Next-to-leading order evolution of color dipole

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JLAB & ODU

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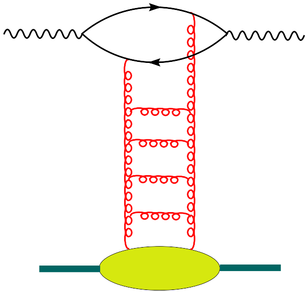
Otuline

- DIS from nucleus at high energy and Wilson line.
- Evolution equation.
- Leading order: BK equation.
- Non linear evolution equation in the NLO.
- NLO kernel.
- $\mathcal{N} = 4$ NLO kernel.
- Conclusions.
- Outlook.

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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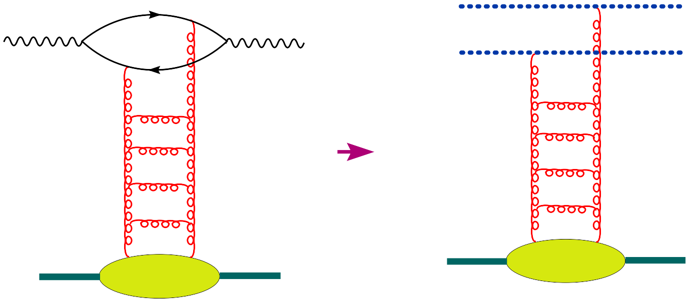

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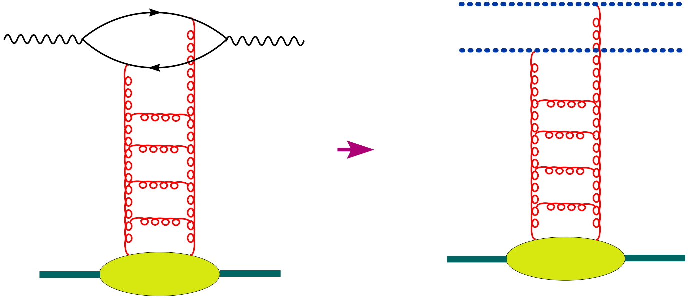

$$A(s) = \int \frac{d^2k_\perp}{4\pi^2} I^A(k_\perp) \langle B|\mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\}|B\rangle + \dots$$

$$U(x_\perp) = P e^{ig\int_{-\infty}^{\infty} du\ n^\mu A_\mu(un+x_\perp)}$$ Wilson line

DIS at high energy

The structure function of a hadron is proportional to a matrix element of this color dipole operator

$$\hat{\mathcal{U}}^{\eta}(x_{\perp}, y_{\perp}) = 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}^{\eta}(x_{\perp})\hat{U}^{\dagger\eta}(y_{\perp})\}$$

switched between the target's states ($N_c = 3$ for QCD). The gluon parton density is approximately:

$$x_B G(x_B, \mu^2 = Q^2) \;\simeq\; \langle p|\, \hat{\mathcal{U}}^{\eta}(x_{\perp}, 0)|p\rangle\Big|_{x_{\perp}^2 = Q^{-2}}$$

where $\eta = \ln \frac{1}{x_B}$

Propagation in the shock wave: Wilson line

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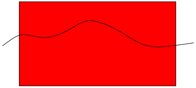

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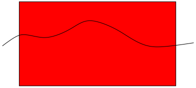

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Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

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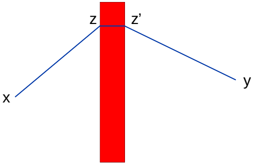

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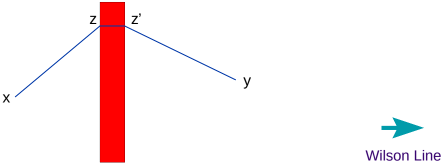

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$$[x, y] = Pe^{ig\int_0^1 du(x-y)^\mu A_\mu(ux+(1-u)y)}$$

Evolution Equation

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

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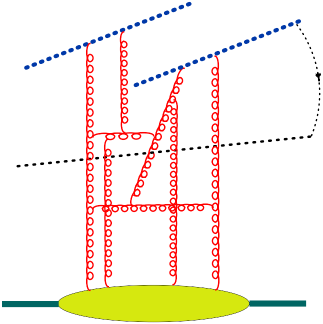

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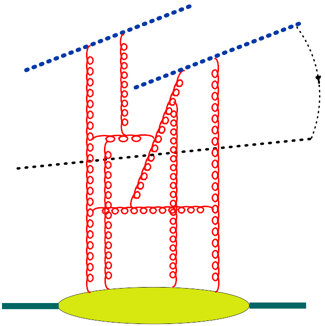

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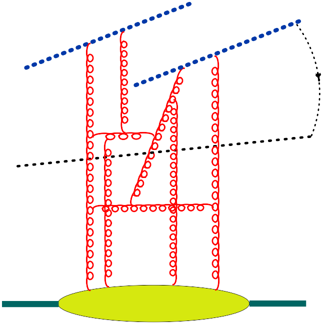
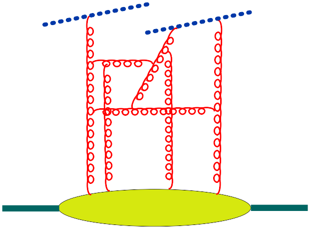


$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \otimes$$



Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$

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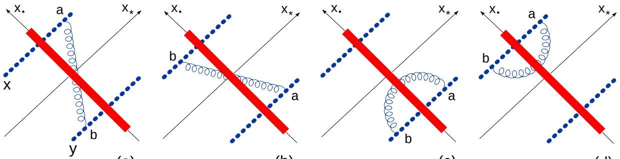

(a) (b) (c) (d)

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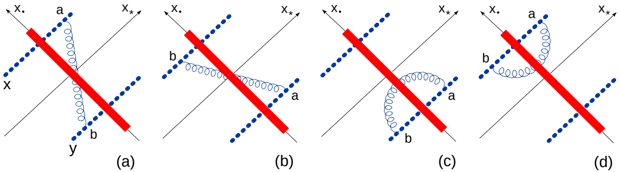

[$x \longrightarrow z$: free propagation]$\times$

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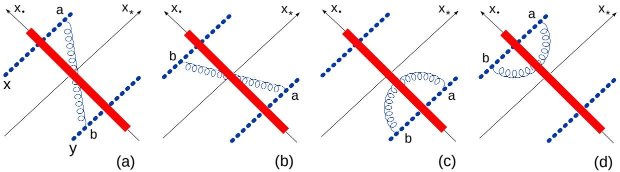

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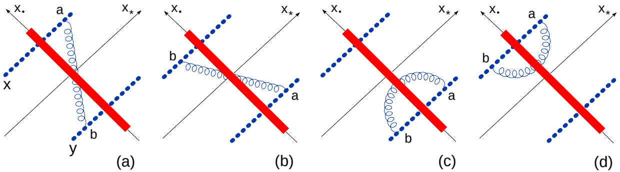

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[$z \longrightarrow y$: free propagation]

Non linear evolution equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

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BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\left\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\}$$

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1,\ \alpha_s^2 A^{1/3} \sim 1$)

(s for semiclassical)

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- To check conformal invariance in $\mathcal{N} = 4$ SYM.

Non linear evolution equation in the NLO

$$\frac{d}{d\eta}Tr\{U_xU_y^\dagger\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s\frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_xU_z^\dagger\}Tr\{U_zU_y^\dagger\} - N_c Tr\{U_zU_y^\dagger\}] +$$
$$\alpha_s^2\int d^2zd^2z'\left(K_4(x,y,z,z')\{U_x,U_{z'}^\dagger,U_z,U_y^\dagger\} + K_6(x,y,z,z')\{U_x,U_{z'}^\dagger,U_{z'},U_z,U_z^\dagger,U_y^\dagger\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K4 and K6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + K_{\mathrm{NLO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \dots$$

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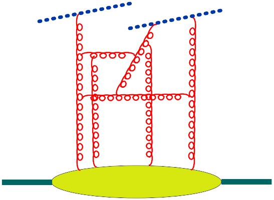
Subtraction of BK^2 contribution $\Longrightarrow$ $\left[\frac{1}{u}\right]_+$ prescription

$$\int_0^1 du\, f(u)\left[\frac{1}{u}\right]_+ \equiv \int_0^1 du\, \frac{f(u)-f(0)}{u}\,, \qquad \int_0^1 du\, f(u)\left[\frac{1}{\bar{u}}\right]_+ \equiv \int_0^1 du\, \frac{f(u)-f(1)}{\bar{u}}$$

Regularization of the rapididty divergence



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

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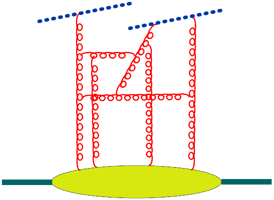

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Regularization by: slope

$$U^\eta(x_\perp) = \mathrm{Pexp}\left\{ig\int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\right\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

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Regularization by:Rigid cut-off

$$U_x^\eta = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\; p_1^\mu A_\mu^\eta(up_1 + x_\perp)\Big]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

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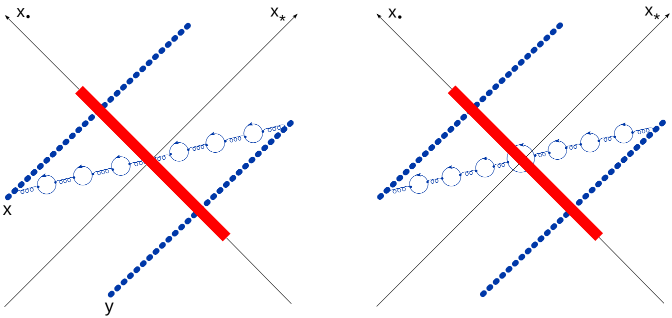
The rigid cut-off leads to (almost) conformal result

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s(?)N_c}{2\pi^2}\int \frac{dz(\vec{x}-\vec{y})^2}{(\vec{x}-\vec{z})^2(\vec{z}-\vec{y})^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

I. Balitsky (2006) **Renormalon-based approach: summation of quark bubbles**

$$-\frac{2}{3}n_f \to b = \frac{11}{3}N_c - \frac{2}{3}n_f$$



Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}]$$

$$\times\Big[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\Big(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\Big) + \frac{1}{Y^2}\Big(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\Big)\Big] + \ldots$$

I.Balitsky; Yu. Kovchegov and H. Weigert (2006)

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}]$$
$$\times\left[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\left(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\right) + \frac{1}{Y^2}\left(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\right)\right] + \ldots$$

I.Balitsky; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s((x-y)^2)}{2\pi^2}\frac{(x-y)^2}{X^2Y^2} \qquad |x-y| \ll |x-z|, |y-z|$$

$$\frac{\alpha_s(X)^2)}{2\pi^2X^2} \qquad |x-z| \ll |x-y|, |y-z|$$

$$\frac{\alpha_s(Y)^2)}{2\pi^2Y^2} \qquad |y-z| \ll |x-y|, |x-z|$$

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} \;=\; \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z\,[\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\}\mathrm{Tr}\{\hat{U}_z\hat{U}^\dagger_y\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}]$$
$$\times\Big[\frac{(x-y)^2}{X^2Y^2} + \frac{1}{X^2}\Big(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)} - 1\Big) + \frac{1}{Y^2}\Big(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)} - 1\Big)\Big] + \ldots$$

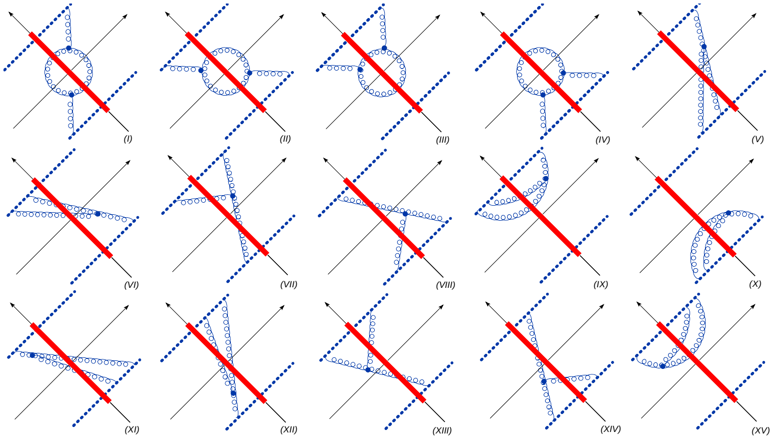
I.Balitsky; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

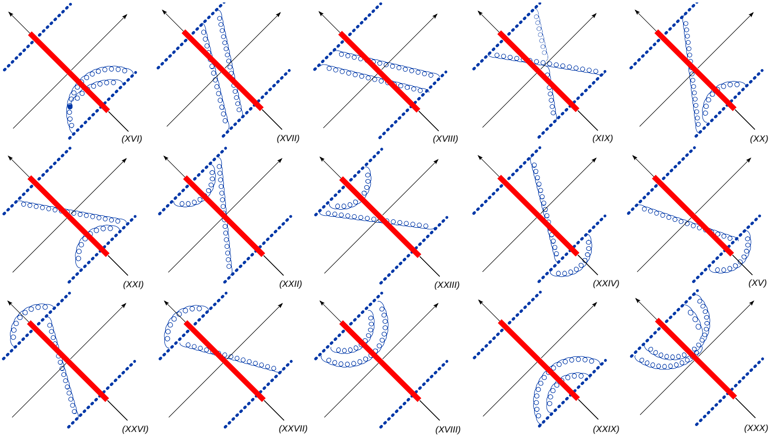
$\frac{\alpha_s((x-y)^2)}{2\pi^2}\frac{(x-y)^2}{X^2Y^2}$	$\lvert x-y\rvert \ll \lvert x-z\rvert, \lvert y-z\rvert$
$\frac{\alpha_s(X)^2)}{2\pi^2X^2}$	$\lvert x-z\rvert \ll \lvert x-y\rvert, \lvert y-z\rvert$
$\frac{\alpha_s(Y)^2)}{2\pi^2Y^2}$	$\lvert y-z\rvert \ll \lvert x-y\rvert, \lvert x-z\rvert$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

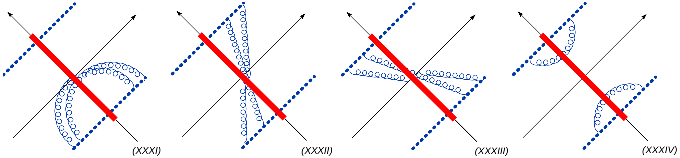
Diagrams with 2 gluons interaction



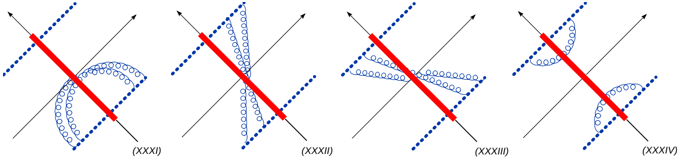
Diagrams with 2 gluons interaction



Diagrams with 2 gluons interaction



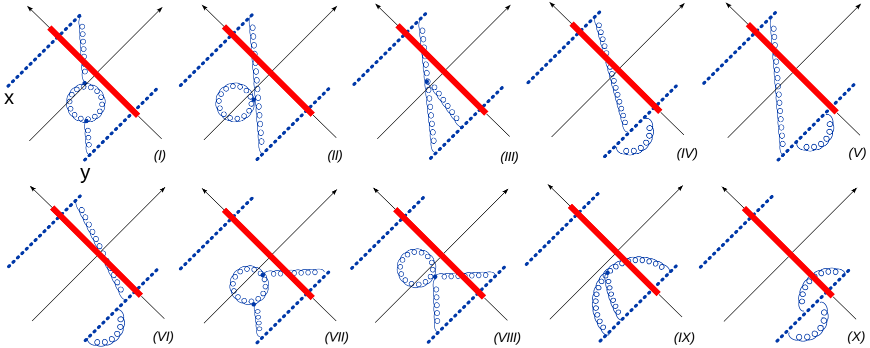
Diagrams with 2 gluons interaction



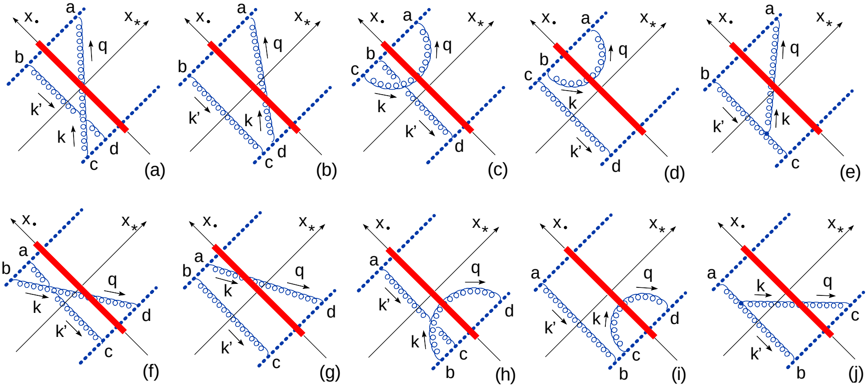
Extracting the UV divergencies

$$\mathrm{Tr}\{t^a U_z t^b U^\dagger_{z'}\} = \mathrm{Tr}\{t^a U_z t^b U^\dagger_{z'} - t^a U_z t^b U^\dagger_z\} + \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\}$$

I.Balitsky 2006; Y. Kovchegov, H. Weigert 2006

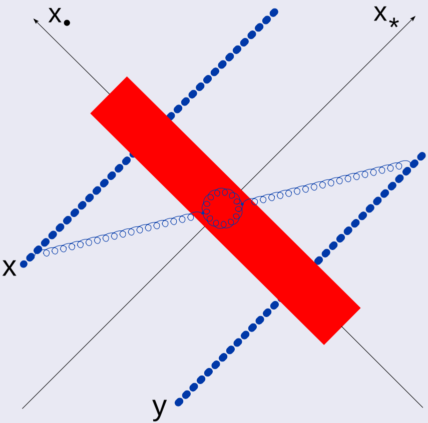

x
y
(I)
(II)
(III)
(IV)
(V)
(VI)
(VII)
(VIII)
(IX)
(X)

1 → 2 dipole transition diagrams



Quark and gluon loop in the shock wave

typical diagram: gluon in the shock wave



Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} \;=\; &\frac{\alpha_s}{2\pi^2}\int d^2z\,\Big(\big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}\big]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]\\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{\big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
&-(z'\to z)\big]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
&+\big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)\big]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}]\\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]\\
& -\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
& +[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)]\\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =
\end{aligned}
$$

Running coupling part

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big(\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}\big] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& -\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} \\
& -(z'\to z)\big]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +\big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)\big] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =
\end{aligned}
$$

Running coupling part + Non-conformal part

$$\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z \Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}]$$
$$\times \Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{4\pi}(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3})\Big]$$
$$-\frac{11}{3}\frac{\alpha_s N_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$
$$+\frac{\alpha_s}{4\pi^2}\int d^2z' \Big\{[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\}$$
$$-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]$$
$$+[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z'\to z)]$$
$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =$$

Running coupling part + Non-conformal part + Conformal "non-analytic" part

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}]$$

$$-(z'\to z)\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]$$

$$+[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)]$$

$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) =$$

Running coupling part $+$ Non-conformal part $+$ Conformal "non-analytic" part $+$ "conformal-analytic" ($\mathcal{N}=4$) part

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} \;=\; &\frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-N_c\mathrm{Tr}\{U_xU^\dagger_y\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3})\Big]\\
&-\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}-\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
&-(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big]\\
&+[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\}-(z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big)+\frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_xU^\dagger_y\}=
\end{aligned}
$$

Our result + Extra term $\Rightarrow$ Agrees with NLO BFKL

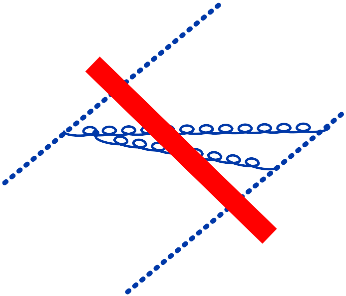
(Comparing the eigenvalue of the forward kernel)

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

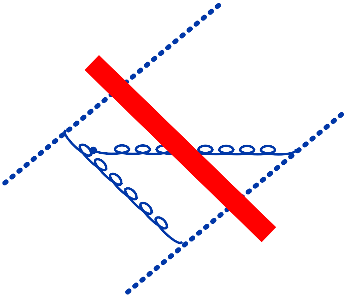
$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}] \\
& \times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(\frac{11}{3}\ln(x-y)^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big] \\
& -\frac{11}{3}\frac{\alpha_sN_c}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2} - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& +\frac{\alpha_s}{4\pi^2}\int d^2z'\,\Big\{[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} \\
& -(z'\to z)]\frac{1}{(z-z')^4}\Big[-2+\frac{X'^2Y^2+Y'^2X^2-4(x-y)^2(z-z')^2}{2(X'^2Y^2-Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big] \\
& +[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)] \\
& \times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\Big) + \frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_xU_y^\dagger\}
\end{aligned}
$$

However the term $\frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{TrU_xU_y^\dagger}$ contradicts the requirement $\frac{d}{d\eta}U_xU_Y^\dagger = 0$ at $x=y$.

Conformal and non-conformal diagrams



Conformal



Non−Conformal

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}$$

$$= \frac{\alpha_s}{2\pi^2}\int d^2z\,\frac{(x-y)^2}{X^2Y^2}\Big\{1+\frac{\alpha_s}{4\pi}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}+(\frac{67}{9}-\frac{\pi^2}{3})N_c-\frac{10}{9}n_f$$

$$-2N_c\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}\,[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\}-N_c\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}]$$

$$+\frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\Big[\Big(-\frac{4}{(z-z')^4}+\Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}$$

$$+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)$$

$$\times\,[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_{z'}^\dagger\}\mathrm{Tr}\{\hat{U}_{z'}\hat{U}_y^\dagger\}-\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\hat{U}_{z'}U_y^\dagger\hat{U}_z\hat{U}_{z'}^\dagger\}-(z'\to z)]$$

$$+\Big\{\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\,\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_{z'}^\dagger\}\mathrm{Tr}\{\hat{U}_{z'}\hat{U}_y^\dagger\}$$

$$+4n_f\Big\{\frac{4}{(z-z')^4}-2\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\mathrm{Tr}\{t^a\hat{U}_xt^b\hat{U}_y^\dagger\}[\mathrm{Tr}\{t^a\hat{U}_zt^b\hat{U}_{z'}^\dagger\}-(z'\to z)]\Big]$$

Gluon-loop in the shock wave

From NLO BK kernl

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi''(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+4\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

Gluon-loop in the shock wave

From NLO BK kernl

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+4\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

From NLO BFKL kernel

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
$$+\frac{\alpha_s b}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]$$
$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+6\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

Gluon-loop in the shock wave

From NLO BK kernl

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
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$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+4\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

From NLO BFKL kernel

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = \frac{\alpha_s N_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)$$
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$$+\frac{\alpha_s N_c}{4\pi}\Big[-\chi"(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+6\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle$$

The coincidence of terms with the nontrivial γ dependence proves that there is no additional $O(\alpha_s)$ correction to the vertex of the gluon-shock wave interaction coming from the small loop inside the shock wave

About the discrepancy

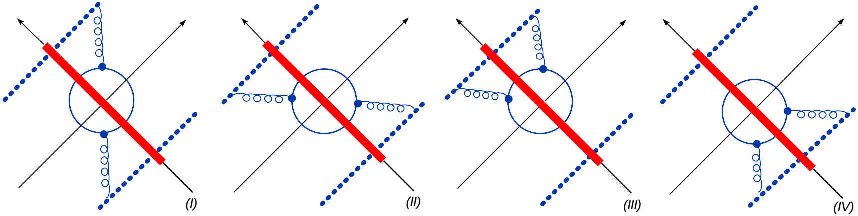
From BK NLO side

The extra term $\frac{\alpha_s^2 N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\mathrm{U}_\mathrm{x}\mathrm{U}_\mathrm{y}^\dagger$ would contradict the requirement $\frac{d}{d\eta}U_x U_Y^\dagger = 0$ at $x = y$

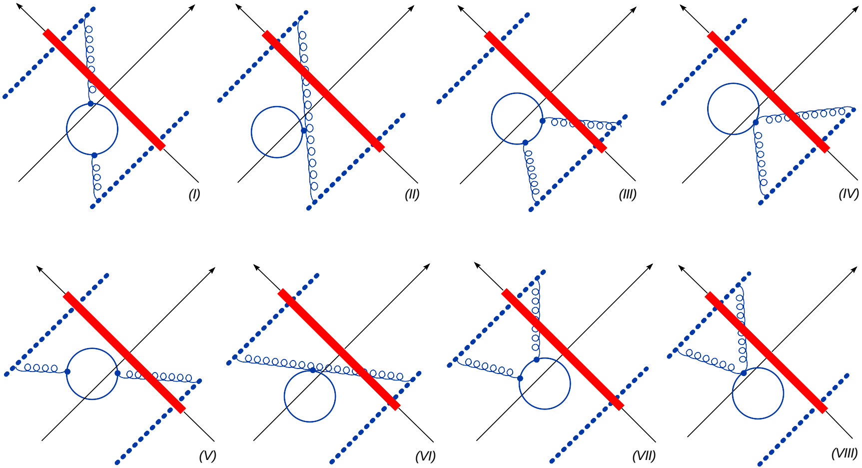
From BFKL NLO side

The coefficient $6\zeta(3)$ agrees with the $j \to 1$ asymptotics of the three-loop anomalous dimensions of leading-twist gluon operators

$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



$\mathcal{N} = 4$ diagrams (scalar and gluino loops)



Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}]\\
\times & \Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{12\pi}(1-\pi^2)\Big] - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
+ & \frac{\alpha_s}{16\pi^2}\int d^2z'\;[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
& -\mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)]\\
\times & \Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)
\end{aligned}
$$

Evolution equation in $\mathcal{N} = 4$ (I. Balitsky and G.A.C)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU^\dagger_y\} &= \frac{\alpha_s}{2\pi^2}\int d^2z\ \Big([\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\} - N_c\mathrm{Tr}\{U_xU^\dagger_y\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{12\pi}(1-\pi^2)\Big] - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{16\pi^2}\int d^2z'\ [\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\{U_{z'}U^\dagger_y\} - \mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
&\qquad -\mathrm{Tr}\{U_xU^\dagger_{z'}U_zU^\dagger_yU_{z'}U^\dagger_z\} - (z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)
\end{aligned}
$$

Conformal scheme-dependent part

Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-N_c\mathrm{Tr}\{U_xU_y^\dagger\}]\\
&\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{12\pi}(1-\pi^2)\Big]-\frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}\\
&+\frac{\alpha_s}{16\pi^2}\int d^2z'\;[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\}-\mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}\\
&\qquad-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\}-(z'\to z)]\\
&\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)
\end{aligned}
$$

Conformal scheme-dependent part $+$ Non-conformal part

Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int d^2z\,\Big([\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]$$

$$\times\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_sN_c}{12\pi}(1-\pi^2)\Big] - \frac{\alpha_sN_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}$$

$$+\frac{\alpha_s}{16\pi^2}\int d^2z'\ [\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}$$

$$-\mathrm{Tr}\{U_xU_{z'}^\dagger U_zU_y^\dagger U_{z'}U_z^\dagger\} - (z'\to z)]$$

$$\times\Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)$$

Conformal scheme-dependent part $+$ Non-conformal part
$+$ Conformal analitic part

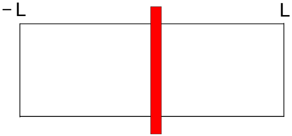
Evolution equation in $\mathcal{N}=4$ (I. Balitsky and G.A.C)

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{U_x U_y^\dagger\} \;=\; & \frac{\alpha_s}{2\pi^2}\int d^2z \,\Big([\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c \mathrm{Tr}\{U_x U_y^\dagger\}] \\
& \times \Big\{\frac{(x-y)^2}{X^2Y^2}\Big[1+\frac{\alpha_s N_c}{12\pi}(1-\pi^2)\Big] - \frac{\alpha_s N_c}{2\pi}\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\} \\
& + \frac{\alpha_s}{16\pi^2}\int d^2z' \;[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_{z'}^\dagger\}\{U_{z'} U_y^\dagger\} - \mathrm{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \\
& \qquad - \mathrm{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \to z)] \\
& \times \Big[\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2 - X'^2Y^2)} + \frac{(x-y)^2}{(z-z')^2X^2Y'^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) + \frac{\alpha_s^2 N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}\{U_x U_y^\dagger\} =
\end{aligned}
$$

Our result $+$ Extra term $\Rightarrow$ Agrees with NLO BFKL in $\mathcal{N}=4$
Lipatov and Kotikov, 2004

(Comparing the forward kernels)

We have checked the possible missing term using gauge/scalar links at infinity



Gauge/scalar links

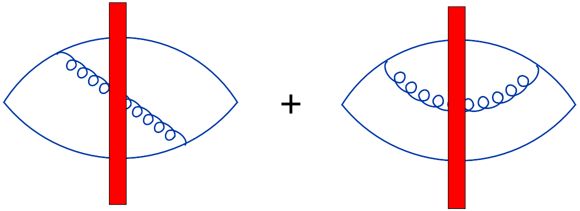
$$\mathrm{Tr}\{U_x U_y^\dagger\} = \lim_{L\to\infty} \mathrm{Tr}\{[Lp_1 + x_\perp, -Lp_1 + x_\perp][-Lp_1 + x_\perp, -Lp_1 + y_\perp] \times [-Lp_1 + y_\perp, Lp_1 + y_\perp][Lp_1 + y_\perp, Lp_1 + x_\perp]\}$$

Conclusions

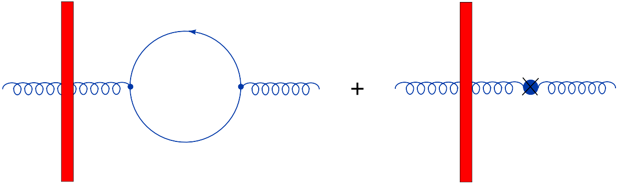
- The NLO kernel for the evolution of the color dipole has been calculated. It consists of three parts: the running-coupling part proportional to β-function, the conformal part describing $1 \to 3$ dipoles transition and the non-conformal term.
- The result agrees with the forward NLO BFKL kernel up to a term proportional $\alpha_s^2\zeta(3)$ times the original dipole.
- For the creation of dipoles in the small-x evolution, the coupling constant is determined by the size of the smallest dipole.
- The NLO evolution kernel depends on the precise definition of the cutoff in the longitudinal momenta.
- With $|\alpha| < \sigma$ cutoff, the NLO-BK and the NLO-BFKL for $\mathcal{N} = 4$ is (almost) conformally invariant in the transverse plane.

Outlook

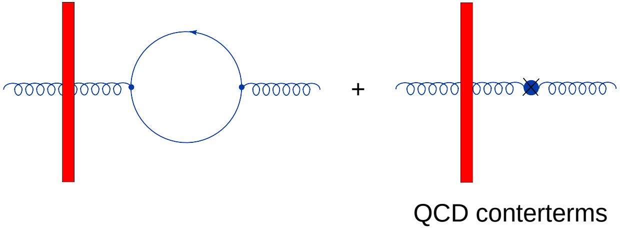
- Comparing the forward kernels also in the QCD case.
- Leading BK is conformal, what about K_{NLO} in the $\mathcal{N} = 4$?
- Checking the conformal invariance for the amplitude to the NLO SYM $\mathcal{N} = 4$.

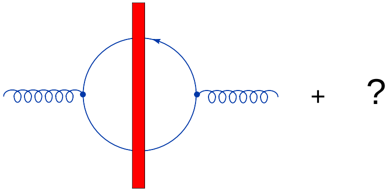


Quark loop in the shock wave



QCD conterterms

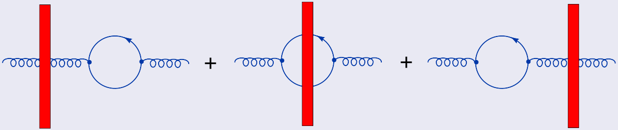




If we simply subtract the divergence using $\overline{MS}$, we could be missing some extra finite terms.

Balitsky 2006

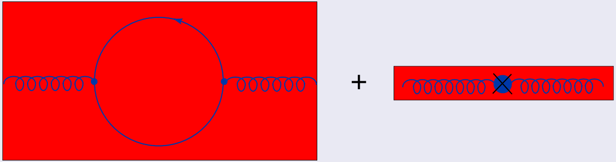
A way to get potential extra terms is to consider the exact calculation of the light-cone expansion of $U_x U_y^\dagger$ at $x_\perp \to y_\perp$ in QCD and compare it with the expansion of the sum of the diagrams:



Exact calculation means:

Consider the quark-loop contribution to the gluon propagator in external field

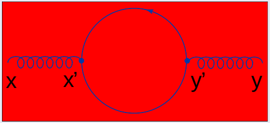
Calculate the light cone expansion of:



meaning...

Quark loop in the shock wave

Calculate the light cone expansion $(x-y)_\perp \to 0$ of:



=

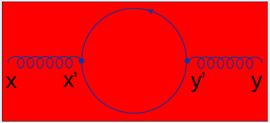
$$\langle A^m_\bullet(x) A^n_\bullet(y)\rangle_A = \int dx'dy' \left(x\left|\frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}}\right|x'\right)^{ma}$$

$$\times \mathrm{Tr}\left\{t^a\gamma_\mu\left(x'\left|\frac{1}{\not{P}}\right|y'\right)t^b\gamma_\nu\left(y'\left|\frac{1}{\not{P}}\right|x'\right)\right\}\left(y'\left|\frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}}\right|y\right)^{bn}$$

$$\mathcal{O}_{\bullet\mu} = \frac{1}{\alpha} D^i G_{i\mu}$$

Quark loop in the shock wave

Calculate the light cone expansion $(x-y)_\perp \to 0$ of:



=

$$\langle A^m_\bullet(x)A^n_\bullet(y)\rangle_A = \int dx'dy' \left(x\left|\frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}}\right|x'\right)^{ma}$$

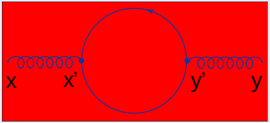
$$\times \mathrm{Tr}\left\{t^a\gamma_\mu\left(x'\left|\frac{1}{\not{P}}\right|y'\right)t^b\gamma_\nu\left(y'\left|\frac{1}{\not{P}}\right|x'\right)\right\}\left(y'\left|\frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}}\right|y\right)^{bn}$$

$$\mathcal{O}_{\bullet\mu} = \frac{1}{\alpha}D^iG_{i\mu}$$

then take the limit $d_\perp \to 2$ (which gives a finite result after adding the conterterms),

Quark loop in the shock wave

Calculate the light cone expansion $(x-y)_\perp \to 0$ of:



=

$$\langle A^m_\bullet(x)A^n_\bullet(y)\rangle_A = \int dx'dy' \left(x\left|\frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}}\right|x'\right)^{ma}$$

$$\times \mathrm{Tr}\left\{t^a\gamma_\mu\left(x'\left|\frac{1}{\not P}\right|y'\right)t^b\gamma_\nu\left(y'\left|\frac{1}{\not P}\right|x'\right)\right\}\left(y'\left|\frac{1}{P^2 g_{\bullet\mu} + 2iG_{\bullet\mu} + \mathcal{O}_{\bullet\mu}}\right|y\right)^{bn}$$

$$\mathcal{O}_{\bullet\mu} = \frac{1}{\alpha}D^i G_{i\mu}$$

then take the limit $d_\perp \to 2$ (which gives a finite result after adding the conterterms), and finally impose the condition that the external field is very narrow by taking the limit $\Delta z_* \to 0$

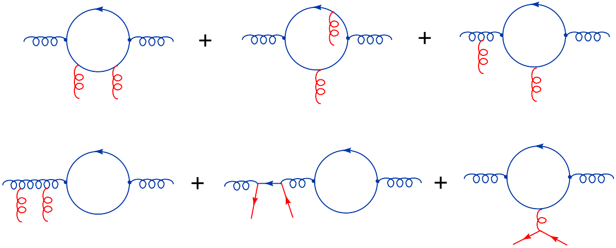
However...

Balitsky (2006) **Commutativity of the two limits**

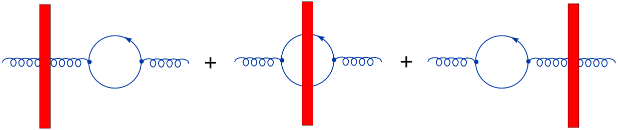
$$\left[d_\perp \to 2\ ,\ \Delta z_* \to 0\right] = 0$$

light-cone expansion

Typical diagrammatic expressin of the light-cone expansion of the quark-loop contribution to the gluon propagator in the external field is



Which coincides with the expansion of



⇒ no additional terms at one loop level.