ENTROPY FLOWS OF A PERFECT FLUID IN (1+1) HYDRODYNAMICS

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Goal

- The formulation of (1+1) hydrodynamics in terms of suitable potentials

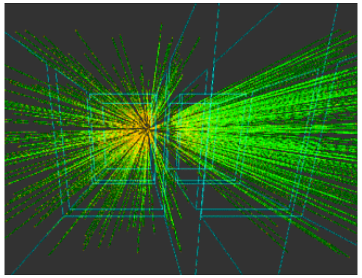
- The extraction of general and exact relations for the entropy flows

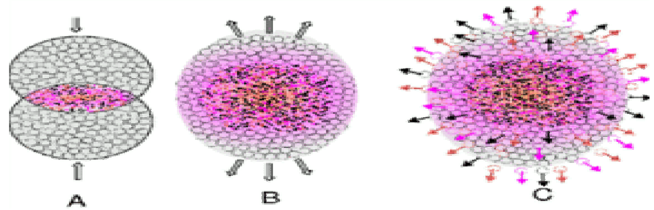
Talk Plan

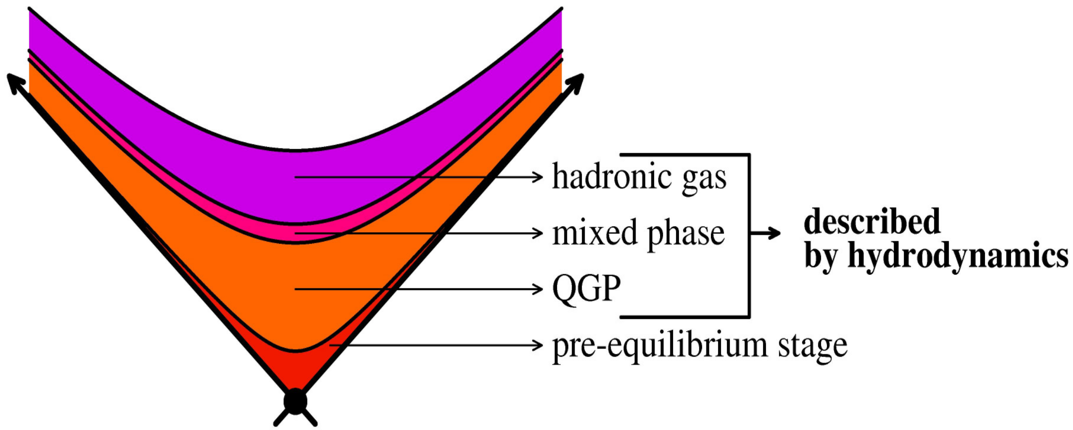
- 1) Introduction (why hydrodynamics?)
- 2) Perfect fluid relativistic hydrodynamics (potential formulation)
- 3) Entropy flow: An exact formula
- 4) General properties of the entropy flow
- 5) Physics of (1+1) hydrodynamics: comparison with experimental data
- 6) Conclusions-Prospects

1) Introduction

- Accumulating evidence that hydrodynamics may be relevant for the description of medium created in heavy-ion collisions







- Perfect fluid (low viscosity)
- Longitudinal motion

2) Perfect Fluid relativistic hydrodynamics

$$T^{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} - p\eta^{\mu\nu},$$

$$\partial_{\mu}T^{\mu\nu} = 0$$

$$p + \varepsilon = Ts$$

$$d\varepsilon = Tds$$

$$dp = sdT$$

$$\frac{dp}{d\varepsilon} = \frac{sdT}{Tds} = c^2(\varepsilon)$$

2) Perfect Fluid (1+1) relativistic hydrodynamics: Formulation

- Use light-cone variables: $z_{\pm} = t \pm z = \tau e^{\pm\eta}$
- Where: $\tau = \sqrt{z^+ z^-}$ the proper time and $\eta = \frac{1}{2}\ln\frac{z^+}{z^-}$ the space-time rapidity
- $u^{\pm} = u^0 \pm u^1 = e^{\pm y} \Rightarrow y = \ln u^+$
- Solutions: For $c^2 = const = 1/g$.
- In the $y = \eta$ case (boost invariant): $T = T_0 \tau^{-1/g}$ $\varepsilon = \varepsilon_0 \tau^{-(g+1)/g}$ [J.Bjorken '83]
- Landau asymptotic flow [Landau '55]
- The generalization of both (harmonic flows: $\partial_+ \partial_- y = 0$) [A.Bialas,R.Janik,R.Peschanski '07]

2) Perfect Fluid (1+1) relativistic hydrodynamics:
Existence of a potential

- Making life simpler! $\theta = \ln \frac{T}{T_0}$

- 1) $\partial_+\left(e^{\theta+y}\right) = \partial_+\left(e^{\theta-y}\right) \Rightarrow \partial_\pm \Phi = e^{\theta \pm y}$ Existence of a potential

- 2) $\partial_+\left(e^{y}s\right) + \partial_-\left(e^{-y}s\right) = 0$ Entropy flow conservation

- 3) Going from (z^+, z^-) to (θ, y)-base: Legendre transform:

$$\chi \equiv \Phi - z^-\left(e^{\theta+y}\right) - z^+\left(e^{\theta-y}\right)$$

- Thus:

$$c_s^2 \partial_\theta^2 \chi(\theta, y) + [1 - c_s^2]\partial_\theta \chi(\theta, y) - \partial_y^2 \chi(\theta, y) = 0$$

Very Good!

Khalatnikov equation [Khalatnikov '54, Landau '55]

2) Perfect Fluid (1+1) relativistic hydrodynamics: Solution

- Solution for $c^2 = const = 1/g$:

$$\chi(\theta, y) = e^{-\left(\frac{g-1}{2}\right)\theta} \int_{\theta}^{-\frac{y}{\sqrt{g}}} I_0\left(\frac{g-1}{2}\sqrt{\theta'^2 - y^2/g}\right) K(\theta - \theta')\, d\theta'$$

the function K carries the information of initial/boundary conditions

3) Entropy flow derivation: fixed-temperature surface

- Interested in $\frac{dS}{dy}(y)$ at fixed temperature T_F : $\frac{dS}{dy}(y) = s_F u^\mu n_\mu \frac{d\lambda}{dy}$

  $d\lambda$: infinitesimal length element along hyper-surface

  n^μ : normal to the hyper-surface

- In (θ, y)-base, the proper time τ and the space-time rapidity η write:

$\tau_F(y) = \tau(\theta_F, y)$ $\eta_F(y) = \eta(\theta_F, y)$

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- Defining first the tangent vector, and then the normal we finally obtain:

$$\frac{dS}{dy}(y) = s_F\left[\tau_F(y)\eta_F{}'(y)\cosh(\eta_F(y) - y) + \tau_F{}'(y)\sinh(\eta_F(y) - y)\right]$$

4) General properties of the entropy flow

- Finally:

$$\frac{dS}{dy}(y) = \frac{s_F c^2}{2} e^{-\theta_F} \left[\partial_\theta^2 \chi(\theta, y) - \partial_\theta \chi(\theta, y) \right] |_{\theta=\theta_F}$$

[G.Beuf, R.Peschanski, E.N.S '08]

Allows for investigation of the general features of the entropy flow of (1+1) hydrodynamic evolution, in terms of temperature and rapidity.

4) Properties of the entropy flow :
Total Entropy- Universal flow properties

- Self consistency test: Total Entropy:

$$S_{tot}\,|_{\theta=\theta_F} = 2\int_0^{Y/2} \frac{dS}{dy}(y)\,|_{\theta=\theta_F}$$

$$\Rightarrow \quad S_{tot}\,|_{\theta=\theta_F} \approx s_F \sqrt{g}\, e^{-g\theta_F}\, \tilde{K}\left(\frac{g-1}{2}\right) = s_0 \sqrt{g}\, \tilde{K}\left(\frac{g-1}{2}\right)$$

since $s_F = s_0\left(\frac{T_F}{T_0}\right)^g \Rightarrow s_F e^{-g\theta_F} = s_0$

$\Rightarrow$ Entropy conservation

4) Properties of the entropy flow:

For flows dominated by hydrodynamic evolution:

$$\Rightarrow \quad \chi(\theta, y) = \frac{\sqrt{g}}{T_0} \int_{\theta}^{-\frac{y}{\sqrt{g}}} I_0\left(\frac{g-1}{2}\sqrt{\theta'^2 - y^2/g}\right) e^{\theta - \left(\frac{g+1}{2}\right)\theta'} d\theta'$$

- So:

$$\frac{dS}{dy}(y) = S_{tot}\frac{g-1}{2(g+1)\sqrt{g}} e^{\left(\frac{g-1}{2}\right)\theta_F}\left[I_0\left(\frac{g-1}{2}\sqrt{\theta_F{}^2 - y^2/g}\right) - I_1\left(\frac{g-1}{2}\sqrt{\theta_F{}^2 - y^2/g}\right)\frac{\theta_F}{\sqrt{\theta_F{}^2 - y^2/g}}\right]$$

5) Physics of (1+1) hydrodynamics:

comparison with experimental data (fixed energy $\sqrt{s}=200$ GeV)

$$\frac{dS}{dy}(y) = S_{tot}\frac{g-1}{2(g+1)\sqrt{g}}e^{\left(\frac{g-1}{2}\right)\theta_F}\left[I_0\left(\frac{g-1}{2}\sqrt{\theta_F^{\,2}-y^2/g}\right)-I_1\left(\frac{g-1}{2}\sqrt{\theta_F^{\,2}-y^2/g}\right)\frac{\theta_F}{\sqrt{\theta_F^{\,2}-y^2/g}}\right]$$

two-parameters: $\theta_F\,,\,g$

- Data: Multiplicity distribution at fixed energy [BRAHMS Collaboration '04]
- (1+1) assumptions: i) $\frac{dN}{dy}\propto\frac{dS}{dy}$ ii) The observed distribution is at fixed θ_F

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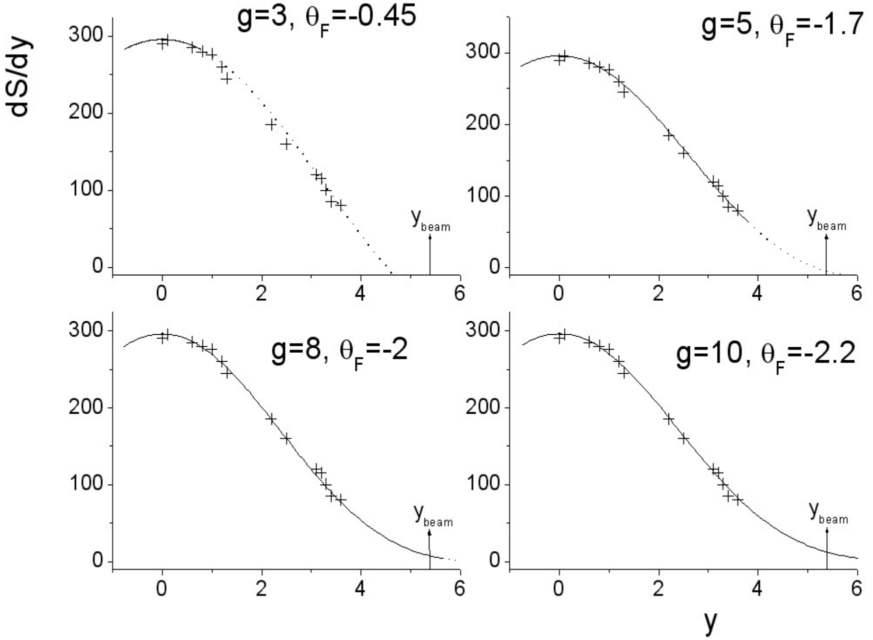

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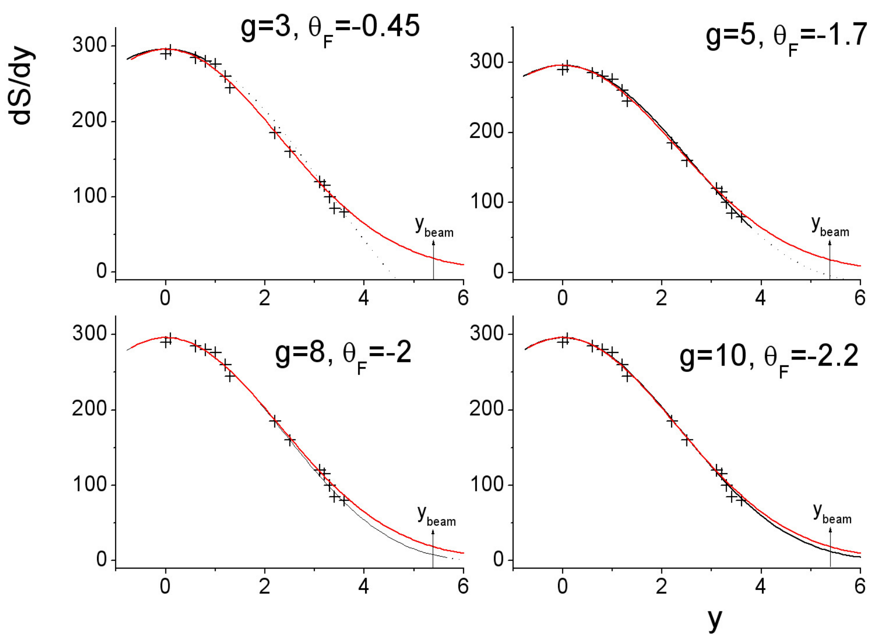

- Gaussian fit: $\frac{dS}{dy}(y)\approx Ae^{-y^2/Y}$ [P. Carruthers M.Van '72]

5) Physics of (1+1) hydrodynamics:
comparison with experimental data (energy dependence)

- Energy (Y) dependence of multiplicity distribution
- For a given pair of θ_F, g : $\frac{dS}{dy}(y) \approx A e^{-y^2/Y}$
- For a given g the $\theta_F - Y$ relation is clearly linear: $\theta_F = -\alpha(Y - Y_0)$

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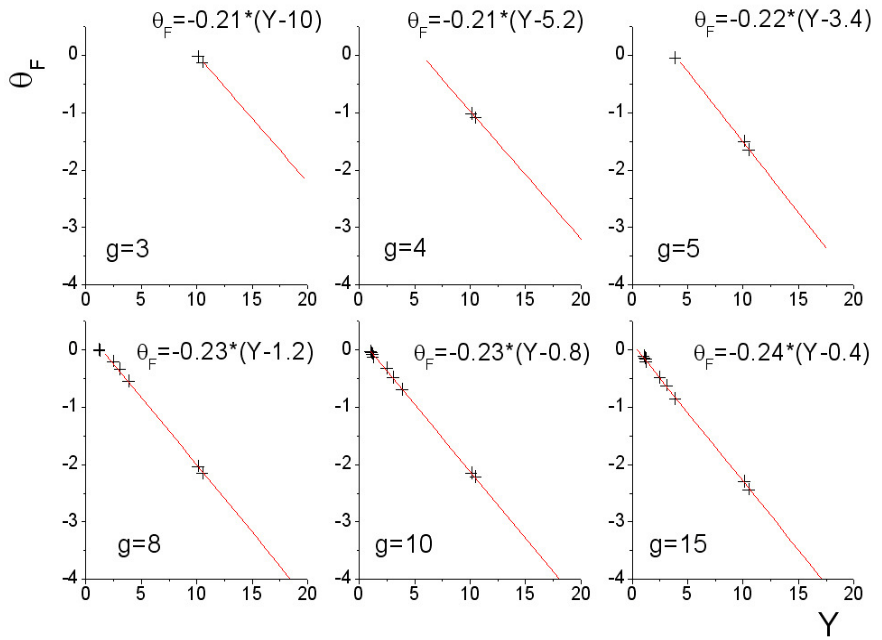

- Experimental data better fitted with smaller sound speed (larger g)
- $\frac{T_0}{T_F} = e^{-\theta_F} = e^{0.22\,(Y - Y_0)} \Rightarrow$

More energy available (larger Y) $\Rightarrow$ more hydrodynamic evolution

At smaller sound speed,the evolution has to occur in a wider temperature interval, to describe the same $\frac{dS}{dy}$

5) Physics of (1+1) hydrodynamics

- Linearity between θ_F and C.o.M energy:
- Allows for comparison between $\frac{dS}{dy}$ and 1-particle inclusive cross-section, for $A+A \to h+X$, and thus to the appropriate scattering amplitudes (not easy to formulate in hydrodynamic formalism).

- Useful to compare with various non-hydrodynamic theories and models applied to high-energy collisions.

Conclusions

- i) Khalatnikov potential χ allows to bypass the difficulties due to non-linearity of the equations [Landau, Khalatnikov '55]. Reformulation in light-cone variables.
- ii) It allows to provide a general, non-boost-invariant, solution for the entropy flow $\frac{dS}{dy}(y,T)$ of an expanding perfect fluid in (1+1)d [G. Beuf, R. Peschanski, E.N.S '08]
- iii) $\frac{dS}{dy}(y,T)$ factorizes the contributions of the hydrodynamic evolution and of
- initial conditions.

  For sufficiently long evolution (large $\frac{T_0}{T_F}$) it displays an explicit universal behavior (depending on $\frac{T_0}{T_F}$ and g)
- iv) $\frac{dS}{dy}(y,T)$ is in agreement with experimental $\frac{dN}{dy}(y,T)$,

  with a relation $\ln\left(\frac{T_0}{T_F}\right) = \alpha(Y - Y_0)$

Open questions

- i) What about entropy flow through other super-surfaces (e.g fixed proper time)?
- ii) What about viscosity and/or varying sound-speed?
- iii) Can we go beyond longitudinal motion?
- iv) Is there a "transition" in hydrodynamics?
- vi) AdS/CFT correspondence of the expanding plasma is for the moment restricted to boost invariance (Bjorken flow).

  Can we introduce the obtained hydrodynamic rapidity dependence in the related Einstein equations?

Appendix1: General solution of Khalatnikov equation

- $\chi(\theta,y)=e^{-\left(\frac{g-1}{2}\right)\theta}Z(\theta,y)\ \Rightarrow$ $\partial_\theta^2 Z - g\,\partial_y^2 Z - \left(\frac{g-1}{2}\right)^2 Z = 0$
- Laplace transforms: $\tilde{Z}(\gamma,y)=\int_{-\infty}^{0} d\theta\, e^{\gamma\theta} Z(\theta,y)$ $Z(\theta,y)=\int_{\gamma_0-i\infty}^{\gamma_0+i\infty}\frac{d\gamma}{2\pi i}e^{-\gamma\theta}\tilde{Z}(\gamma,y)$
- $\Rightarrow\ g\,\partial_y^2\tilde{Z}=\left[\gamma^2-\left(\frac{g-1}{2}\right)^2\right]\tilde{Z}$
- General solution: $\tilde{Z}(\gamma,y)=e^{-\frac{y}{\sqrt{g}}\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}\dfrac{\tilde{K}(\gamma)}{\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}$
- $$\Rightarrow\ \chi(\theta,y)=\int_{\gamma_0-i\infty}^{\gamma_0+i\infty}\frac{d\gamma}{2\pi i}\left[e^{-\left(\gamma+\frac{g-1}{2}\right)\theta-\frac{y}{\sqrt{g}}\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}\right]\frac{\tilde{K}(\gamma)}{\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}\ \Rightarrow$$
- $$h_1(\theta)=\int_{\gamma_0-i\infty}^{\gamma_0+i\infty}\frac{d\gamma}{2\pi i}e^{-\gamma\theta}\tilde{K}(\gamma)=\Theta(-\theta)K(\theta)$$

$$h_2(\theta)=\int_{\gamma_0-i\infty}^{\gamma_0+i\infty}\frac{d\gamma}{2\pi i}\frac{1}{\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}\left[e^{-\gamma\theta-\frac{y}{\sqrt{g}}\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}\right]=\Theta\left(-\theta-|y|/\sqrt{g}\right)I_0\left(\frac{g-1}{2}\sqrt{\theta^2-y^2/g}\right)$$

- $$\chi(\theta,y)=e^{-\left(\frac{g-1}{2}\right)\theta}\int_{\theta}^{-\frac{y}{\sqrt{g}}} I_0\left(\frac{g-1}{2}\sqrt{\theta'^2-y^2/g}\right)K(\theta-\theta')\,d\theta'$$

Appendix2: General solution of Khalatnikov equation

- $$\partial_\theta^2 Z - g\,\partial_y^2 Z - \left(\frac{g-1}{2}\right)^2 Z = 0$$
  $$\alpha = -\theta + \frac{y}{\sqrt{g}}\ ,\ \beta = -\theta - \frac{y}{\sqrt{g}}$$
  $$\Rightarrow\ \partial_\alpha \partial_\beta Z(\alpha,\beta) - \left(\frac{g-1}{4}\right)^2 Z(\alpha,\beta) = 0$$
- Green functions: $\partial_\alpha \partial_\beta \overline{G}(\alpha,\beta) - \left(\frac{g-1}{4}\right)^2 \overline{G}(\alpha,\beta) = \delta(\alpha)\delta(\beta)$
- $$\Rightarrow\ \overline{G}(\alpha,\beta) = \Theta(\alpha)\Theta(\beta)\, I_0\!\left(\frac{g-1}{2}\sqrt{\alpha\beta}\right)$$
- Using: $\delta(\alpha)\delta(\beta) = \delta(-\theta + \frac{y}{\sqrt{g}})\delta(-\theta - \frac{y}{\sqrt{g}}) = \sqrt{g}\,\delta(\theta)\delta(y)$

$$G(\theta,y) = \frac{1}{4\sqrt{g}}\overline{G}(\alpha,\beta) = \frac{1}{4\sqrt{g}}\Theta(-\theta + \frac{y}{\sqrt{g}})\Theta(-\theta - \frac{y}{\sqrt{g}})\, I_0\!\left(\frac{g-1}{2}\sqrt{\theta^2 - \frac{y^2}{g}}\right)$$

- Thus, we construct the general solution inserting distribution of sources: $F(\tilde{\theta},\tilde{y})$:

$$\chi(\theta,y) = e^{-\left(\frac{g-1}{2}\right)\theta}\int d\tilde{y}\int d\tilde{\theta}\, G(\theta-\tilde{\theta},\, y-\tilde{y})\, F(\tilde{\theta},\tilde{y}) = \frac{e^{-\left(\frac{g-1}{2}\right)\theta}}{4\sqrt{g}}\int d\tilde{y}\int_{\theta+|y-\tilde{y}|/\sqrt{g}}^{+\infty} d\tilde{\theta}\, F(\tilde{\theta},\tilde{y})\, I_0\!\left(\frac{g-1}{2}\sqrt{(\theta-\tilde{\theta})^2 - \frac{(y-\tilde{y})^2}{g}}\right)$$

- Evolution dominated solutions: $F(\tilde{\theta},\tilde{y}) = 4\sqrt{g}\,K(\tilde{\theta})\Theta(-\tilde{\theta})\delta(\tilde{y})$
- $\theta' \equiv \theta - \tilde{\theta}$

$$\Rightarrow\ \chi(\theta,y) = e^{-\left(\frac{g-1}{2}\right)\theta}\int_{\theta}^{-\frac{y}{\sqrt{g}}} I_0\!\left(\frac{g-1}{2}\sqrt{\theta'^2 - y^2/g}\right) K(\theta-\theta')\, d\theta'$$

Appendix3: Entropy flow derivation: fixed-temperature surface

- Fixed temperature T_F : $\frac{dS}{dy}(y) = s_F u^\mu n_\mu \frac{d\lambda}{dy}$ (1)

  $d\lambda$: infinitesimal length element along hyper-surface
  n^μ : normal to the hyper-surface [A.Bialas,R.Janik,R.Peschanski '07]

- $\left.\begin{array}{l} \tau_F(y) = \tau(\theta_F, y) \\ \eta_F(y) = \eta(\theta_F, y) \end{array}\right\}$ Fixed temperature surface

- Tangent vector: $\begin{cases} V^+(y) \equiv z_F^+{}'(y) = (\tau_F{}' + \eta_F{}'\tau_F)e^{\eta_F} \\ V^-(y) \equiv z_F^-{}'(y) = (\tau_F{}' - \eta_F{}'\tau_F)e^{-\eta_F} \end{cases}$

- $\Rightarrow$ Perpendicular vector: $\begin{cases} n^+(y)n^-(y) = 1 \\ \frac{1}{2}\left[n^+(y)V^-(y) + n^-(y)V^+(y)\right] = 0 \end{cases}$

  $\Bigg\}$ $n^+(y)e^{-\eta_F}(\eta_F{}'\tau_F - \tau_F{}') = n^-(y)e^{\eta_F}(\eta_F{}'\tau_F + \tau_F{}')$

- $\Rightarrow \begin{cases} n^+(y) = \sqrt{\frac{\eta_F{}'\tau_F + \tau_F{}'}{\eta_F{}'\tau_F - \tau_F{}'}}e^{\eta_F} \\ n^-(y) = \sqrt{\frac{\eta_F{}'\tau_F - \tau_F{}'}{\eta_F{}'\tau_F + \tau_F{}'}}e^{-\eta_F} \end{cases}$ (2)

- $d\lambda^\mu d\lambda_\mu = -dz_F^+ dz_F^- = -(\tau_F'^2 - \tau_F^2\eta_F'^2)(dy)^2 \Rightarrow d\lambda = \sqrt{\tau_F^2\eta_F'^2 - \tau_F'^2}\,dy$ (3)

- (2),(3) $\rightarrow$ (1) $\Rightarrow$ $\frac{dS}{dy}(y) = s_F\left[\tau_F(y)\eta_F{}'(y)\cosh(\eta_F(y) - y) + \tau_F{}'(y)\sinh(\eta_F(y) - y)\right]$

Appendix4:
Non-linear equations of motion

- Use light-cone variables: $z_{\pm} = t \pm z = \tau e^{\pm\eta}$
- Where: $\tau = \sqrt{z^+ z^-}$ the proper time and $\eta = \frac{1}{2}\ln\frac{z^+}{z^-}$ the space-time rapidity
- $u^{\pm} = u^0 \pm u^1 = e^{\pm y}$

$$\left(\frac{e^{2y}-1}{2}\right)\partial_+(\varepsilon+p)+e^{2y}(\varepsilon+p)\partial_+ y+\left(\frac{1-e^{-2y}}{2}\right)\partial_-(\varepsilon+p)+e^{-2y}(\varepsilon+p)\partial_- y+\partial_+ p-\partial_- p=0$$

$$\left(\frac{e^{2y}+1}{2}\right)\partial_+(\varepsilon+p)+e^{2y}(\varepsilon+p)\partial_+ y+\left(\frac{1+e^{-2y}}{2}\right)\partial_-(\varepsilon+p)-e^{-2y}(\varepsilon+p)\partial_- y-\partial_+ p+\partial_- p=0$$

- Solutions: Only for $c^2 = const = 1/g$ in the boost invariant case: $y = \eta$:

$T = T_0 \tau^{-1/g}$ $\quad$ $\varepsilon = \varepsilon_0 \tau^{-(g+1)/g}$ [J.Bjorken '83] or in the harmonic flow

[A.Bialas,R.Janik,R.Peschanski '07]

Appendix5:
General properties of the entropy flow

- Using the general solution for $\chi(\theta, y)$ our formula gives:

$$\frac{dS}{dy}(y)\Big|_{\theta=\theta_F} = \frac{s_F}{2g}\int_{\gamma_0-i\infty}^{\gamma_0+i\infty}\frac{d\gamma}{2\pi i}\, e^{-\left(\gamma+\frac{g+1}{2}\right)\theta_F}\left[(\gamma+g/2)^2-1/4\right]\tilde{K}(\gamma)\,\frac{e^{-\frac{y}{\sqrt{g}}\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}}{\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}$$

- Kernel: $Q(\gamma, y) = \dfrac{e^{-\frac{y}{\sqrt{g}}\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}}{\sqrt{\gamma^2-\frac{(g-1)^2}{4}}}$ (entropy flow due to hydrodynamic evolution)

- Coefficient function: $\left[(\gamma+g/2)^2-1/4\right]\tilde{K}(\gamma)$ (initial conditions of the entropy flow)

Allows for investigation of the general features of the entropy flow of any (1+1) hydrodynamic evolution, in terms of temperature and rapidity.

Appendix6:

$$z^{+} = \frac{1}{2} e^{y-\theta} \left(\partial_y \chi - \partial_\theta \chi \right) \qquad z^{-} = -\frac{1}{2} e^{-y-\theta} \left(\partial_y \chi + \partial_\theta \chi \right)$$

Good!