High Energy Behavior in $N = 4$ SYM and the BDS formula

J.Bartels
II.Inst.f.Theor.Physik, Univ.Hamburg

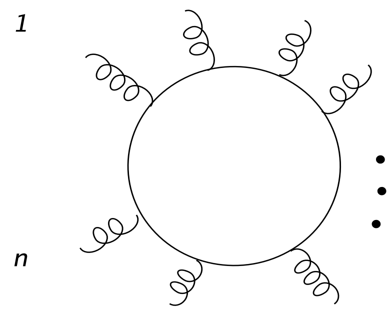
Low-x meeting, $K o \lambda \upsilon \mu \pi \alpha \rho i$, July 2008

- Introduction
- High Energy Behavior in Yang Mills Theories
- Comparison with the BDS formula (Bern, Dixon, Smirnov, Phys.Rev.D 72, 085001 (2005))
- Outlook: tasks

Based upon:
JB, Lev Lipatov, Agustin Sabio Vera, arXiv:0802.2065[hep-th]; 0807.0894[hep-th]

Introduction

MSYM Yang Mills (N=4, $SU(N_c)$: dual to AdS$_5$ String theory
proposal for n point amplitude, leading order N_c, all-order coupling constant, maximal helicity violating
Bern, Dixon, Kosover, Phys.Rev.D 72, 085001 (2005)



If correct: major breakthrough in QFT, even if not valid for QCD.
Could help, e.g., to compute NLLO correction to BFKL kernel.

Proposal of BDS:

Large-N_c (planar), MHV amplitudes.
Remove color factors

$$tr(T^{a_1}...T^{a_n}) + noncycl.perm$$

factor out tree amplitude:

$$A_n = A_n^{tree} \cdot M_n(\epsilon)$$

$$\ln M_n = \sum_l a^l \Big[\Big(f^{(l)}(\epsilon) I_n(l\epsilon) + F_n(0) \Big) + C^{(l)} + E_n^{(l)}[\epsilon] \Big]$$

$$a = \frac{N_c \alpha}{2\pi} (4\pi e^{-\gamma})^\epsilon, \;\; d = 4 - 2\epsilon$$

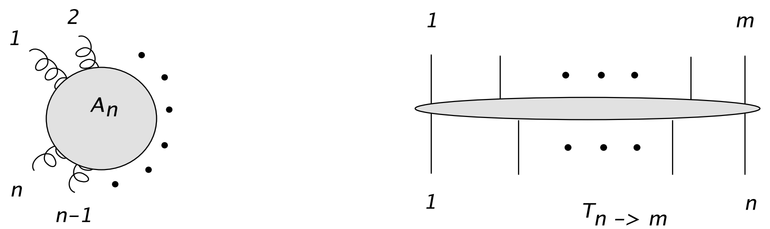
(based upon universality of IR singularities (=poles in ϵ) and 1-loop calculation).

On-shell amlitudes, infrared singular.

Goal: comparison of (known) high energy behavior of SYM with BDS formula:
discrepancy for $n > 5$, beyond one loop (also: Alday,Maldacena...)
related to imaginary parts

Restrict to leading logarithmic approximation: no distinction between (pure) $SU(N_c)$ and SYM Yang Mills (N=4, $SU(N_c)$, dual to AdS$_5$ String theory).

Simplest high energy limits: multiregge limit ($\longrightarrow$ total cross section).

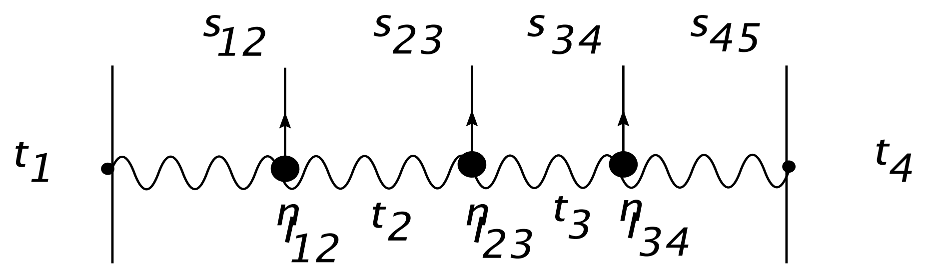


Simple ordering in rapidity. Analytic structure relatively simple.
This talk: high energy behavior in Yang-Mills, then few remarks on comparison with BDS.

Deeper into the theory: see Lev Lipatov's talk.

High energy behavior

Leading logarithmic approximation is real, e.g.:



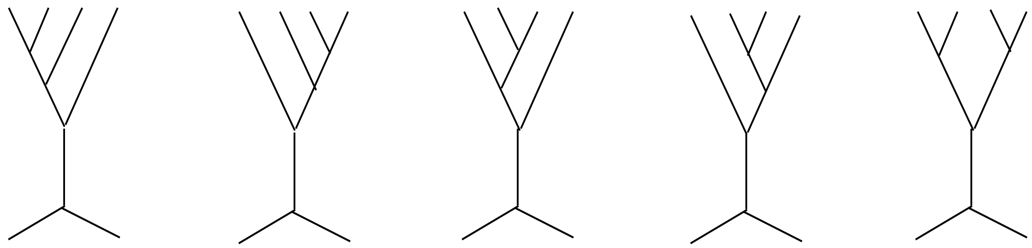
$$A_{2\to5} = 2s\,\beta^{(0)}(t_1)\,\delta_{\lambda_{A},\lambda_{A'}}\frac{s_{12}^{\omega(t_1)}}{t_1}\,\Gamma^{(0)}(t_1,t_2,\eta_{12})\,\frac{s_{23}^{\omega(t_2)}}{t_2}\ldots\frac{s_{45}^{\omega(t_4)}}{t_4}\,\beta^{(0)}(t_4)\delta_{\lambda_{B},\lambda_{B'}}$$

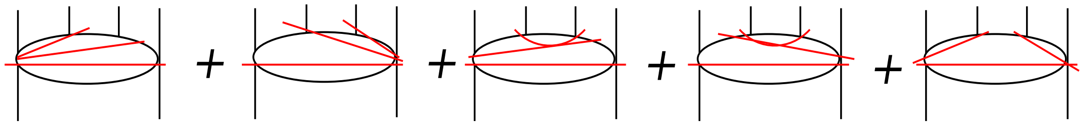
Simple factorization (exponentiation):

$$\ln M_7 = \ln\Gamma(t_1) + \omega(t_1)\ln s_{12} + \ln\Gamma(t_1,t_2,\eta) + \omega(t_2)\ln s_{23} + \ldots \ln\Gamma(t_4)$$

Problem of BDS resides in the imaginary parts - energy discontinuities:
independent energy variables .

Steinmann relations: 'no simultaneous discontinuities in overlapping channels'
Example: $2 \longrightarrow 4$, physical region (all energies positive)



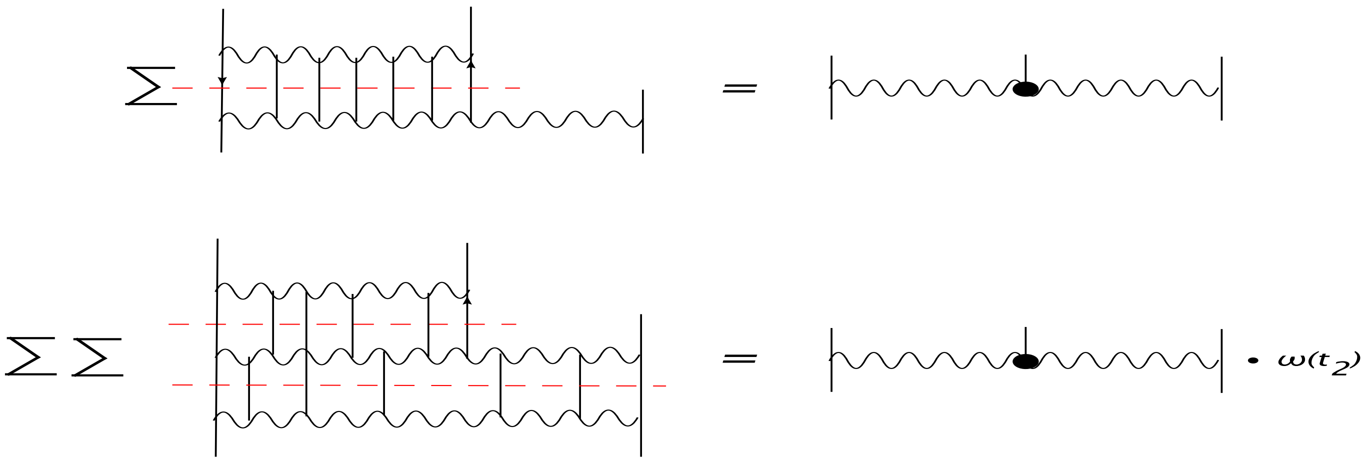
$A_6 =$ 

Amplitude $=$ sum of different triple discontinuties .
Analytic representation: all phases are in energy and signature factors.

Similarly $3 \longrightarrow 3$: 5 terms.

Number of terms grows: $2 \longrightarrow 5$: 14 terms etc.

Analytic representation can be used to compute all terms from (multiple) discontinuities.
(JB, Nucl.Phys.B 151 and B 175; Fadin,Lipatov, Nucl.Phys.406). Example:



Bootstrap relations: known from BFKL. Hold for inelastic amplitudes.

Bootstrap relations are valid beyond leading order.

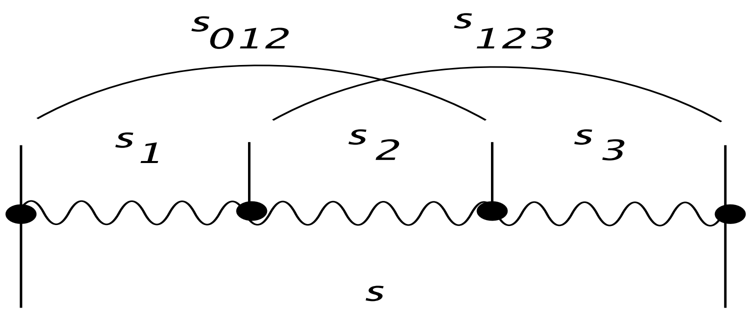
High degree of selfconsistency.

Results for QCD: five partial waves, e.g. the first term

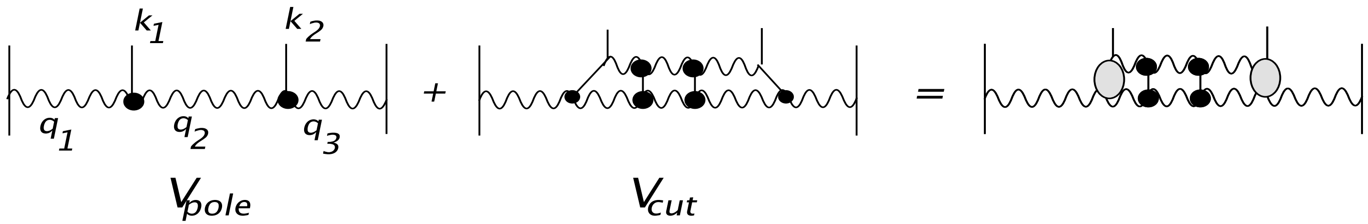
$$\frac{g^2 s}{t_1 t_2 t_3} \Big[\left(\frac{s_{12}}{\mu^2}\right)^{\omega(t_1)-\omega(t_2)} \left(\frac{s_{123}}{\mu^2}\right)^{\omega(t_2)-\omega(t_3)} \left(\frac{s}{\mu^2}\right)^{\omega(t_3)} \xi(t_1, t_2)\xi(t_2, t_3)\xi(t_3)\cdot$$

$$\frac{\omega(t_3)}{4} \left(\frac{a}{\epsilon} + \omega(t_1) - \omega(t_2) - a \ln \frac{\kappa_{12}}{\mu^2}\right) \cdot \left(\frac{a}{\epsilon} + \omega(t_2) - \omega(t_3) - a \ln \frac{\kappa_{23}}{\mu^2}\right) \Big]$$

Belongs to Regge pole picture:



New feature appears for terms 3 and 4:
contains not only gluon Regge pole but also Regge cut:



Combine the two contributions:

$$disc_{s_2} A_6 \sim s_2^{\omega(t_2)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s_2}{\mu^2}\right)^{\omega} \widetilde{f}_2(\omega),$$

$$\widetilde{f}_2(\omega) = \hat{\alpha}_\epsilon \, \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k \, d^{2-2\epsilon}k' \, \Phi_1(\boldsymbol{k}, \boldsymbol{q}_2, \boldsymbol{q}_1) \, \widetilde{G}_\omega(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q}_2) \, \Phi_3(\boldsymbol{k}', \boldsymbol{q}_2, \boldsymbol{q}_3) \, .$$

- Regge cut piece violates factorization
- Regge cut piece is present in several discontinuities, e.g. in total energy s, but not in all discontinuities.
- Regge cut piece present in all A_n with $n > 5$, e.g. $3 \to 3$.

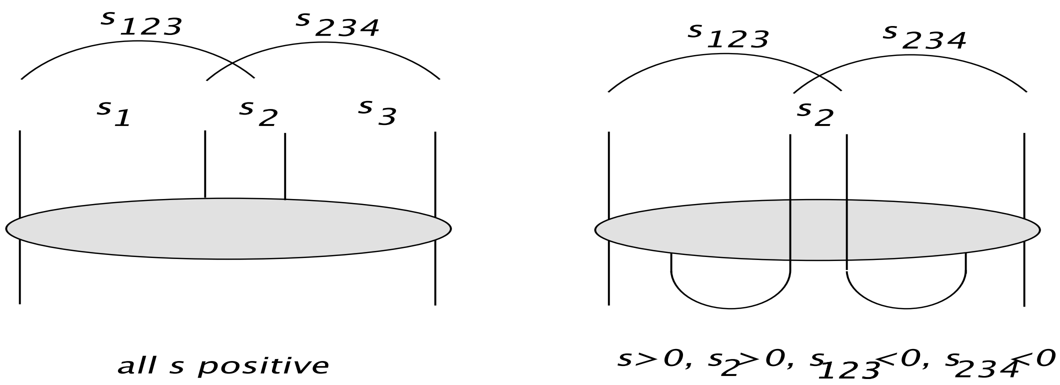
Sum the 5 different pieces and obtain the full scattering amplitudes A_n:

Leading order: many cancellations, real-valued expression factorizes (see above).

Sum of all imaginary parts ($=$ sum of discontinuities in different variables):
again substantial cancellations:

- in physical region (where all energies are positive),
  the Regge cut piece cancels, simple factorizing structure is valid .
- But: in another physical region $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$
  the cancellation of all imaginary parts is incomplete,
  Regge cut piece appears, factorization is violated .

Planar approximation: has only right hand cuts.
But still allows different physical regions:



Comparison with BDS formula

General strategy:
our analysis has been done for $\ln M$, discarding terms which vanish as $\epsilon \longrightarrow 0$.
Start from region where all invariants are negative, take multiregge limit.
Then, by analytic continuation, compare with previous result in different physical regions (all at large-N_c, MHV).

The four point amplitude:

$$\ln M_4 = 2 \ln \Gamma(t) + \omega(t) \ln(-s)/\mu^2$$

$$M_4 = \Gamma(t) \left(\frac{-s}{\mu^2} \right)^{\omega(t)} \Gamma(t)$$

- No squares of $\ln s$
- one loop expression for Γ
  and two-loop expression for $\omega(t)$ agree with explicit calculations
- exact: can also be written in 'dual' t-channel form (no high energy approximation).

The five point amplitude:

In $\ln M_5$: terms with squares of logarithms cancel. New production vertex:

$$M_{2\to 3} = \Gamma(t_1) \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma(t_2, t_1, \ln -\kappa) \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma(t_2)$$

with

$$-\kappa = \frac{(-s_1)(-s_2)}{(-s)}$$

Representation is exact.
Analytic continuation to positive energies:

$$-s \to e^{-i\pi} s, \quad \ln(-\kappa) \to \ln \kappa - i\pi, \quad \kappa = \boldsymbol{k}^2$$

Amplitude can be written in the analytic form:

$$\frac{M_{2\to 3}}{\Gamma(t_1)\Gamma(t_2)} = \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)-\omega(t_2)} \left(\frac{-s}{\mu^2}\right)^{\omega(t_2)} c_1 + \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)-\omega(t_1)} \left(\frac{-s}{\mu^2}\right)^{\omega(t_1)} c_2,$$

with real-valued functions c_1, c_2. Consistency check: the region $s_{12}, s_{23} < 0$.

The six point amplitude: $T_{2\to 4}$

In the unphysical region (all energies negative):

$$\frac{M_{2\to 4}}{\Gamma(t_1)\Gamma(t_3)} = \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma(t_2, t_1, \ln -\kappa_{12}) \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma(t_3, t_2, \ln -\kappa_{23}) \left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}$$

with

$$-\kappa_{12} = \frac{(-s_1)(-s_2)}{-s_{012}}, \quad -\kappa_{23} = \frac{(-s_2)(-s_3)}{-s_{123}}.$$

The same functions $\Gamma(t)$ and $\Gamma(t_1, t_2, \kappa)$ as before.

Analytic continuation: inconsistency appears .
Can be seen in several different ways:

(a) attempt to write as a sum of five terms with real-valued functions c_i (use also the other physical region: $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$): no solution for the c_i.

(b) comparison with the earlier QCD results: in the region $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$, one should see the Regge cut piece. The BDS formula yields:

$$C = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi \left(\ln\frac{(-t_1)(-t_3)}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right)\right]$$

$$\approx 1 + i\pi a\left(\ln\frac{(-t_1)(-t_3)}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right).$$

agrees with the one loop approximation to the Regge cut piece,
but BDS cannot reproduce the full Regge cut structure

Outlook: results and tasks

What has been achieved, by comparison with explicit QCD calculations:

- BDS ok for 4 and 5 point amplitude. Regge limit is even exact.
- subtle disagreement for M_n for $n \geq 6$ beyond one loop.
- in general, expect no simple exponential form. What instead?

Can we correct the formula? Reasons for being optimistic:

- many features of the BDS formula seem already to be correct (infrared and beyond)
- structure seen in the Regge limit may not be too far from general kinematics
- experience from analyzing QCD in Regge limit: structures seen in leading log (bootstrap, unitarity properties) may survive in higher order

$\epsilon\nu\ \alpha\rho\chi\grave{\eta}\ \tilde{\eta}\nu\ o\ \lambda\acute{o}\gamma o\sigma$: BDS

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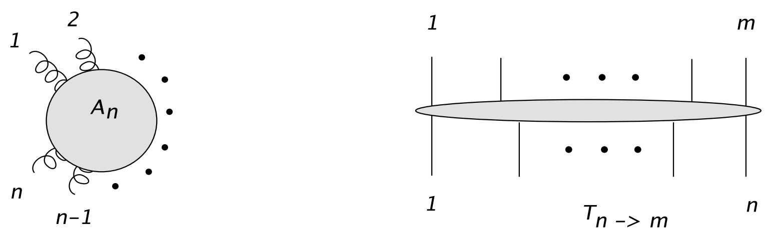
Introduction

Goal: comparison of (known) high energy behavior of SYM with BDS formula: discrepancy

Restrict to leading logarithmic approximation: no distinction between (pure) $SU(N_c)$ and SYM Yang Mills (N=4, $SU(N_c)$, dual to AdS$_5$ String theory).

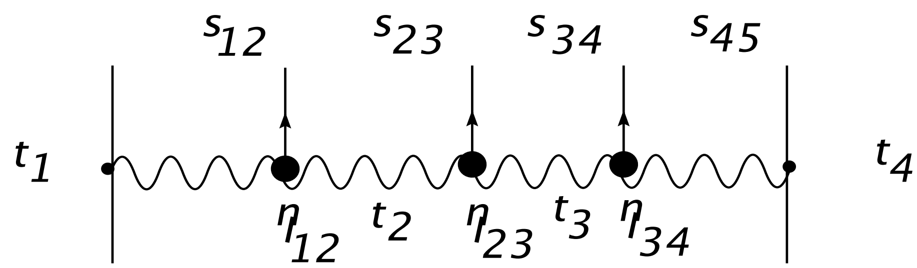
Simplest high energy limits: multiregge limit ($\longrightarrow$ total cross section).



Simple ordering in rapidity. Analytic structure relatively simple. Also: triple Regge limit.
This talk: first high energy behavior in Yang-Mills, then comparison with BDS.
Notation: scattering amplitude A_n, after removal of Born approximation M_n.

High energy behavior

Leading logarithmic approximation is real, e.g.:



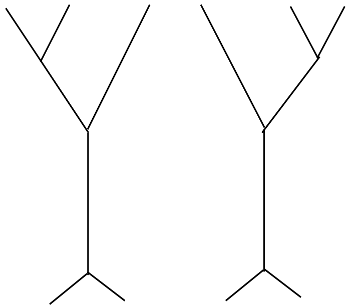
$$A_{2\to5} = 2s\,\beta^{(0)}(t_1)\,\delta_{\lambda_A,\lambda_{A'}}\frac{s_{12}^{\omega(t_1)}}{t_1}\,\Gamma^{(0)}(t_1,t_2,\eta_{12})\,\frac{s_{23}^{\omega(t_2)}}{t_2}\ldots\frac{s_{45}^{\omega(t_4)}}{t_4}\,\beta^{(0)}(t_4)\delta_{\lambda_B,\lambda_{B'}}$$

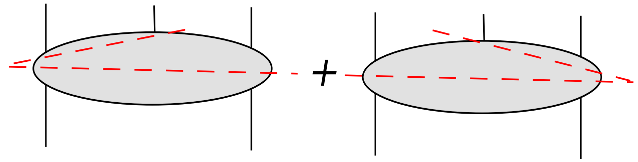
Simple factorization (exponentiation):

$$\ln M_7 = \ln\Gamma(t_1) + \omega(t_1)\ln s_{12} + \ln\Gamma(t_1,t_2,\eta) + \omega(t_2)\ln s_{23} + \ldots \ln\Gamma(t_4)$$

What about imaginary parts - energy discontinuities (belong still to leading log): independent energy variables?

Steinmann relations: 'no simultaneous discontinuities in overlapping channels'
Example:
$2 \longrightarrow 3$, in double Regge limit, in physical region $s \gg s_{12}, s_{23} > 0$, color octet exchange:



$A_5 =$ 

Singularities decouple at high energies.

History:
Axiomatic field theory; B_5 Veneziano amplitudes, scalar field theory, proper partial wave decomposition
(Steinmann; Brower et al, Gribov, W.Zakrzewski et al, A.White,....).

Analytic representation for positive energies:

$$A_5 = 2sg\beta(t_1)\delta_{\lambda\lambda'}\left(\frac{s_{12}^{\omega(t_1)-\omega(t_2)}s^{\omega(t_2)}\xi(t_1,t_2)\xi(t_2)}{t_2}V_R(t_1,t_2,\kappa)+\frac{s_{23}^{\omega(t_2)-\omega(t_1)}s^{\omega(t_1)}\xi(t_2,t_1)\xi(t_1)}{t_1}V_L(t_1,t_2,\kappa)\right)g\beta(t_2)\delta_{\lambda}$$

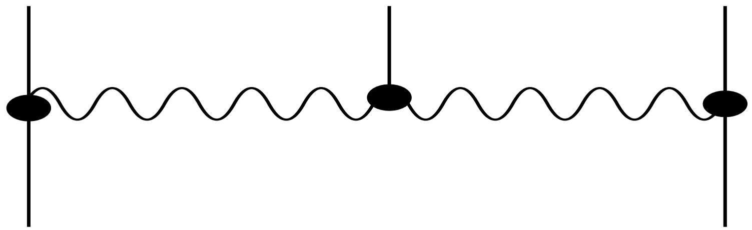
$$\xi(t) = 1+e^{-i\pi\omega(t)},\;\; \xi(t_1,t_2) = \frac{1+e^{-i\pi(\omega(t_1)-\omega(t_2))}}{(\omega(t_1)-\omega(t_2))}$$

Decomposition into sum of double discontinuties . All vertex functions are real-valued.

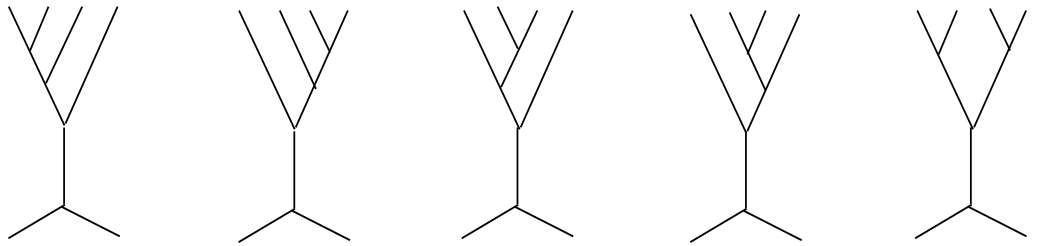
Can also be written in a factorized form:

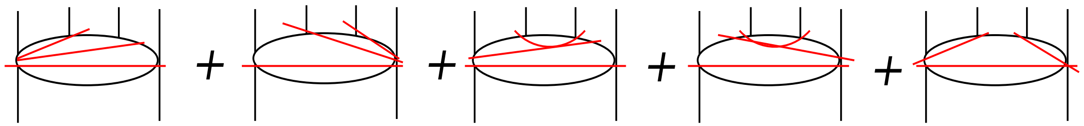
$$A_5 = 2sg\beta(t_1)\delta_{\lambda\lambda'}\left(\frac{|s_1|}{\mu^2}\right)^{\alpha(t_1)}\xi(t_1)\,V(t_1,t_2,\kappa)\,\left(\frac{|s_2|}{\mu^2}\right)^{\alpha(t_2)}\xi(t_2)\beta(t_2)\delta_{\lambda\lambda'}$$

In this representation there are phases inside the production vertex function V.



Second example: $2 \to 4$, physical region (all energies positive)



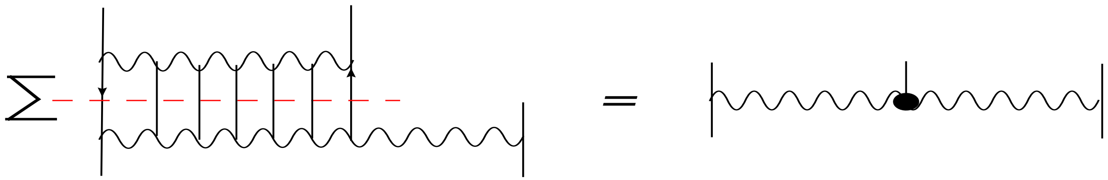
$A_6 =$ 

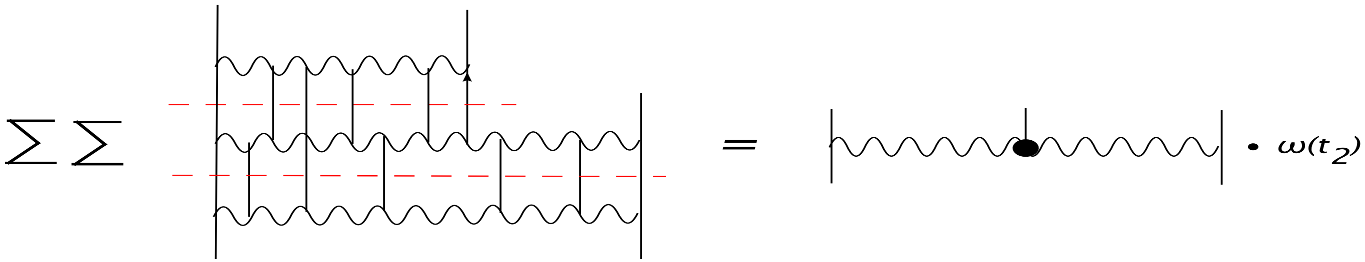
Again: sum of double discontinuties .
Analytic representation: all phases are in energy and signature factors.
Similarly $3 \to 3$: 5 terms.

Number of terms grows: $2 \to 5$: 14 terms etc.

Systematics: hexagraphs (A.White).

Analytic representation can be used to compute all terms from (multiple) discontinuities. (JB, Nucl.Phys.B 151 and B 175; Fadin,Lipatov, Nucl.Phys.406). Example:





Bootstrap relations: known from BFKL. Hold for inelastic amplitudes.

Bootstrap relations are valid beyond leading order.

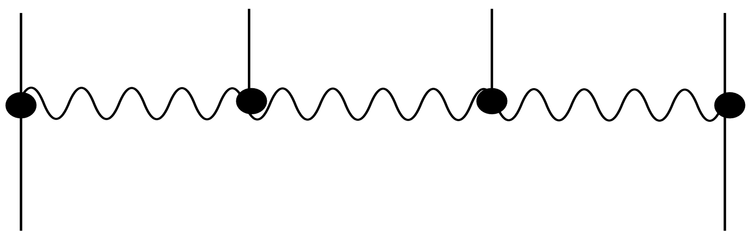
High degree of selfconsistency.

Results for QCD: five partial waves, e.g. the first term

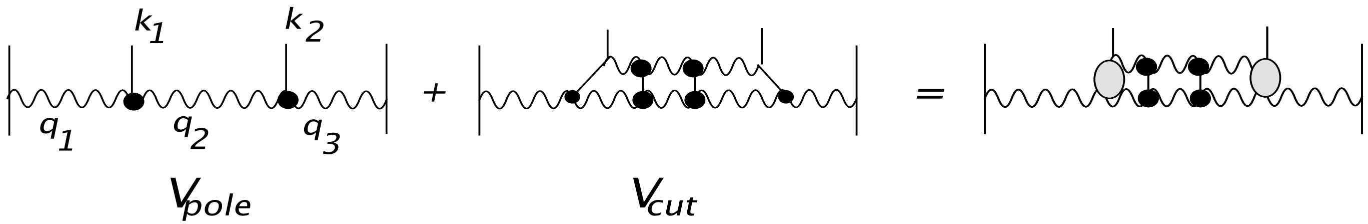
$$\frac{g^2 s}{t_1 t_2 t_3}\Big[\,\left(\frac{s_{12}}{\mu^2}\right)^{\omega(t_1)-\omega(t_2)}\left(\frac{s_{123}}{\mu^2}\right)^{\omega(t_2)-\omega(t_3)}\left(\frac{s}{\mu^2}\right)^{\omega(t_3)}\xi(t_1,t_2)\xi(t_2,t_3)\xi(t_3)\cdot$$

$$\frac{\omega(t_3)}{4}\left(\frac{a}{\epsilon}+\omega(t_1)-\omega(t_2)-a\ln\frac{\kappa_{12}}{\mu^2}\right)\cdot\left(\frac{a}{\epsilon}+\omega(t_2)-\omega(t_3)-a\ln\frac{\kappa_{23}}{\mu^2}\right)\Big]$$

Belongs to Regge pole picture:



New feature appears for terms 3 and 4:
contains not only gluon Regge pole but also Regge cut:



Combine the two contributions:

$$s_2^{\omega(t_2)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s_2}{\mu^2}\right)^{\omega} \widetilde{f}_2(\omega),$$

$$\widetilde{f}_2(\omega) = \hat{\alpha}_\epsilon\, \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k\, d^{2-2\epsilon}k'\, \Phi_1(\boldsymbol{k}, \boldsymbol{q}_2, \boldsymbol{q}_1)\, \widetilde{G}_\omega(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q}_2)\, \Phi_3(\boldsymbol{k}', \boldsymbol{q}_2, \boldsymbol{q}_3)\,.$$

$$\widetilde{f}_2 = \frac{a}{2}\left(\ln\frac{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}{(\boldsymbol{k}_1+\boldsymbol{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right) + \frac{a^2}{2}\,\ln s_2\,\ln\frac{|q_1-q_3|^2|q_2|^2}{|q_1|^2|k_2|^2}\,\ln\frac{|q_1-q_3|^2|q_2|^2}{|q_3|^2|k_1|^2} + \quad \ldots\,.$$

Note: only the one-loop approximation is singular (important for comparison with BDS)

Exact solution of the octet BFKL equation:

$$G_\omega(\vec{k},\vec{k}';\vec{q}) = \frac{1}{2\pi^2}\frac{|q|^2}{|k|^2||q-k|^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} d\nu\,\frac{f^*_{\nu n}(\vec{k}',\vec{q}')\,f_{\nu n}(\vec{k},\vec{q})}{\omega-\omega(\nu,n)}\,,$$

$$\omega_n(\nu,n) = \frac{g^2\,N_c}{2\pi^2}\left(2\psi(1) - \Re\,\psi\left(1+i\nu+\frac{n}{2}\right) + \Re\psi\left(1+i\nu-\frac{n}{2}\right)\right)\,.$$

Leading eigenvalue (at $\nu=0$): $\omega(0,n=1)=4\ln 2-2>0$ ($\longrightarrow$ Odderon).
(Singular term: leading eigenvalue $\omega(0,0)$ =0.)
Möbius invariance in dual variables ($\longrightarrow$ dual conformal symmetry?).

Comments:

- Regge cut piece violates factorization
- Regge cut piece is present in several discontinuities, e.g. in total energy s, but not in all discontinuities.
- Regge cut piece present in all A_n with $n>5$, e.g. $3\rightarrow 3$.

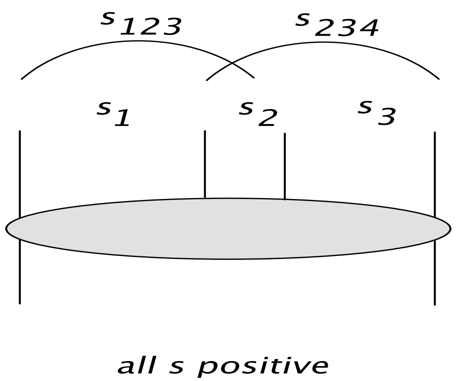
Sum the 5 different pieces and obtain the full scattering amplitudes A_n:

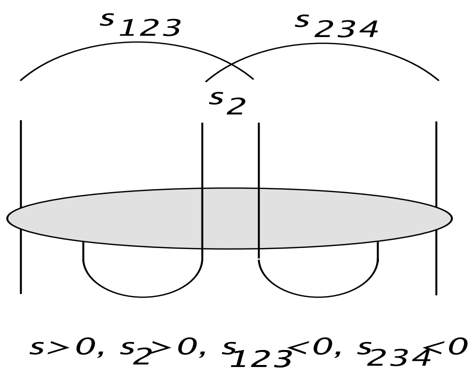
Leading order: many cancellations, real-valued expression factorizes (see above).

Sum of all imaginary parts ($=$ sum of discontinuities in different variables):
again substantial cancellations:

- in physical region (where all energies are positive),
  the Regge cut piece cancels, simple factorizing structure is valid .
- But: in another physical region $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$
  the cancellation of all imaginary parts is incomplete,
  Regge cut piece appears, factorization is violated .

Planar approximation: has only right hand cuts.
But still allows different physical regions:





Comparison with BDS formula

After removal of color factors from the scattering amplitude

$$tr(T^{a_1}...T^{a_n})A_n + noncycl.perm,$$

factor out the tree amplitude:

$$A_n = A_n^{tree} \cdot M_n(\epsilon)$$

Conjecture:

$$\ln M_n = \sum_l a^l \Big[\Big(f^{(l)}(\epsilon) I_n(l\epsilon) + F_n(0) \Big) + C^{(l)} + E_n^{(l)}[\epsilon] \Big]$$

$$a = \frac{N_c \alpha}{2\pi}(4\pi e^{-\gamma})^{\epsilon}, \;\; d = 4 - 2\epsilon$$

(based upon universality of IR singularities (=poles in ϵ) and unitarity, verified in 1-loop).

General strategy:

our analysis has been done for $\ln M$, discarding terms which vanish as $\epsilon \longrightarrow 0$.

Start from region where all invariants are negative, take multiregge limit.

Then, by analytic continuation, compare with previous result in different physical regions (all at large-N_c, MHV).

All our results for the scattering amplitude M_n are valid up to a factor

$$M_n = \ldots.. \left(1 + \mathcal{O}(\epsilon)\right)$$

(important for comparison with fixed order NLO calculations).

The four point amplitude: (Korchemsky,...)

$$\ln M_4 = 2 \ln \Gamma(t) + \omega(t) \ln(-s)/\mu^2$$

$$M_4 = \Gamma(t) \left(\frac{-s}{\mu^2} \right)^{\omega(t)} \Gamma(t)$$

- No squares of $\ln s$
- one loop expression for Γ
  and two-loop expression for $\omega(t)$ agree with explicit calculations
- exact: can also be written in 'dual' t-channel form (no high energy approximation).

The five point amplitude:

In $\ln M_5$: terms with squares of logarithms cancel. New production vertex:

$$M_{2\to3} = \Gamma(t_1)\,\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\,\Gamma(t_2,t_1,\ln-\kappa)\,\left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)}\,\Gamma(t_2)$$

with

$$-\kappa = \frac{(-s_1)(-s_2)}{(-s)}$$

Representation is exact.

Analytic continuation to positive energies:

$$-s \to e^{-i\pi}s, \quad \ln(-\kappa) \to \ln\kappa - i\pi, \quad \kappa = \boldsymbol{k}^2$$

Amplitude can be written in the analytic form:

$$\frac{M_{2\to3}}{\Gamma(t_1)\Gamma(t_2)} = \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)-\omega(t_2)}\left(\frac{-s}{\mu^2}\right)^{\omega(t_2)}c_1 + \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)-\omega(t_1)}\left(\frac{-s}{\mu^2}\right)^{\omega(t_1)}c_2,$$

with real-valued functions c_1, c_2. Consistency check: the region $s_{12}, s_{23} < 0$.

The six point amplitude: $T_{2\to4}$

In the unphysical region (all energies negative):

$$\frac{M_{2\to4}}{\Gamma(t_1)\Gamma(t_3)} = \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma(t_2, t_1, \ln -\kappa_{12}) \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma(t_3, t_2, \ln -\kappa_{23}) \left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}$$

with

$$-\kappa_{12} = \frac{(-s_1)(-s_2)}{-s_{012}}, \quad -\kappa_{23} = \frac{(-s_2)(-s_3)}{-s_{123}}.$$

The same functions $\Gamma(t)$ and $\Gamma(t_1, t_2, \kappa)$ as before.

Analytic continuation: inconsistency appears .
Can be seen in several different ways:

(a) attempt to write as a sum of five terms with real-valued functions c_i (use also the other physical region: $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$): no solution for the c_i.

(b) comparison with the earlier QCD results: in the region $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$, one should see the Regge cut piece. The BDS formula yields:

$$C = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi \left(\ln\frac{(-t_1)(-t_3)}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right)\right]$$

$$\approx 1 + i\pi a \left(\ln\frac{(-t_1)(-t_3)}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right).$$

agrees with the one loop approximation to the Regge cut piece,
but BDS cannot reproduce the full Regge cut structure

Important: the higher order terms in the Regge cut are not singular in ϵ and are not in conflict with the infrared structure of the BDS formula.

Outlook: results and tasks

What has been achieved, by comparison with explicit QCD calculations:

- BDS ok for 4 and 5 point amplitude. Regge limit is even exact.
- subtle disagreement for M_n for $n \geq 6$ beyond one loop.
- in general, expect no simple exponential form. What instead?

Can we correct the formula? Reasons for being optimistic:

- many features of the BDS formula seem already to be correct (infrared and beyond)
- structure seen in the Regge limit may not be too far from general kinematics
- experience from analyzing QCD in Regge limit: structures seen in leading log (bootstrap, unitarity properties) may survive in higher order