\text{unresolved}} = \int \mathrm{d}\boldsymbol{\Phi}_{\text{unresolved}}\left(\theta\left(u_1^0 - u_2^0\right) + \theta\left(u_2^0 - u_1^0\right)\right)$$

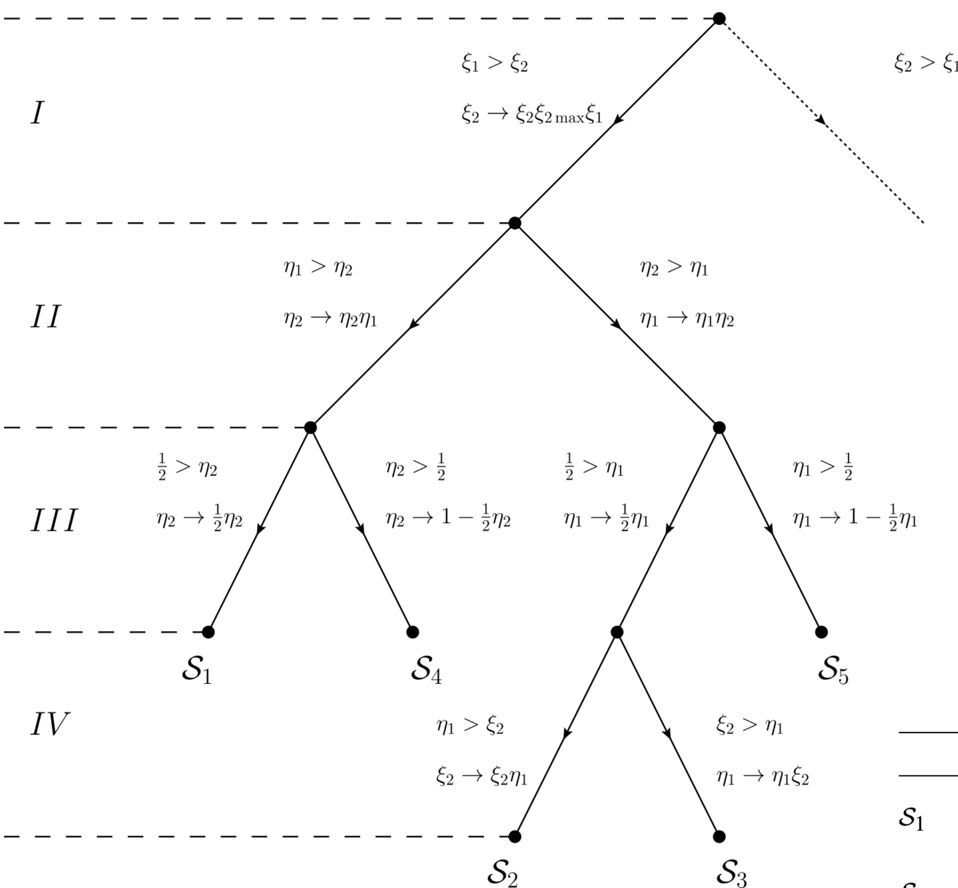
$$u_1^0 = E_{\max}\,\hat{\xi}_1, \qquad u_2^0 = E_{\max}\,\hat{\xi}_2,$$

$$\cos\theta_1 = 1 - 2\hat{\eta}_1, \qquad \cos\theta_2 = 1 - 2\hat{\eta}_2, \qquad \cos\phi_2 = \frac{1 - 2\eta_3 - (1 - 2\hat{\eta}_1)(1 - 2\hat{\eta}_2)}{4\sqrt{(1-\hat{\eta}_1)\hat{\eta}_1(1-\hat{\eta}_2)\hat{\eta}_2}}.$$

$$\eta_3 = \frac{\hat{u}_1 \cdot \hat{u}_2}{2} = \frac{1 - \cos\theta_{12}}{2} = \frac{(\hat{\eta}_1 - \hat{\eta}_2)^2}{\hat{\eta}_1 + \hat{\eta}_2 - 2\hat{\eta}_1\hat{\eta}_2 - 2(1 - 2\zeta)\sqrt{\hat{\eta}_1(1-\hat{\eta}_1)\hat{\eta}_2(1-\hat{\eta}_2)}}.$$

Phase space parameterization

Triple unresolved parameterization



$$\int \mathrm{d}\boldsymbol{\Phi}_{\text{unresolved}}\,\theta\left(u_1^0 - u_2^0\right) = \frac{E_{\max}^4}{(2\pi)^6}\left(\frac{\pi\mu_R^2 e^{\gamma_\mathrm{E}}}{8E_{\max}^2}\right)^{2\epsilon} \int_{\mathcal{S}_1^{1-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}(\phi_1,\rho_1,\ldots) \int_{\mathcal{S}_1^{-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}(\sigma_1,\sigma_2,\ldots) \times \int_0^1 \mathrm{d}\zeta\,\left(\zeta(1-\zeta)\right)^{-\frac{1}{2}-\epsilon} \iiiint_0^1 \mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\xi_1\mathrm{d}\xi_2 \sum_{i=1}^{5}\mu_{\mathcal{S}_i},$$

	$\mu_{\mathcal{S}_i}$
$\mathcal{S}_1$	$\eta_1^{1-2\epsilon}\eta_2^{-\epsilon}\xi_1^{3-4\epsilon}\xi_2^{1-2\epsilon}\left((1-\eta_1)(2-\eta_1\eta_2)\right)^{-\epsilon}\left(\frac{\eta_{31}(\eta_1,\eta_2)}{2-\eta_2}\right)^{1-2\epsilon}\xi_{2\,\max}^{2-2\epsilon}$
$\mathcal{S}_2$	$\eta_1^{2-3\epsilon}\eta_2^{1-2\epsilon}\xi_1^{3-4\epsilon}\xi_2^{1-2\epsilon}\left((1-\eta_2)(2-\eta_1\eta_2)\right)^{-\epsilon}\left(\frac{\eta_{31}(\eta_2,\eta_1)}{2-\eta_1}\right)^{1-2\epsilon}\xi_{2\,\max}^{2-2\epsilon}$
$\mathcal{S}_3$	$\eta_1^{-\epsilon}\eta_2^{1-2\epsilon}\xi_1^{3-4\epsilon}\xi_2^{2-3\epsilon}\left((1-\eta_2)(2-\eta_1\eta_2\xi_2)\right)^{-\epsilon}\left(\frac{\eta_{31}(\eta_2,\eta_1\xi_2)}{2-\eta_1\xi_2}\right)^{1-2\epsilon}\xi_{2\,\max}^{2-2\epsilon}$
$\mathcal{S}_4$	$\eta_1^{1-2\epsilon}\eta_2^{1-2\epsilon}\xi_1^{3-4\epsilon}\xi_2^{1-2\epsilon}\left((1-\eta_1)(2-\eta_2)(2-\eta_1(2-\eta_2))\right)^{-\epsilon}\eta_{32}^{1-2\epsilon}(\eta_1,\eta_2)\,\xi_{2\,\max}^{2-2\epsilon}$
$\mathcal{S}_5$	$\eta_1^{1-2\epsilon}\eta_2^{1-2\epsilon}\xi_1^{3-4\epsilon}\xi_2^{1-2\epsilon}\left((1-\eta_2)(2-\eta_1)(2-\eta_2(2-\eta_1))\right)^{-\epsilon}\eta_{32}^{1-2\epsilon}(\eta_2,\eta_1)\,\xi_{2\,\max}^{2-2\epsilon}$

Phase space parameterization

Extra dimensions for soft poles: 5 at NLO, 6 at NNLO

$$\int_{\mathcal{S}_1^{-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}(\rho_1,\ldots)$$

$$= \frac{(4\pi)^{-\epsilon}\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_{-1}^{+1} \mathrm{d}\cos\rho_1 \Bigg(\delta(1-\cos\rho_1) + \delta(1+\cos\rho_1)$$

$$- 2\epsilon \frac{4^\epsilon \Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \left[\frac{1}{(1-\cos^2\rho_1)^{1+\epsilon}} \right]_+ \Bigg).$$

$$\int_{\mathcal{S}_1^{-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}(\sigma_1,\sigma_2,\ldots)$$

$$= \frac{(4\pi)^{-\epsilon}\Gamma(1-\epsilon)}{2\Gamma(1-2\epsilon)} \int_{-1}^{+1} \mathrm{d}\cos\sigma_1$$

$$\times \int_{-1}^{+1} \mathrm{d}\cos\sigma_2 \Bigg(\big(\delta(1-\cos\sigma_1) + \delta(1+\cos\sigma_1)\big)\big(\delta(1-\cos\sigma_2) + \delta(1+\cos\sigma_2)\big)$$

$$- 2\epsilon \frac{4^\epsilon \Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \left[\frac{1}{(1-\cos^2\sigma_1)^{1+\epsilon}} \right]_+ \big(\delta(1-\cos\sigma_2) + \delta(1+\cos\sigma_2)\big)$$

$$- \frac{2+4\epsilon}{\pi} \left[\frac{1}{(1-\cos^2\sigma_1)^{1+\epsilon}} \right]_+ \left[\frac{1}{(1-\cos^2\sigma_2)^{\frac{3}{2}+\epsilon}} \right]_+ \Bigg).$$

Generation of subtraction terms

Example: real radiation at NLO

Behavior of the phase space at the boundaries: $\iint_0^1 \mathrm{d}\eta\, \mathrm{d}\xi\, \eta^{-\epsilon}\, \xi^{1-2\epsilon}$

Worst case behavior of the matrix element: $\frac{1}{\eta}\frac{1}{\xi^2}$

Write cross section as:

$$\hat{\sigma}^{\mathrm{R}} = \sum_{ik} \iint_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} \frac{\mathrm{d}\xi}{\xi^{1+2\epsilon}} f_{i,k}(\eta,\xi)$$

$$f_{i,k}(\eta,\xi) = \frac{E_{\max}^2}{16\pi^3 \hat{s} N_{ab}} \left(\frac{\pi \mu_R^2 e^{\gamma_{\mathrm{E}}}}{4E_{\max}^2(1-\eta)}\right)^{\epsilon} \int_{\mathcal{S}_1^{1-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}(\phi,\rho_1,\ldots)$$

$$\times \int \mathrm{d}\boldsymbol{\Phi}_n(p_1+p_2-u)\, \mathcal{S}_{i,k}\left[\eta\,\xi^2 \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(0)} \rangle\right] \mathrm{F}_{n+1}$$

Apply formula:

$$\frac{1}{x^{1+a\epsilon}} = -\frac{1}{a\epsilon}\delta(x) + \left[\frac{1}{x^{1+a\epsilon}}\right]_+ \qquad \int_0^1 \mathrm{d}x \left[\frac{1}{x^{1+a\epsilon}}\right]_+ f(x) = \int_0^1 \mathrm{d}x\, \frac{f(x)-f(0)}{x^{1+a\epsilon}}$$

Calculate limits:

$$\lim_{\eta\to 0}\left[\eta\,\xi^2 \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(0)} \rangle\right], \qquad \lim_{\xi\to 0}\left[\eta\,\xi^2 \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(0)} \rangle\right],$$

$$\lim_{\eta\to 0}\lim_{\xi\to 0}\left[\eta\,\xi^2 \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(0)} \rangle\right].$$

Generation of subtraction terms

Collection of matrix elements required

$$\langle \mathcal{M}_n^{(0)} | \mathcal{M}_n^{(0)} \rangle, \quad \langle \mathcal{M}_n^{(0)} | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_n^{(0)} \rangle, \quad \langle \mathcal{M}_n^{(0)} | \lambda_i \rangle \langle \lambda_i' | \mathcal{M}_n^{(0)} \rangle,$$
$$\langle \mathcal{M}_n^{(0)} | \{ \mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_k \cdot \mathbf{T}_l \} | \mathcal{M}_n^{(0)} \rangle, \quad \langle \mathcal{M}_n^{(0)} | f^{abc} T_i^a T_j^b T_k^c | \mathcal{M}_n^{(0)} \rangle,$$
$$\langle \mathcal{M}_n^{(0)} | \mathbf{T}_i \cdot \mathbf{T}_j | \lambda_k \rangle \langle \lambda_k' | \mathcal{M}_n^{(0)} \rangle, \quad \langle \mathcal{M}_n^{(0)} | \lambda_i \lambda_j \rangle \langle \lambda_i' \lambda_j' | \mathcal{M}_n^{(0)} \rangle,$$
$$\langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(0)} \rangle, \quad \langle \mathcal{M}_{n+1}^{(0)} | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_{n+1}^{(0)} \rangle, \quad \langle \mathcal{M}_{n+1}^{(0)} | \lambda_i \rangle \langle \lambda_i' | \mathcal{M}_{n+1}^{(0)} \rangle,$$
$$\langle \mathcal{M}_{n+2}^{(0)} | \mathcal{M}_{n+2}^{(0)} \rangle, \quad \langle \mathcal{M}_n^{(0)} | \mathcal{M}_n^{(1)} \rangle, \quad \langle \mathcal{M}_n^{(0)} | \mathbf{T}_i \cdot \mathbf{T}_j | \mathcal{M}_n^{(1)} \rangle,$$
$$\langle \mathcal{M}_n^{(0)} | \lambda_i \rangle \langle \lambda_i' | \mathcal{M}_n^{(1)} \rangle, \quad \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(1)} \rangle,$$
$$\langle \mathcal{M}_n^{(1)} | \mathcal{M}_n^{(1)} \rangle, \quad \langle \mathcal{M}_n^{(0)} | \mathcal{M}_n^{(2)} \rangle.$$

Average over azimuthal angles

When it works as expected

$$\left[\int_{\mathcal{S}_1^{1-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}\, 1\right]^{-1} \int_{\mathcal{S}_1^{1-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}\,(\phi, \rho_1, \rho_2, \ldots)\, \frac{u_\perp^\mu u_\perp^\nu}{u_\perp^2} = \frac{1}{2(1-\epsilon)}\left(g^{\mu\nu} - \frac{r^\mu \bar{r}^\nu + r^\nu \bar{r}^\mu}{r \cdot \bar{r}}\right)$$

When there are subtleties

$$\left[\int_0^\pi \mathrm{d}\tilde{\phi}_2\, |\tan\tilde{\phi}_2|^{-2\epsilon} \int_{\mathcal{S}_1^{-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}\, 1\right]^{-1} \int_0^\pi \mathrm{d}\tilde{\phi}_2\, |\tan\tilde{\phi}_2|^{-2\epsilon} \int_{\mathcal{S}_1^{-2\epsilon}} \mathrm{d}\boldsymbol{\Omega}\,(\sigma_1, \sigma_2, \ldots)\, \frac{u_{3\perp}^\mu u_{3\perp}^\nu}{u_{3\perp}^2}$$

$$= \frac{1}{2}\left(g^{\mu\nu} - \frac{u_1^\mu \bar{u}_1^\nu + u_1^\nu \bar{u}_1^\mu}{u_1 \cdot \bar{u}_1}\right) - \epsilon\, u_{3\perp}^\mu(\tilde{\phi}_2 = 0)\, u_{3\perp}^\nu(\tilde{\phi}_2 = 0),$$

Separation of finite contributions

At NLO (just the real and virtual corrections)

$$\hat{\sigma}^{\mathrm{R}} = \hat{\sigma}^{\mathrm{R}}_{\mathrm{F}} + \hat{\sigma}^{\mathrm{R}}_{\mathrm{U}}, \qquad \hat{\sigma}^{\mathrm{V}} = \hat{\sigma}^{\mathrm{V}}_{\mathrm{F}} + \hat{\sigma}^{\mathrm{V}}_{\mathrm{U}}$$

$$\hat{\sigma}^{\mathrm{R}}_{\mathrm{F}} = \frac{1}{2\hat{s}}\frac{1}{N}\int \mathrm{d}\boldsymbol{\Phi}_{n+1}\left[\langle\mathcal{M}^{(0)}_{n+1}|\mathcal{M}^{(0)}_{n+1}\rangle \mathrm{F}_{n+1} + \text{subtraction terms}\right]$$

$$\hat{\sigma}^{\mathrm{V}}_{\mathrm{F}} = \frac{1}{2\hat{s}}\frac{1}{N}\int \mathrm{d}\boldsymbol{\Phi}_{n}\, 2\,\mathrm{Re}\langle\mathcal{M}^{(0)}_{n}|\mathcal{F}^{(1)}_{n}\rangle \mathrm{F}_{n},$$

$$\hat{\sigma}^{\mathrm{V}}_{\mathrm{U}} = \frac{1}{2\hat{s}}\frac{1}{N}\int \mathrm{d}\boldsymbol{\Phi}_{n}\, 2\,\mathrm{Re}\langle\mathcal{M}^{(0)}_{n}|\mathbf{Z}^{(1)}|\mathcal{M}^{(0)}_{n}\rangle \mathrm{F}_{n}$$

At NNLO (only double real radiation)

$$\hat{\sigma}^{\mathrm{RR}} = \hat{\sigma}^{\mathrm{RR}}_{\mathrm{F}} + \hat{\sigma}^{\mathrm{RR}}_{\mathrm{SU}} + \hat{\sigma}^{\mathrm{RR}}_{\mathrm{DU}}$$

$$\hat{\sigma}^{\mathrm{RR}}_{\mathrm{F}} = \frac{1}{2\hat{s}}\frac{1}{N}\int \mathrm{d}\boldsymbol{\Phi}_{n+2}\left[\langle\mathcal{M}^{(0)}_{n+2}|\mathcal{M}^{(0)}_{n+2}\rangle \mathrm{F}_{n+2} + \text{subtraction terms}\right]$$

Complete list

LO	$\hat{\sigma}^{B}$
NLO	$\hat{\sigma}^{\mathrm{R}}_{\mathrm{F}}, \quad \hat{\sigma}^{\mathrm{V}}_{\mathrm{F}}, \quad \hat{\sigma}_{\mathrm{U}} = \hat{\sigma}^{\mathrm{R}}_{\mathrm{U}} + \hat{\sigma}^{\mathrm{V}}_{\mathrm{U}} + \hat{\sigma}^{\mathrm{C}}$
NNLO	$\hat{\sigma}^{\mathrm{RR}}_{\mathrm{F}}, \quad \hat{\sigma}^{\mathrm{RV}}_{\mathrm{F}}, \quad \hat{\sigma}^{\mathrm{VV}}_{\mathrm{F}}, \quad \hat{\sigma}_{\mathrm{FR}} = \hat{\sigma}^{\mathrm{RV}}_{\mathrm{FR}} + \hat{\sigma}^{\mathrm{VV}}_{\mathrm{FR}} + \hat{\sigma}^{\mathrm{C2}}_{\mathrm{FR}}, \quad \hat{\sigma}_{\mathrm{SU}} = \hat{\sigma}^{\mathrm{RR}}_{\mathrm{SU}} + \hat{\sigma}^{\mathrm{RV}}_{\mathrm{SU}} + \hat{\sigma}^{\mathrm{C1}}_{\mathrm{SU}},$
	$\hat{\sigma}_{\mathrm{DU}} = \hat{\sigma}^{\mathrm{RR}}_{\mathrm{DU}} + \hat{\sigma}^{\mathrm{RV}}_{\mathrm{DU}} + \hat{\sigma}^{\mathrm{VV}}_{\mathrm{DU}} + \hat{\sigma}^{\mathrm{C1}}_{\mathrm{DU}} + \hat{\sigma}^{\mathrm{C2}}_{\mathrm{DU}}$

Separation of finite contributions

The separation into single- and double-unresolved does not work naively because of

$$\hat{\sigma}^{\mathrm{RR}}_{\mathrm{SU}} \qquad \hat{\sigma}^{\mathrm{RR}}_{\mathrm{DU}}$$

They must be made separately finite, because they have Different dimensionality phase spaces in HV regularization

The most obscure feature of the scheme

List of corrections provided in the paper

‘t Hooft-Veltman regularization

Facts:

1) The finiteness does not depend on the functional form of the matrix elements

2) The integral over epsilon-dimensional volume of a finite function must be of order epsilon

Consequences:

1) Drop all epsilon expansion terms of the matrix elements

2) Restrict the phase space with delta-functions

't Hooft-Veltman regularization

Phase space restriction achieved with the measurement function

$$\mathrm{F}_n \longrightarrow \mathrm{F}_n \left(\frac{\mu_R^2 e^{\gamma_E}}{4\pi}\right)^{-(n-1)\epsilon} \left[\prod_{i=1}^{n-1} (2\pi)^{-2\epsilon} \delta^{(-2\epsilon)}(q_i)\right]$$

Non-trivial consequences:

Maximally unresolved contributions:

$$\delta^{(-2\epsilon)}(r+u) = \left(r^0+u^0\right)^{2\epsilon} \delta^{(-2\epsilon)}(\hat{r})$$

for the single-collinear sector,

$$\delta^{(-2\epsilon)}(r+u_1+u_2) = \left(r^0+u_1^0+u_2^0\right)^{2\epsilon} \delta^{(-2\epsilon)}(\hat{r})$$

for the triple-collinear sector,

$$\delta^{(-2\epsilon)}(r_1+u_1)\,\delta^{(-2\epsilon)}(r_2+u_2) = \left[\left(r_1^0+u_1^0\right)\left(r_2^0+u_2^0\right)\right]^{2\epsilon} \delta^{(-2\epsilon)}(\hat{r}_1)\,\delta^{(-2\epsilon)}(\hat{r}_2)$$

for the double-collinear sector.

Single unresolved contributions in the triple-collinear sector

$\delta^{(-2\epsilon)}(r+u_1)\,\delta^{(-2\epsilon)}(u_2)$ for the collinear pole in η_1 in sector $\mathcal{S}_3$,

$\delta^{(-2\epsilon)}(r+u_2)\,\delta^{(-2\epsilon)}(u_1)$ for the collinear pole in η_2 in sector $\mathcal{S}_1$,

$\delta^{(-2\epsilon)}(r)\,\delta^{(-2\epsilon)}(u_1+u_2)$ for the collinear pole in η_1 in sector $\mathcal{S}_5$, and in η_2 in sector $\mathcal{S}_4$,

$\delta^{(-2\epsilon)}(r)\,\delta^{(-2\epsilon)}(u_1)$ for the soft pole in ξ_2 in sectors $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_4$ and $\mathcal{S}_5$.

It does work

Example: gg -> tt~ + ng, n=0,1,2

	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
$\tilde{\sigma}_{\text{DU}}^{\text{VV}}$	0.0321959	0.135003	0.177418	0.04517	−0.1242
$\tilde{\sigma}_{\text{DU}}^{\text{RV}}$	−0.0724423(9)	−0.456495(4)	−1.196150(11)	−1.81962(4)	−2.8562(1)
$\tilde{\sigma}_{\text{DU}}^{\text{RR}}$	0.0402448(2)	0.321486(1)	1.045064(6)	1.61821(4)	1.3065(3)
$\tilde{\sigma}_{\text{DU}}^{\text{C1}}$		−0.154649(4)	−0.447655(20)	0.09385(8)	1.8313(2)
$\tilde{\sigma}_{\text{DU}}^{\text{C2}}$		0.154650	0.421336	0.06247	−0.1878
$\tilde{\sigma}_{\text{DU}}^{\text{CDR}}$	−0.0000016(9)	−0.000005(6)	0.000013(24)	0.00007(9)	−0.0304(4)

	$1/\epsilon^4$	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
$\tilde{\sigma}_{\text{DU}}^{\text{VV}}$	0.0321959	0.086177	0.021985	−0.03200	0
$\tilde{\sigma}_{\text{DU}}^{\text{RV}}$	−0.0724415(9)	−0.346630(3)	−0.702124(8)	−1.04640(3)	−2.3910(1)
$\tilde{\sigma}_{\text{DU}}^{\text{RR}}$	0.0402447(2)	0.260452(1)	0.706469(6)	1.06119(3)	1.8461(2)
$\tilde{\sigma}_{\text{DU}}^{\text{C1}}$		−0.154646(4)	−0.283008(15)	0.08326(5)	0.5144(1)
$\tilde{\sigma}_{\text{DU}}^{\text{C2}}$		0.154650	0.256668	−0.06603	0
$\tilde{\sigma}_{\text{DU}}^{\text{HV}}$	−0.0000009(9)	0.000003(6)	−0.000010(17)	0.00002(6)	−0.0304(2)

It does work

Example: gg -> tt~ + ng, n=0,1,2

	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{RR}}$	0.064772(4)	0.42742(3)	1.0623(3)
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{RV}}$	−0.064780(6)	−0.31419(4)	−0.6044(2)
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{C1}}$		−0.11329(3)	−0.1999(1)
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{A}}$			−0.00737(2)
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{CDR}}$	−0.000008(8)	−0.00006(6)	0.2506(3)

	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{RR}}$	0.064780(5)	0.25429(3)	0.2584(2)
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{RV}}$	−0.064770(7)	−0.14096(2)	0
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{C1}}$		−0.11329(2)	0
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{A}}$			−0.00734(1)
$\tilde{\sigma}_{\mathrm{SU}}^{\mathrm{HV}}$	0.000011(8)	0.00004(4)	0.2511(2)

…even for distributions

Resolving the Tevatron top quark forward-backward asymmetry puzzle

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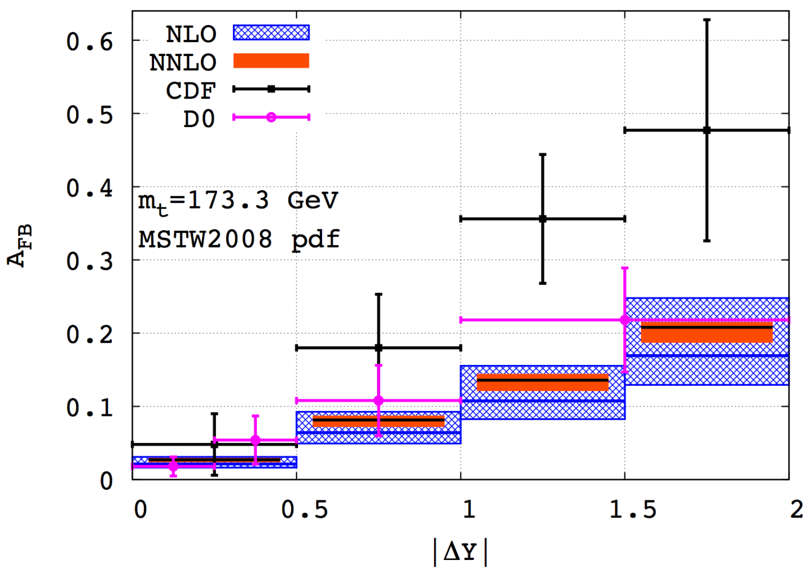
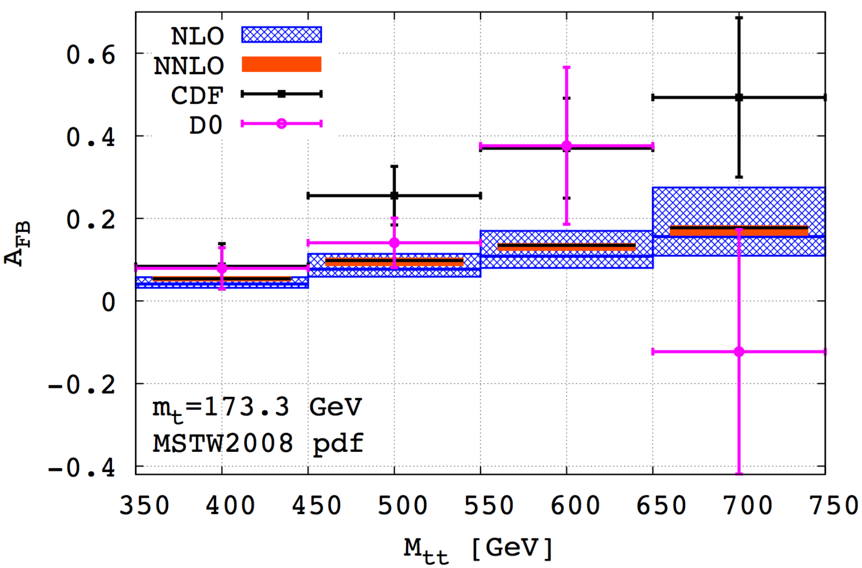
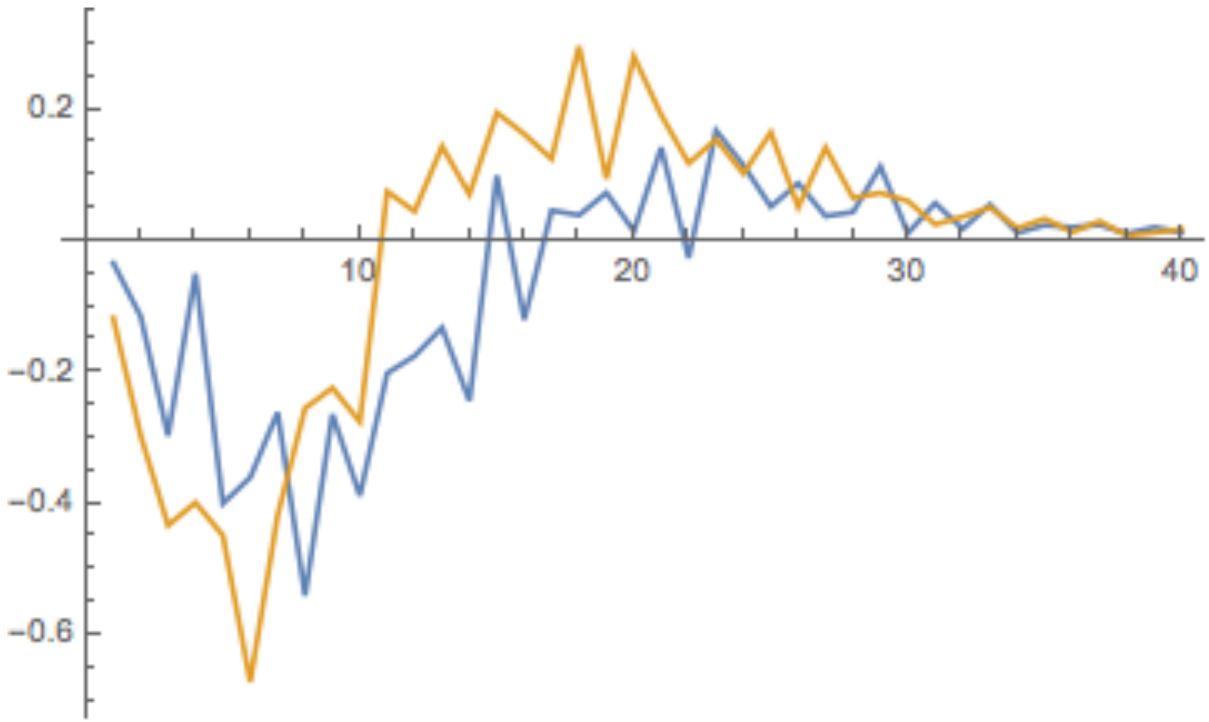




Secret plot



Current project

C++ software with the complete implementation of the scheme

Born level amplitudes contained thanks to a FORTRAN95 library
by Andreas van Hameren

The user must provide the one-loop amplitudes

- (n+1)-point function
- n-point function spin and color correlated
- n-point function squared (note below)
- on the technical side:
  1) scale dependence separately
  2) everything as finite remainders

Would be a dream to have a standard software built in, but beware of high requirements on stability (much higher than at NLO) of the (n+1)-point function

The user may provide the two-loop finite remainder
However: the two-loop amplitude and the one-loop squared have no kinematic singularities at NNLO, therefore can be treated separately of our software

Current project

Many lessons learned from

HELAC-DIPOLES, MC, Papadopoulos, Worek '09

Current project

Goodies already available:

- Decays, lepton colliders, lepton-hadron colliders, hadron-hadron colliders
- Simultaneous evaluation for all PDFs and scales (fixed, dynamic)
- 1d, 2d, variable bin size histograms with/without smearing
- Monte Carlo over processes
- Monte Carlo over polarizations
  - the scheme uses factorization of polarized amplitudes
- Improvements of convergence,
  - in particular missed binning avoidance in integrated subtraction terms

Planned:

- Narrow width approximation decays of top quarks
  - Interface partly built-in already

To be made publicly available next year!!!