Production amplitudes and integrability in $N = 4$ SUSY

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Content

1 Gluon reggeization

Regge kinematics of scattering amplitudes

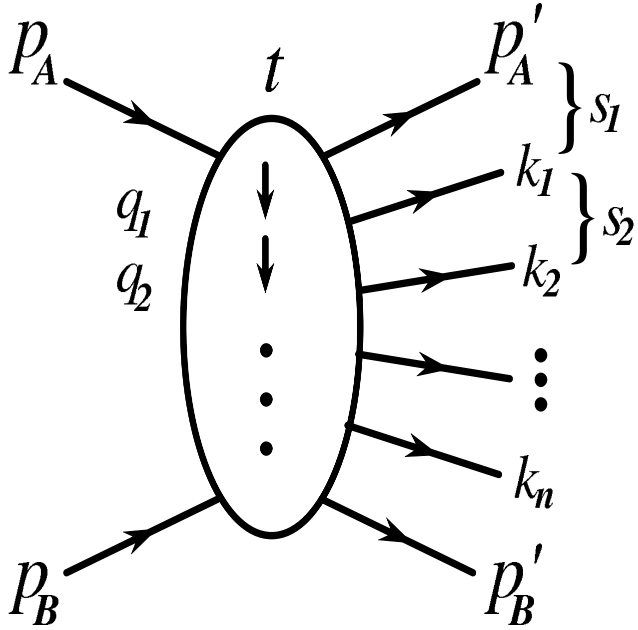
$$s = 4E^2 \gg -t = |q|^2 \approx E^2\,\theta^2$$

Elasic amplitude in LLA of QCD

$$M_{AB}^{A'B'}(s,t)|_{LLA} = 2\,g\,T^c_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\,\frac{s^{1+\omega(t)}}{t}\,g\,T^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2k\,\frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}$$



Multi-Regge amplitudes (F.,K.,L. (1975))

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, g T^{d_1}_{c_2 c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... C(q_n,q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2},$$

$$C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\;\; \sigma_t = \sum_n \int d\Gamma_n \left|M_{2\to 1+n}\right|^2$$

2 Effective action approach

Gluon and Reggeized gluon fields

$$v_\mu(x) = -iT^a v_\mu^a(x)\,,\;\; A_\pm(x) = -iT^a A_\pm^a(x)$$

Local gauge transformations

$$\delta v_\mu(x) = \frac{1}{g}[D_\mu, \chi(x)],\; \delta\psi(x) = -\chi(x)\,\psi(x),\; \delta A_\pm(x) = 0$$

Effective action for reggeized gluons (L., 1995)

$$S = \int d^4x \left(L_0 + L_{ind}^{GR}\right)\,,\;\; L_0 = i\bar{\psi}\hat{D}\psi + \frac{1}{2}\,Tr\,G_{\mu\nu}^2$$

$$L_{ind}^{GR} = -\frac{1}{g}\partial_+\,P\exp\left(-g\frac{1}{2}\int_{-\infty}^{x^+} v_+(x')d(x')^+\right)\,\partial_\sigma^2 A_- + (+\to -)$$

3 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\; \sigma_t \sim s^{\Delta}\;,\;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln\,|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^*$$

$$+\frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 - 4\psi(1)\,,\;\; \rho_{12} = \rho_1 - \rho_2$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,$$

$$m = \gamma + n/2\,,\;\; \widetilde{m} = \gamma - n/2\,,\;\; \gamma = 1/2 + i\nu$$

4 BKP equation (1980)

Bartels-Kwiecinski-Praszalowicz equation

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)\ ,\ \ H = \sum_{k<l} \frac{\vec{T}_k \vec{T}_l}{-N_c}\,H_{kl}$$

Holomorphic separability at large N_c (L. (1988))

$$H = h + h^*\ ,\ \ h = \ln p_1 + \ln p_2 + \frac{1}{p_1}(\ln \rho_{12})\,p_1 + \frac{1}{p_2}(\ln \rho_{12})\,p_2 - 2\psi(1)$$

Holomorphic factorization of wave functions

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\,\Psi_r(\rho_1, ..., \rho_n)\,\Psi_s(\rho_1^*, ..., \rho_n^*)$$

5 Integrability at $N_c \to \infty$

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \; T(u) = A(u) + D(u)\,,$$

$$[T(u), T(v)] = [T(u), h] = 0\,, \; L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix}$$

Yang-Baxter equation (L. (1993))

$$t_{r'_1}^{s_1}(u)\, t_{r'_2}^{s_2}(v)\, l_{r_1 r_2}^{r'_1 r'_2}(v-u) = l_{s'_1 s'_2}^{s_1 s_2}(v-u)\, t_{r_2}^{s'_2}(v)\, t_{r_1}^{s'_1}(u)\,, \; \hat{l} = u\,\hat{1} + i\,\hat{P}$$

Duality symmetry (L. (1999))

$$p_r \to \rho_{r+1,r} \to p_{r+1}$$

6 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\;\hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1)\;-\;\frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

7 Maximal transcedentality for γ

Three loop anomalous dimension (K.,L.,O.,V (2002-2004))

$$\gamma(j) = \hat{a}\gamma_1(j) + \hat{a}^2\gamma_2(j) + \hat{a}^3\gamma_3(j) + \ldots\,,\;\; \gamma_1(j+2) = -4S_1(j)$$

$$\frac{\gamma_2(j+2)}{8} = 2S_1\left(S_2 + S_{-2}\right) - 2S_{-2,1} + S_3 + S_{-3}$$

$$\gamma_3(j+2)/32 = -12\,\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right)$$

$$+6\,\left(S_{-4,1} + S_{-3,2} + S_{-2,3}\right) - 3\,S_{-5} - 2\,S_3\,S_{-2} - S_5$$

$$-2\,S_1^2\left(3\,S_{-3} + S_3 - 2\,S_{-2,1}\right) - S_2\left(S_{-3} + S_3 - 2\,S_{-2,1}\right)$$

$$+24\,S_{-2,1,1,1} - S_1\left(8S_{-4} + S_{-2}^2 + 4S_2S_{-2} + 2S_2^2\right)$$

$$-S_1\left(3S_4 - 12S_{-3,1} - 10S_{-2,2} + 16S_{-2,1,1}\right)\,,$$

$$S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j}\frac{1}{m^a}\,S_{b,c,\cdots}(m)\,,\;\; S_{-a,b,\cdots}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a}\,S_{b,\cdots}(m),$$

Integrability of the RG equations in $N=4$ SUSY (L. (1997))

8 Maximal helicity violation

BDS amplitudes in $N=4$ SUSY at $N_c \gg 1$ (2005)

$$A^{a_1,...,a_n} = \sum_{\{i_1,...,i_n\}} Tr\, T^{a_{i_1}} T^{a_{i_2}} ... T^{a_{i_n}}\, f(p_{i_1}, p_{i_2}, ..., p_{i_n})\,,\;\; f = f_B\, M_n$$

Invariant amplitudes

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) \left(-\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + F_n^{(1)}(0) \right) + C^{(l)} \right)\,,$$

$$a = \frac{\alpha N_c}{2\pi} \left(4\pi e^{-\gamma}\right)^{\epsilon}\,,\;\; C^{(1)} = 0\,,\;\; C^{(2)} = -\zeta_2^2/2\,,\;\; f^{(l)}(\epsilon) = \sum_{k=0}^{2} \epsilon^k\, f_k^{(1)}$$

Cusp anomalous dimension

$$f_0^{(l)} = \frac{1}{4}\gamma_K^{(l)}\,,\;\; f_1 = \beta(f) = -a\zeta_3/2 + a^2(2\zeta_5 + 5\zeta_2\zeta_3/3) + ...$$

9 Elastic BDS amplitude

Its Regge asymptotics at $s/t \to \infty$

$$M_{2\to 2} = \Gamma(t) \left(\frac{-s}{\mu^2}\right)^{\omega(t)} \Gamma(t)$$

Reggeized gluon trajectory

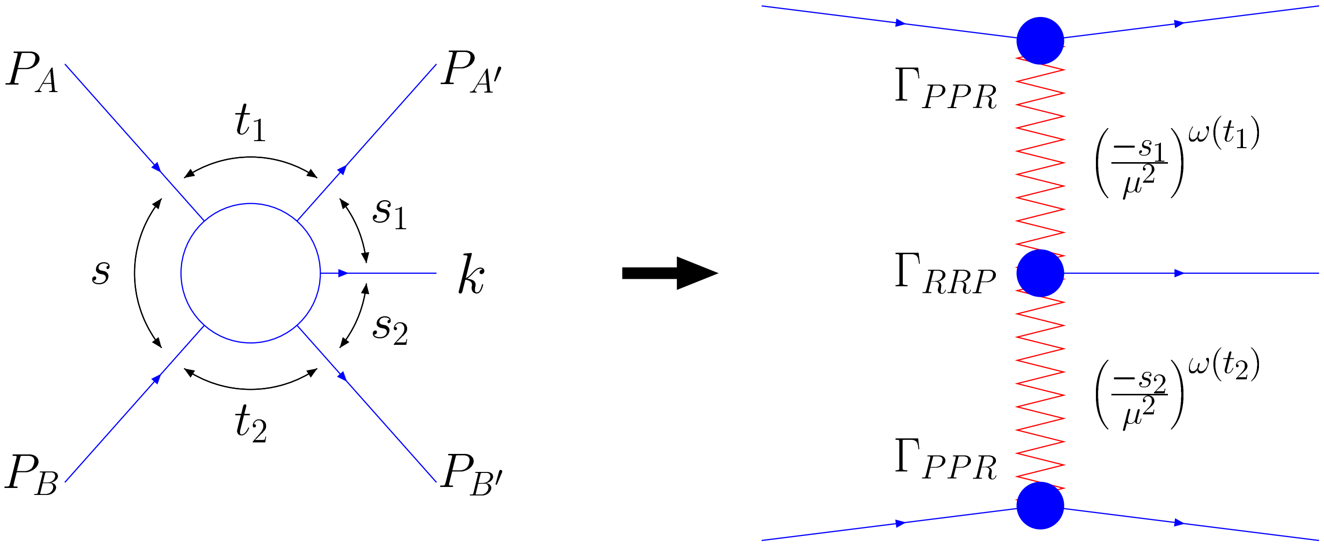
$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a')\right)$$

Reggeon residues

$$\ln \Gamma(t) = \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2}\right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2}\zeta_2$$

$$- \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a')\right)$$

10 One particle production



$$\ln \Gamma_{\kappa=s_1 s_2/s} = -\frac{1}{2}\left(\omega(t_1)+\omega(t_2) - \int_0^a \frac{da'}{a'}\left(\frac{\gamma_K(a')}{4\epsilon}+\beta(a')\right)\right)\ln\frac{-\kappa}{\mu^2} - \frac{\gamma_K(a)}{16}\left(\ln^2\frac{-\kappa}{\mu^2} - \ln^2\frac{-t_1}{-t_2} - \zeta_2\right) - \frac{1}{2}\int_0^a \frac{da'}{a'}\ln\frac{a}{a'}\left(\frac{\gamma_K(a')}{4\epsilon^2}+\frac{\beta(a')}{\epsilon}+\delta(a')\right)$$

11 Steinman relations

Overlapping channels

$$(s_1, s_2)\ (2 \to 3)\,;\ \ (s_1, s_2),\ (s_2, s_3),\ (s_{012}, s_2),\ (s_{123}, s)\ (2 \to 4)$$

Dispersion representation for $M_{2\to 3}$ in the Regge anzatz

$$M_{2\to 3} = c_1(-s)^{j(t_2)}(-s_1)^{j(t_1)-j(t_2)} + c_2(-s)^{j(t_1)}(-s_2)^{j(t_2)-j(t_1)}$$

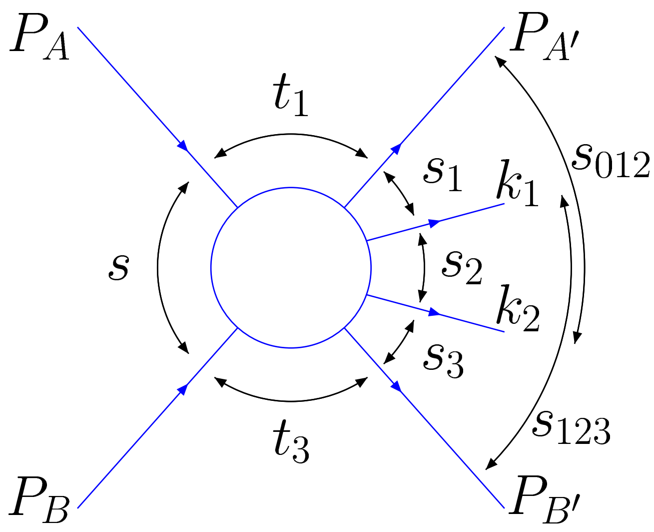
Violation of the dispersion representation for $M_{2\to 4}$

$$M_{2\to 4} \neq d_1(-s)^{j_3}(-s_{012})^{j_2-j_3}(-s_1)^{j_1-j_2} + d_2(-s)^{j_1}(-s_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$
$$+d_3(-s)^{j_3}(-s_{012})^{j_1-j_3}(-s_2)^{j_2-j_1} + d_4(-s)^{j_1}(-s_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}$$
$$+d_5(-s)^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2}\,,\ \ j_r = j(t_r)$$

Important relations

$$\Phi \equiv \frac{(-s)(-s_2)}{(-s_{012})(-s_{123})}\,,\ \ Li_2(1-\Phi)_{\Phi\to\exp(2\pi i)} = \ln(1-\Phi) \approx \ln\frac{(\vec{k}_1+\vec{k}_2)^2}{s_2}$$

12 Regge factorization violation



$$M_{2\to 4}|_{s_2>0;\, s_1,s_3<0} = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi\, \left(\ln \frac{t_1 t_2}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right)\right]$$

$$\times\Gamma(t_1)\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma(t_2,t_1)\left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma(t_3,t_2)\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)} \Gamma(t_3)$$

13 Mandelstam cuts in j_2-plane

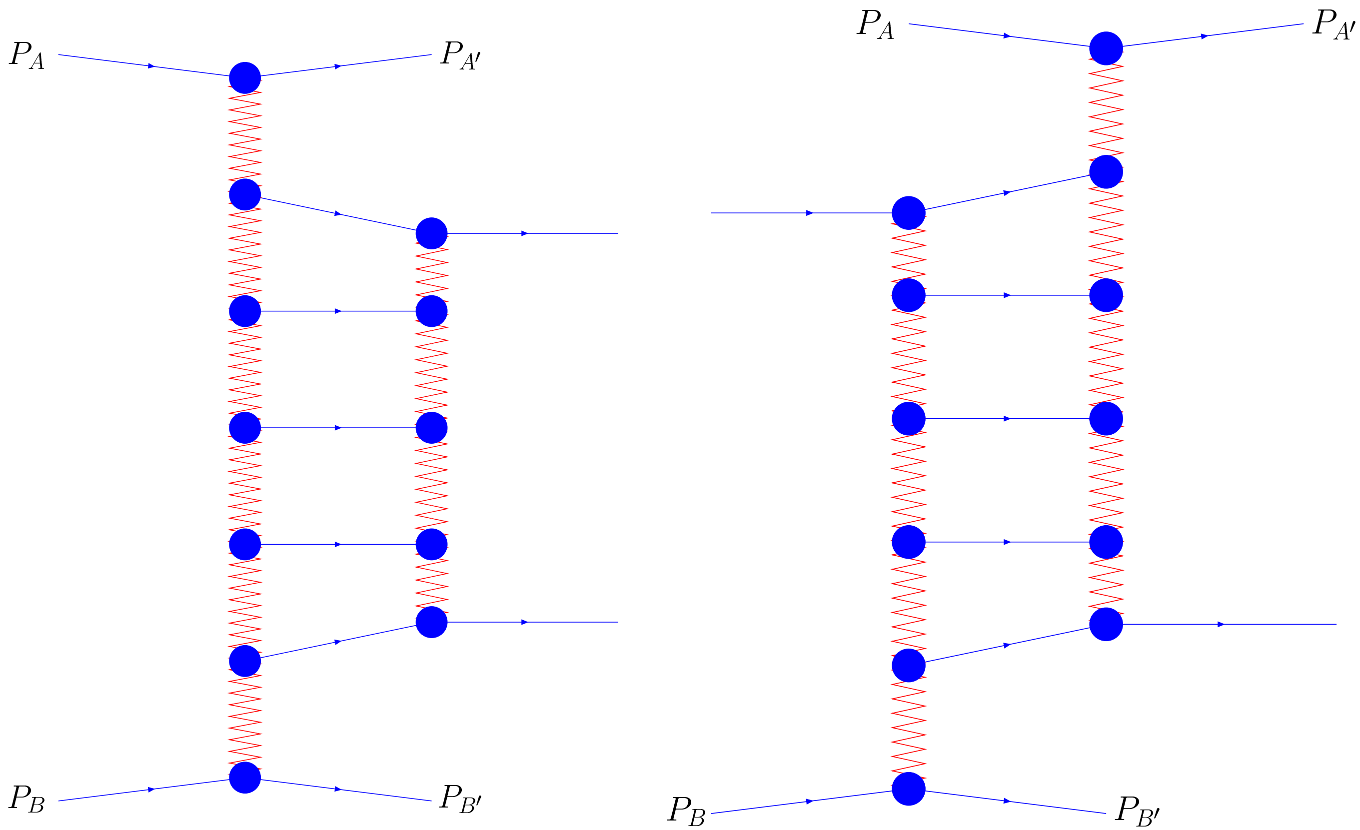


Figure 1: BFKL ladders in $M_{2\to4}$ and $M_{3\to3}$

14 BFKL equation in octet channels

Factorization of infrared divergencies in LLA

$$\lim_{\epsilon \to 0} M_{2\to 4}^{LLA} = f_{2\to 4}^{LLA} \lim_{\epsilon \to 0} M_{2\to 4}^{BDS} \,,$$

Renormalization of the intercept in the s_2-channel

$$\Delta_2 = -a \left(E + \ln \frac{t_2}{\mu^2} - \frac{1}{\epsilon} \right)$$

BFKL hamiltonian for the partial wave f_{j_2}

$$H = \ln \frac{|p_1 p_2|^2}{|p_1 + p_2|^2} + \frac{1}{2}\, p_1 p_2^* \ln |\rho_{12}|^2 \frac{1}{p_1 p_2^*} + \frac{1}{2}\, p_1^* p_2 \ln |\rho_{12}|^2 \frac{1}{p_1^* p_2} + 2\gamma$$

Eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left(\frac{p_1}{p_2} \right)^{i\nu + n/2} \left(\frac{p_1^*}{p_2^*} \right)^{i\nu - n/2} , \; E_{n,\nu} = 2Re\, \psi(i\nu + \frac{|n|}{2}) - 2\psi(1)$$

15 Multi-gluon states in octet channels

Equation for n-gluon composite states

$$\frac{1}{2}(h+h^*)\Psi = E\Psi\,,\;\; h = \ln(Z_1^2\partial_1) - 2\psi(1) + \ln\partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}$$

Pair hamiltonian of the spin chain

$$h_{1,2} = \ln(Z_{12}^2\partial_1) + \ln(Z_{12}^2\partial_2) - 2\ln Z_{12} - 2\psi(1)$$

Monodromy matrix

$$t(u) = L_1(u)L_2(u)...L_{n-1}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Integrals of motion and Baxter equation for the open spin chain

$$[D(u), h] = 0\,,\;\; D(u)Q(u) = (u-i)^{n-1}Q(u-i)$$

16 Three-gluon composite state

Integrals of motion for three gluons

$$\hat{\lambda} = -i(Z_1 p_1 + Z_2 p_2)\,,\ \hat{q} = -Z_1 Z_{12} p_1 p_2$$

Two independent eigenfunctions

$$\Psi^{\pm}_{\lambda\, q} = Z_2^{\lambda} \sum_{\mu-\lambda_{\pm}=0}^{\infty} \frac{\Gamma(\mu)\Gamma(\mu-\lambda)\,(Z_1/Z_2)^{\mu}}{\Gamma(\mu-\lambda_{+}+1)\Gamma(\mu-\lambda_{-}+1)}\,,\ \lambda_{\pm} = \frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4}+q}$$

Holomorphic factorization of the wave function

$$\Psi = \sum_{r=\pm} C_r \Psi^{r}_{\lambda\, q}(Z_1, Z_2)\Psi^{r}_{\widetilde{\lambda}\, \widetilde{q}}(Z_1^*, Z_2^*)$$

Quantization of λ and q from the single-valuedness for Ψ

$$\lim_{Z_1/Z_2 \to 1} \Psi\,,\ \lim_{Z_1/Z_2 \to \infty} \Psi$$

17 Discussion

1. Effective action for reggeized gluons.
2. Integrability of BFKL dynamics in LLA.
3. Remarkable properties of NLLA in $N = 4$ SUSY.
4. NNLLA corrections to the BFKL kernel.
5. BDS amplitudes in the multi-Regge kinematics.
6. Breakdown of the Regge factorization.
7. Mandelstam cuts in the planar amplitudes M_n for $n > 5$.
8. Integrable open spin chain for scattering amplitudes.